1

- $G=(V,E)$, $|V|=n$, $\mathcal{U}=\{u_1,\dots,u_n\}\subseteq S^{n-1}$ is <u>orthonormal representation</u> if $\{i,j\}\notin E \Rightarrow u_i^T u_j = 0$.
- Theta($\mathcal{U}$): $\vartheta(\mathcal{U}) := \min\limits_{\|c\|=1} \max\limits_{i=1}^{n} \dfrac{1}{(c^T u_i)^2}$ || $\vartheta(G) :=$ smallest $\vartheta(\mathcal{U})$, $\mathcal{U}$ orthonormal r.

$\boxed{\text{Thm (Lovász)} \quad 2^{\sigma(G)} = S(G) \le \vartheta(G).}$

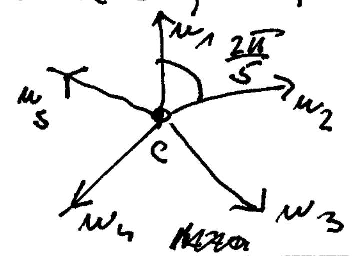
<u>We know</u>: $S(C_5) \geq \sqrt{5}$

Shannon Conjecture follows from

$\boxed{\text{Lemma (Lovász)}}$ $\vartheta(C_5) \le \sqrt{5}$

<u>Need</u>: orthonormal representation of C_5 with value $\le \sqrt{5}$.

- Do in S^2, lift to S^4 by $(\dots 0 0)$. $c := (0,0,1)$



Fold umbrella $\rightsquigarrow$

$$(*)\quad u_i = \frac{\left(\cos\frac{2\pi i}{5}, \sin\frac{2\pi i}{5}, z\right)}{\left\|\left(\cos\frac{2\pi i}{5}, \sin\frac{2\pi i}{5}, z\right)\right\|}, \quad i=1,\dots,5$$

$$0 = u_5^T u_2 \Leftrightarrow (1,0,z)\begin{pmatrix}\cos\frac{4\pi}{5}\\ \sin\frac{4\pi}{5}\\ z\end{pmatrix} = 0 \Rightarrow$$

$$z = \sqrt{-\cos\frac{4\pi}{5}} \qquad u_5 = \dots \quad \boxtimes$$

Folding leads to orthonormal repres. since first $\measuredangle(u_1,u_2) = \frac{4\pi}{5} > \frac{\pi}{2}$ and finally this $\measuredangle(u_1,u_4) = 0$. The value is $\le \sqrt{5}$ [not hard (*)]

SDP for the Theta function

Thm. $\vartheta(G)$ is the value of

$$(*)\quad \begin{aligned} &\text{Min } t\\ &y_{ij} = -1, \quad \{i,j\} \in \overline{E}\\ &y_{ii} = t-1, \quad i = 1,\dots,n\\ &Y \succeq 0 \end{aligned}$$

$\boxed{\text{Proof}}$ Let the value of $(*)$ be $\vartheta'(G)$.

(A) $\underline{\vartheta'(G) \le \vartheta(G)}$:

(u, c) optimal representation. Let $\tilde{Y} \in SYM_{nn}$ be: $\tilde{y}_{ij} := \dfrac{u_i^T u_j}{(c^T u_i)(c^T u_j)} - 1, \; i \ne j$

$\tilde{y}_{ii} := \vartheta(G) - 1, \quad i = 1,\dots,n$

- If $\boxed{\tilde{Y} \succeq 0}$ $\Rightarrow (\tilde{Y}, \tilde{t} = \vartheta(G))$ is a feasible sol. of $(*)$.

$\Rightarrow \vartheta'(G) \le \vartheta(G)$

$\tilde{Y} = D + U^T U$, D diagonal ≥ 0 and

$U^i = c - u_i / c^T u_i \rightarrow$ i-th column of U

Thus $\tilde{Y} \succeq 0$.

(B) $\underline{\vartheta'(G) \ge \vartheta(G)}$ a FACT

$\boxed{\text{Theorem}}$ $G = (V, E)$ $\quad \underline{\omega(\overline{G}) \le \vartheta(G) \le \chi(\overline{G})}$

We had last time: $\alpha(G) \le \vartheta(G)$

ω: size of a largest clique (complete subgraph) of G

α: size of a largest indep. set

$\chi(G)$: chromatic number.

$\vartheta(G) = 1 \Rightarrow G = K_n \Rightarrow \boxtimes$

$\vartheta(G) > 1 \Rightarrow \chi(\overline{G}) \ge 2$.

$\boxed{\text{Rescale } (*)}$

$$\begin{aligned} &\text{Min } t\\ &y_{ij} = -1/(t-1), \quad \{i,j\} \notin E\\ &y_{ii} = 1 \quad i = 1,\dots,n\\ &Y \succeq 0 \end{aligned}$$

Definition

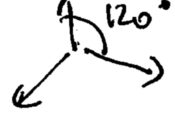
$k \in \mathbb{R}$, vector k-coloring of G is $f: V \to S^{n-1}$:

$$f(v)^T f(w) = -\frac{1}{k-1}, \quad \{v,w\} \in E$$

Lemma. If G has k-coloring then it has vector k-coloring.

[FACT]

$k = 3$



$\boxed{\text{Perfect Graphs}}$ G perfect if $\omega(G') = \chi(G')$ for each induced subgraph G'.

<u>Corollary</u>. • G perfect $\Rightarrow$ $\omega(G) = \vartheta(\bar{G}) = \chi(G)$

• G perfect $\Rightarrow$ $\chi(G)$ polynomial !! by SDP:

(*) suffices to solve with accuracy $\varepsilon < \frac{1}{2}$ since value is integer.

$\boxed{\text{The only known polynomial method}}$