$\boxed{\text{Max-Cut Problem}}$

$G = (V, E)$, $w : E \longrightarrow \mathbb{Q}$

$A \subseteq E \Rightarrow w(A) \overset{def}{=} \sum_{e \in A} w(e)$

$V' \subseteq V \Rightarrow \{e \in E;\ |e \cap V'| = 1\}$ is <u>edge-cut</u>

Find edge-cut of max weight.

- A basic NP-complete problem [unweighted]
- $w : E \to \mathbb{Q}^{-} \Rightarrow$ polynomial [max flow - min cut]
- Polynomial for graphs embeddable on any fixed surface and the weights polynomially bounded (in $|G|$)
- Seminal approximation alg. by Semi Def. Programming

Structure of edge-cuts

$G = (V, E)$ graph

I_G incidence matrix (over $\mathbb{F}_2 = (\{0,1\}, +, \cdot \bmod 2)$)

$$\begin{array}{cc} & \begin{array}{ccc} & \{x,y\} = e & \quad E \end{array} \\ V & \begin{array}{c|ccc|} & & 0 & \\ x & & 1 & \\ & & 0 & \\ y & & 1 & \\ & & 0 & \end{array} \end{array}$$

$A \subseteq E$ __even__ if (V, A) has all degrees even

Cycle Space (Prostor Cyklů)

$\mathcal{E}_G = \{A \subseteq E;\ A \text{ even}\}$

- $\mathcal{E}_G = \operatorname{Ker}_2(I_G)$
- $\mathcal{C}_G = \operatorname{Im}_2(I_G)$
- $\dim \mathcal{E}_G = |E| - |V| + \#\text{components}$

Cut Space (Prostor Řezů)

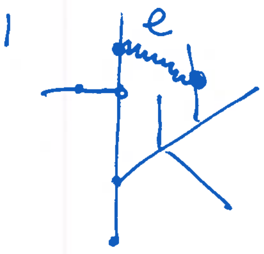
$\mathcal{C}_G = \{A \subseteq E;\ A \text{ edge-cut}\}$

- $\dim \mathcal{C}_G = |V| - \#\text{components}$

<u>basis of $\mathcal{E}_G$</u> : T : spanning tree of G, $e \in E \setminus T$,

C_e : unique cycle of $T \cup \{e\}$.

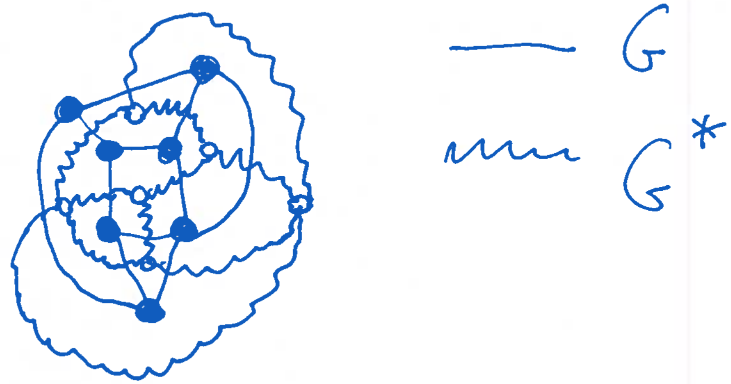
$B_T = \{ C_e \,;\; e \in E \setminus T \}$.



<u>basis of $\mathcal{C}_G$</u> : $B_0 = \{ \{ e \,;\; v \in e \} \,;\; v \in V \}$

G planar, G^* geometric dual $\Rightarrow$

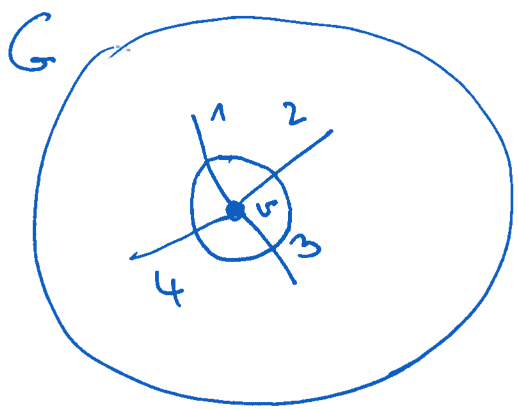
$$\mathcal{C}_G \simeq \mathcal{E}_{G^*}$$

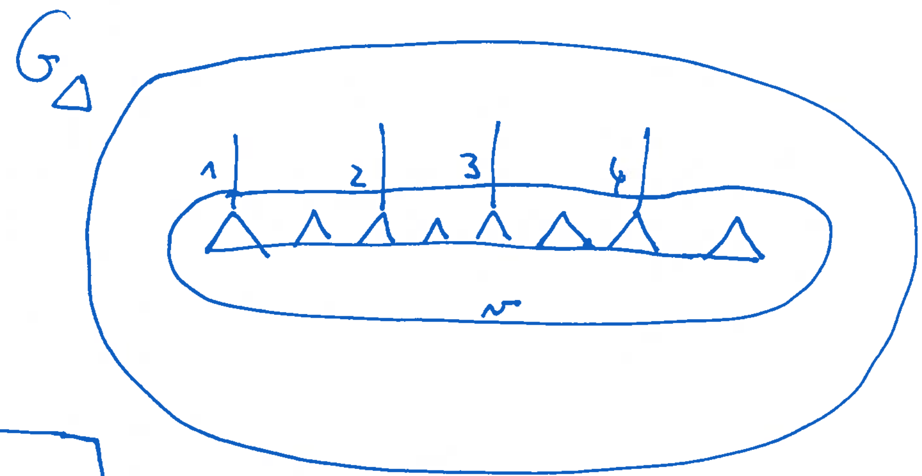


G planar $\Rightarrow \mathcal{E}_G$ generated by face-boundaries

- Max-Cut NP-complete
- Max Even Set Polynomial

Reduction to weighted perfect matching [Fischer 60 physicist]





weights of the gadget edges $\equiv 0$
All other weights unchanged.

<u>Thm</u>. $A \subseteq E$ even iff A can be uniquely extended to a perfect matching of G_Δ by the gadget edges.

<u>Proof</u>: simple

Why was a physisist Fisher interested in perfect matchings, edge-cuts (in 1960)?

$\boxed{\text{Statistical Physics}}$

perfect matching = dimer arrangement

edge-cut = state of Ising model

<u>40's Ising</u>: understanding of Ising model on planar graphs

<u>50's, 60's</u>: Attempts to understand Onsager's result combinatorially: Kac, Ward, Feynman, Fischer, Kastelyn, Sherman ...

$\boxed{\text{Ising Model}}$ on $G=(V,E)$

<u>state</u>: $s : V \to \{1,-1\}$ spin

<u>energy</u>: $E(s) = \sum_{\{i,j\}\in E} + s_i \cdot s_j \cdot w(ij)$

Each state defines edge-cut: $\{e;\ s(i) \neq s(j)\}$

$C(s)$

$w(C(s)) = \qquad (*)$

$[-E(s) + w(E)]/2$

MAX CUT on surfaces

Optimisation : Polynomial reduction to weighted perfect matching
!! Planar graphs !!

Enumeration

$$\mathcal{C}(G, x) = \sum_{\substack{E' \subseteq E \\ \text{max cut}}} x^{w(E')}$$

Gives :

- number and weight of Max Cuts
- the statistics of all cuts

Important for statistical physics

Partition fction of Ising model on graph G :

$$Z(G, w) = \sum_{\sigma : V \to \{1,-1\}} x^{\sum_{\{i,j\} \in E} \sigma(i)\sigma(j) w(ij)} \approx \mathcal{C}(G, x)$$

"energy of σ"

Duality of enumeration

$\mathcal{C}$: cut space $\qquad \mathcal{E}$: cycle space

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{E} \\ \rotatebox{90}{$\subseteq$} & & \rotatebox{90}{$\subseteq$} \\ \mathrm{Im}_2 I_G & \xrightarrow{\boxed{\text{dual}}} & \mathrm{Ker}_2 I_G \end{array}$$

$\mathcal{C}$, $\mathcal{E}$ dual binary linear codes

$\mathcal{C}(G,x)$ weight enumerator of $\mathcal{C}$

$\mathcal{E}(G,x)$ weight enumerator of $\mathcal{E}$

Theorem (Mc Williams, Van der Waerden)

There is a formula giving weight enumerator of the dual code from the weight enumerator of primal code

Enumeration of cuts for Planar Graphs

$$\mathcal{C}(G, x) = \sum_{E' \text{ edge-cuts}} x^{w(E')}$$

$\|\|$ $\boxed{\text{duality of enum.}}$

$$\mathcal{E}(G, x) = \sum_{E' \text{ even}} x^{w(E')}$$

$\|\|$ $\boxed{\text{Fisher construction}}$

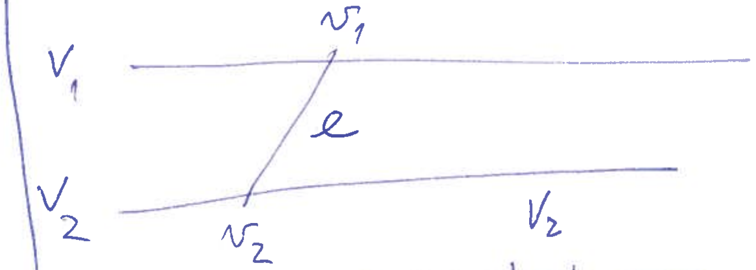
$$\mathcal{P}(G_\Delta, x) = \sum_{E' \text{ perf. matching}} x^{w(E')}$$

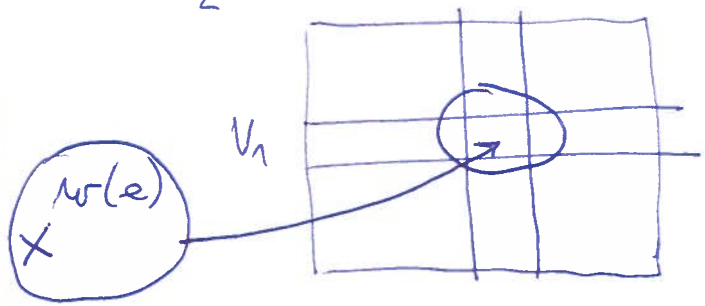
$\boxed{G \text{ genus } g \Rightarrow G_\Delta \text{ genus } g}$

$\mathcal{P}(G, x)$ $\boxed{\text{hard}}$

G bipartite $\Rightarrow$

$\mathcal{P}(G, x)$ is $\boxed{\text{permanent}}$





$$per(A) = \sum_{\pi} \prod a_{i\pi(i)}$$

no sign

Method of Kasteleyn orientations

* 60's core of theoretical physics * Valiant (2000's): holographic algorithms

* 1916 Polya: when can we find signs for A_{ij}'s so that

$$\text{per } A = \det A' \quad [A' : \text{signing of } A]$$

$\equiv$ given directed graph, does it have even dicycle?

Polynomial [Robertson, Seymour, Thomas]

$$\text{Pfaf}(D, x) = \sum_{E' \text{ perf. matching}} \text{sign}(D, E')\, x^{w(E')}$$

; D orientation of G

M_0 fixed perf. m.

$\text{sign}(D, E') = (-1)^{\#}$

$\#$: no. of even alternating cycles of $M_0 \triangle E'$.

M_0

E'

even (4)

• D Pfaffion if all signs the same

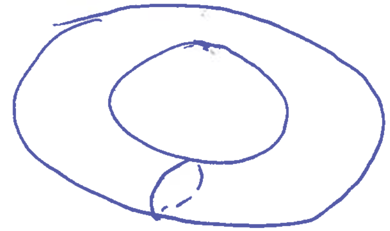
Kasteleyn

• $\text{Pfaf}(D, x)$ polynomial [determinant-type]

• G planar $\Rightarrow \exists D$ Pfaffion:

"make all inner faces clockwise odd"

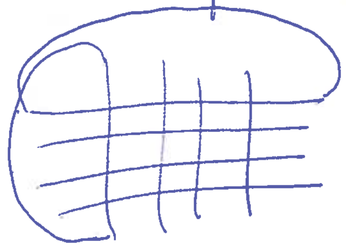
Toroidal and higher genus graphs



- No strongly polynomial alg. for MAX-CUT known
- G ~~[illegible]~~ genus $g \Rightarrow \mathcal{C}(G,x) = \frac{1}{2^g} \sum_{i=1}^{4^g} \pm \, \mathrm{Pfaf}(D_i, x)$

  [Arf-invariant formula] Algebraic topology, QFT

- This provides the only known (weakly) polynomial algorithm for max cut in toroidal graphs [4 Pfaffians]



Semidefinite Programming Algorithm for Max-Cut

$G = (V, E)$; $V = \{1, \dots, n\}$

- Variables $x_1, \dots, x_n \in \{1, -1\}$
- Assignment of values from $\{1, -1\}$ to $x_1, \dots, x_n$ encodes a cut $(S, V \setminus S)$, where $S = \{i \in V;\ x_i = 1\}$.
- $\frac{1 - x_i x_j}{2}$ is the contribution of edge $\{i, j\}$ to this cut.
- $\boxed{\text{Max Cut}}$ $\mathrm{Opt}(G)$: $\max \sum_{ij \in E} \frac{1 - x_i x_j}{2}$; $x_i \in \{1, -1\}$, $i = 1, \dots, n$.
- Replace x_i by $w_i \in S^{n-1} = \{x \in \mathbb{R}^n;\ \|x\| = 1\}$

$$\max \sum_{ij \in E} \frac{1 - w_i^T w_j}{2} \ ; \quad w_i \in S^{n-1}$$

is $\boxed{\text{upper bound}}$ to Opt G.

$\boxed{\text{vector program}}$

(*) Semidefinite Programming solves efficiently to any desired accuracy

Substitute: $x_{ij} := u_i^T u_j$

$\boxed{SDP(G)}$ $\max \sum_{\{i,j\} \in E} \frac{1 - x_{ij}}{2}$; $x_{ii} = 1$, $i = 1, 2, \dots, n$

$X \succeq 0$ positive semidefinite

$X = U^T U$, the columns of U, $u_1, \dots, u_n$ form feasible solution of the vector program.

$M \in Sym_n$ [symmetric matrix] The following is equivalent

1. M positive semidefinite, i.e., all eigenvalues non-negative
2. $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$
3. There is $U \in \mathbb{R}^{n \times n}$, $M = U^T U$ [Cholesky Factorization]

$\boxed{\text{Columns of } U \text{ unit vectors iff } X_{ii} = 1,\ i = 1, \dots, n}$

(*) We can find in polynomial time matrix $X^* \succeq 0$, $X^*_{ii} = 1$, $i = 1, \dots, n$ s.t. $\sum_{ij \in E} \frac{1 - X^*_{ij}}{2} \geq SDP(G) - \varepsilon$, for every $\varepsilon > 0$.

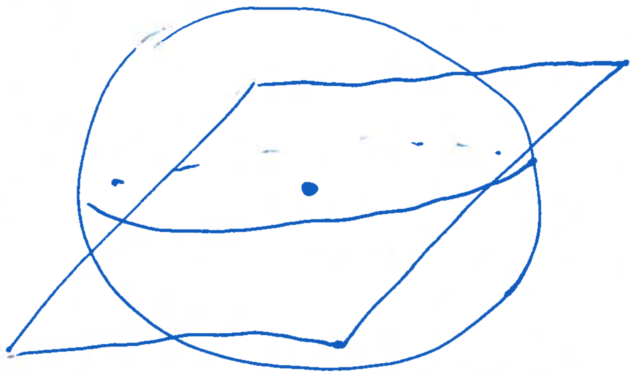
(*) We compute U^* s.t. $X^* = (U^*)^T U^*$ up to a tiny error.

Hence (U^* has columns $u_1^*, u_2^*, \dots, u_n^*$)

$$\sum_{\{i,j\}\in E} \frac{1 - u_i^{*T} u_j^*}{2} \geq SDP(G) - \varepsilon \geq Opt(G) - \varepsilon .$$

Rounding the Vector Solution

We want to map S^{n-1} back to S^0 so that we do not lose too much.



cut by a random hyperplane

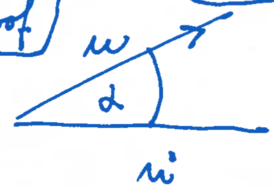
randomised rounding

- We take p uniformly at random from S^{n-1}

$$u \longrightarrow \begin{cases} 1 & \text{if } p^T u \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

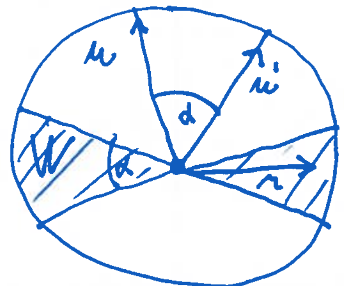
- $u, u' \in S^{n-1}$. Probability that u, u' get different values is $\frac{1}{\pi} \arccos u^T u'$

Proof



$\alpha = \arccos u^T u' \in [0, \pi]$

since $\cos \alpha = u^T u' \in [-1, 1]$



- u, u' go differently iff projection r of p lies in W. r uniformly distributed in $[0, 2\pi]$

$\Rightarrow$ Probability is $\frac{\alpha}{\pi}$. $\boxtimes$

The expected number of edges in the resulting cut equals

$$\sum_{\{i,j\}\in E} \frac{\arccos u_i^{*T} u_j^*}{\pi}$$

$\boxed{\text{Lemma}}$ For all $x \in [-1,1]$ $\quad \frac{\arccos(x)}{\pi} \geq 0{,}8785672\,\frac{1-x}{2}$

$\boxed{\text{Hence}}$

$$\sum_{\{i,j\}\in E} \frac{\arccos u_i^{*T} u_j^*}{\pi} \geq 0{,}8785672 \sum_{\{i,j\}\in E} \frac{1-u_i^{*T} u_j^*}{2} \geq 0{,}8785672\,(\mathrm{Opt}(G)-\varepsilon)$$

$$\geq 0{,}878\,\mathrm{Opt}(G)$$

if $\varepsilon \leq 5\cdot 10^{-4}$.

Approximation Algorithms

P optimization problem (maximization)

- I : set of instances ; $i \in I \Rightarrow F(i)$ set of feasible solutions
- $s \in F(i)$ has a non-negative real value $w(s) \geq 0$.
- $Opt(i) = \sup_{s \in F(i)} w(s) \in \mathbb{R}_+ \cup \{-\infty, \infty\}$.

- A algorithm to solve P is Δ-approximation algorithm for P if
  $[\Delta : \mathbb{N} \longrightarrow \mathbb{R}_+]$
  - there is polynomial p s.t. for all $i \in I$, the runtime of A on i is $\leq p(|i|)$
  - $i \in I \Rightarrow w(A(i)) \geq \Delta(|i|) \cdot Opt(i)$

$\boxed{\text{Randomised algorithms}}$ A randomised Δ-approximation algorithm has $\boxed{\text{expected}}$ poly runtime and must satisfy $E[w(A(i))] \geq \Delta(|i|)\, Opt(i)$

[$w(A(i))$ is random variable]

A randomized 0,5- approximation algorithm for Max-Cut

$G=(V,E)$; Pick S a random subset of V [each vertex $v \in V$ included in S with probability 1/2, independent of all other vertices.

Thm This is a randomised 0,5-approximation algorithm for Max Cut.

Proof $$\mathbb{E}[w(A(G))] = \mathbb{E}[|E(S, V\setminus S)|] = \sum_{e\in E} \mathrm{Prob}[e \in E(S, V\setminus S)] =$$

$$\sum_{e\in E} \frac{1}{2} = \frac{1}{2}|E| \geq \frac{1}{2}\mathrm{Opt}(G).$$

$e \in E(S, V\setminus S)$ iff exactly one of the two end-points of e ends up in S.

LP

$$\max \ c^T x$$
$$Ax = b$$
$$x \geq 0$$

$c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$

X <u>feasible</u> solution if $X \in SYM_n$, $A(X) = b$, $X \succeq 0$.

<u>the value</u> is

$$\sup\{C \bullet X : A(X) = b, X \succeq 0\}$$

Semidefinite Programming

$$SYM_n = \{X \in \mathbb{R}^{n \times n};\ x_{ij} = x_{ji},\ 1 \leq i < j \leq n\}$$

[replaces $x \in \mathbb{R}$]

$A : SYM_n \longrightarrow \mathbb{R}^m$ linear [replaces $A \in \mathbb{R}^{m \times n}$]

$$X \bullet Y := \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} y_{ij}, \quad X, Y \in SYM_n$$

$$X \bullet Y = \mathrm{Tr}(X^T Y)$$

$X \succeq 0$ (being positive semidefinite) [replaces $x \geq 0$]

$$\max \ C \bullet X$$
$$\left.\begin{array}{l} A_1 \bullet X = b_1 \\ \vdots \\ A_m X = b_m \end{array}\right\} \quad A(X) = b$$
$$X \succeq 0$$

where $b = (b_1, \ldots, b_m)$

$A : SYM_n \longrightarrow \mathbb{R}^m$ linear

$$A = (a_{ijk})_{i,j=1}^{n} \Big|_{k=1}^{m}$$

WARNINGS

1. Finite value is not necessarily attained:

$$\begin{aligned} &\text{Max } -x_{11} \\ &x_{12} = 1 \\ &X \succeq 0 \end{aligned}$$

Feasible solutions: all

$X \succeq 0$, $X = \begin{pmatrix} x_{11} & 1 \\ 1 & x_{22} \end{pmatrix}$

$$\Updownarrow$$

$x_{11} \geq 0$, $x_{22} \geq 0$, $x_{11} x_{22} \geq 1$

$$\Updownarrow$$

$x_{11} > 0$ and $x_{22} \geq \frac{1}{x_{11}}$

Value 0 but no opt. solution

(*) All known eff. algorithms return only $\boxed{\text{approximately optimal solutions}}$ [ellipsoid method]

(*) Cholesky factorisation also can be constructed only approximately

Back to Max-Cut Approximation Algorithms

Approximation Ratio of Goemans-Williamson Alg. : $\alpha_{GW} \approx 0.87856720..$

$\boxed{\text{Better}}$

(1) dense graphs (cn^2 edges, $c>0$ fixed) : Polynomial-time approximation scheme [use of regularity lemma]

(2) bounded max degree

But it is conjectured that no improvement possible for All graphs [Unique Games Conj.]

Theorem (Håstad) • Assuming $P \neq NP$, no polynomial-time algorithm can approximate Max Cut better than $16/17 \approx 0{,}94$.

• Given a 3-SAT formula φ, Håstad's construction yields graph $G = G(\varphi)$ s.t.

(1) Max Cut $\geq x$ if φ satisfiable (x eff. computed from φ)

(2) Max Cut $\leq 16/17\, x$ if φ is unsatisfiable.

Approximation Ratio and Integrality Gap

$\alpha_{GW} = \inf_{G} \frac{Algo(G)}{Opt(G)}$; $Algo(G)$: expected size of cut found by random hyperplane rounding in GW algorithm

$SDP(G) = \max_{\cancel{\{i,j\} \in E}} \left\{ \sum_{\{i,j\} \in E} \frac{1 - v_i^T v_j}{2} \; ; \; \|v_1\| = \dots = \|v_n\| = 1 \right\}$ semidefinite relaxation

Integrality Gap

$$\boxed{Gap := \sup_{G} \frac{SDP(G)}{Opt(G)}}$$

Metatheorem $\frac{1}{Gap} \geq$ Approximation Ratio

[for maximisation algorithms based on semidefinite or linear progr.]

Theorem (Feige, Schechtman) The integrality gap of the Goemans-Williamson SDP relaxation of Max Cut satisfies

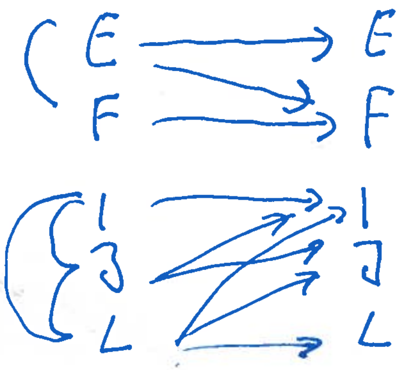
$$\text{Gap} \geq \frac{1}{\alpha_{GW}} \approx 1.1382 \; ; \; \forall \varepsilon > 0 \; \exists G : \frac{SDP}{Opt} \geq \frac{1}{\alpha_{GW}} - \varepsilon$$

<u>We had</u> : $\text{Algo}/SDP \geq \alpha_{GW}$ and $Opt \geq \text{Algo}$ $\Rightarrow$

$$GAP = \sup \left(SDP/Opt \right) \leq \frac{1}{\alpha_{GW}}$$

<u>Hence</u> : $GAP = \dfrac{1}{\alpha_{GW}}$

Sending messages over noisy chanel



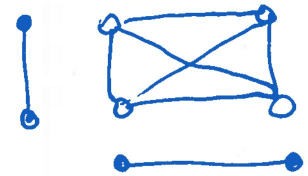
$(v_1, \dots, v_k)$ <u>similar to</u> $(v_1', \dots, v_k')$ if $(\forall i)$(v_i similar to v_i')

Similarity Free Dictionary set of pairwise non-similar words

? Largest similarity free dict. ?

For words of size k it is $\alpha(G^k)$

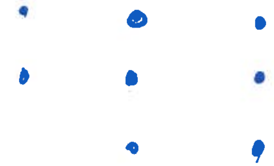
$G \boxtimes G = G^2$:



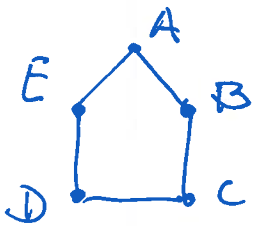
- Using 1-letter words we can transmit $\alpha(G)$ messages:
  1 letter carries [roughly] $\log \alpha(G)$ bits of information
- k-letter words $\Rightarrow$ 1 letter carries [roughly] $\frac{1}{k} \log \alpha(G^k)$ bits of info

Shannon capacity of noisy chanel $\Theta(G) = \sup \{ \frac{1}{k} \log \alpha(G^k) ; k \in \mathbb{N} \}$

- $\alpha(G^{k+\ell}) \geq \alpha(G^k)\,\alpha(G^\ell)$



- <u>strict</u> : $\alpha(C_5)=2$, $\alpha(C_5^2) \geq 5 > \alpha(C_5)^2 = 4$



AA, BC, CE, DE, E[illegible]

pairwise non-similar

Shannon 56

???

$s(C_5) = \frac{1}{2}\log 5$

<u>YES</u>: Lovász 79

$S(G) = 2^{s(G)} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}$

$s(G) = \sup\{\frac{1}{k}\log\alpha(G^k);\ k\in\mathbb{N}\}$.

- $s(G)$ bounded; $s(G) = \lim\{\frac{1}{k}\log\alpha(G^k);\ k\in\mathbb{N}\}$:
  - $s(G) \leq \frac{1}{k}\log\alpha(G^k) \leq \frac{1}{k}\log|V|^k = \log|V|$
  - let $x_k = \log\alpha(G^k) \Rightarrow x_{k+\ell} \geq x_k + x_\ell$

$\boxed{\text{sequence } (x_k)_{k\in\mathbb{N}} \text{ superadditive}}$

Lemma (Fekete)

$(x_k)_{k\in\mathbb{N}}$ superaditive

$\Downarrow$

$\left(\frac{x_k}{k}\right)_{k\in\mathbb{N}}$ converges to its supremum.

Lovász Theta Function

$G=(V,E) \Rightarrow \bar{E} = \binom{V}{2} \setminus E$

- An <u>orthonormal representation</u> of G, $|V|=n$, is $\mathcal{U} = (u_1, \dots, u_n)$ s.t. $u_i \in S^{n-1}$ [unit vectors] and $u_i^T u_j = 0$ if $\{i,j\} \in \bar{E}$.

[example: $e_1, \dots, e_n$]

- $\vartheta(\mathcal{U}) \overset{def}{:=} \min_{\|c\|=1} \max_{i=1}^{n} \frac{1}{(c^T u_i)^2}$

$\boxed{c \text{ called handle of } \mathcal{U}}$

<u>exists</u>: min of continuous fction over a compact set S^{n-1} $\cup$ open set cont. $\{x;\ x^T u_i = 0\}$

$\boxed{\text{Definition}}$ $\vartheta(G)$ is <u>minimum</u> $\vartheta(\mathcal{U})$ over all orthonormal repr. of G

<u>exists</u>: min of a cont. fction over compact set

<u>Examples</u>: • $G = K_n \Rightarrow$ every sequence of unit vectors is orthonormal repr.

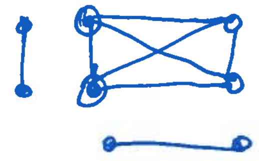
$c \in S^{n-1} \Rightarrow c$ handle for $\{u_i = c,\ i = 1, \dots, n\} \Rightarrow \boxed{\vartheta(G) = 1}$

- G independent $[E(G) = \emptyset] \Rightarrow \{u_i\}$ must form an orthonormal basis of $\mathbb{R}^n$;

A.s. $u_i = e_i \Rightarrow$ handle is $c = \sum_{i=1}^{n} u_i/\sqrt{n} \Rightarrow c^T u_i = 1/\sqrt{n}\ (\forall i) \Rightarrow$

$\boxed{\vartheta(G) = n}$

$G \cdot H$



Lema 1 $\vartheta(G\cdot H) \le \vartheta(G)\cdot\vartheta(H)$

Lema 2 $\alpha(G) \le \vartheta(G)$

Theorem. $(\forall G)\big(S(G) \le \vartheta(G)\big)$

Proof. $\alpha(G^k) \le \vartheta(G^k) \le \vartheta(G)^k$

$$\Downarrow$$

$$\sqrt[k]{\alpha(G^k)} \le \vartheta(G) \Rightarrow$$

$$S(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)} \le \vartheta(G).$$

$\boxtimes$

Proof of lema 2. I max indep. of G, $\mathcal{U} = (u_1, \dots, u_n)$ optimal repr. with handle c

- $\{u_i;\ i\in I\}$ pairwise orthogonal $\Rightarrow$

$$c^Tc \ge \sum_{i\in I} (c^Tu_i)^2$$ Hence,

$$1 = c^Tc \ge \sum_{i\in I} (c^Tu_i)^2 \ge |I| \cdot \min_{i\in I}(c^Tu_i)^2 =$$

$\alpha(G) \min_{i\in I} (c^Tu_i)^2$. Hence

$$\alpha(G) \le \frac{1}{\min_{i\in I}(c^Tu_i)^2} = \max_{i\in I} \frac{1}{(c^Tu_i)^2} \le$$

$$\max_{i\in V} \frac{1}{(c^Tu_i)^2} = \vartheta(G).$$

$\boxed{\text{Proof of lema 1}}$ $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n \Rightarrow$ tensor product $x\otimes y = (x_1y_1 \dots x_1y_n x_2y_1 \dots x_my_n)$

- $x\otimes y \in \mathbb{R}^{mn}$ • $(x\otimes y)^T(x'\otimes y') = (x^Tx')(y^Ty')$
- $\mathcal{W}, \mathcal{V}$ optimal repres. of $G, H \Rightarrow$ $\mathcal{W}\otimes\mathcal{V}$ orthonormal repr. of $G\cdot H$

c handle of $\mathcal{W}$, d handle of $\mathcal{V} \Rightarrow$

$$\vartheta(\mathcal{W}\otimes\mathcal{V}) \le \max_{i\in V, j\in W} \frac{1}{((c\otimes d)^T(u_i\otimes v_j))^2} = \max_{i\in V, j\in W} \frac{1}{(c^Tu_i)^2(d^Tv_j)^2} = \vartheta(G)\,\vartheta(H)$$

V vertices of G

W vertices of H

$\boxtimes$

- $G=(V,E)$, $|V|=n$, $\mathcal{U}=\{u_1,\dots,u_n\}\subseteq S^{n-1}$ is <u>orthonormal representation</u> if $\{i,j\}\notin E \Rightarrow u_i^T u_j = 0$.
- Theta($\mathcal{U}$) : $\vartheta(\mathcal{U}) := \min\limits_{\|c\|=1} \max\limits_{i=1}^{n} \frac{1}{(c^T u_i)^2}$ || $\vartheta(G) :=$ smallest $\vartheta(\mathcal{U})$, $\mathcal{U}$ orthonormal r.

$\boxed{\text{Thm (Lovász)} \quad 2^{\rho(G)} = S(G) \le \vartheta(G).}$

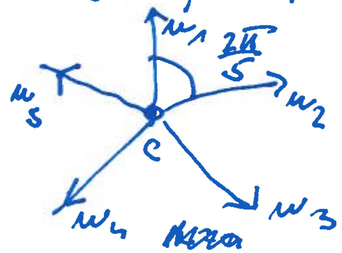
<u>We know</u> : $S(C_5) \geq \sqrt{5}$

Shannon Conjecture follows from

$\boxed{\text{Lemma (Lovász)}}$ $\vartheta(C_5) \le \sqrt{5}$

<u>Need</u> : orthonormal representation of C_5 with value $\le \sqrt{5}$.

• Do in S^2, lift to S^4 by $(\dots 0\,0)$. $c:=(0,0,1)$



fold umbrella $\rightsquigarrow$

(*) $u_i = \frac{(\cos\frac{2\pi i}{5}, \sin\frac{2\pi i}{5}, z)}{\|(\cos\frac{2\pi i}{5}, \sin\frac{2\pi i}{5}, z)\|}$, $i=1,\dots,5$

$0 = u_5^T u_2 \Leftrightarrow (1,0,z)\begin{pmatrix}\cos\frac{4\pi}{5}\\ \sin\frac{4\pi}{5}\\ z\end{pmatrix} = 0 \Rightarrow$

$z = \sqrt{-\cos\frac{4\pi}{5}}$ $\quad u_5 = \dots$ ⊠

Folding leads to orthonormal repres. since first $\measuredangle(u_1,u_4) = \frac{4\pi}{5} > \frac{\pi}{2}$ and finally this $\measuredangle(u_1,u_4)=0$. The value is $\le\sqrt{5}$ [not hard] (*)

SDP for the Theta function

(2)

Thm. $\vartheta(G)$ is the value of

$$(*)\quad \begin{aligned} &\text{Min } t \\ &y_{ij} = -1, \quad \{i,j\} \in \overline{E} \\ &y_{ii} = t-1, \quad i = 1,\dots,n \\ &Y \succeq 0 \end{aligned}$$

Proof Let the value of $(*)$ be $\vartheta'(G)$.

(A) $\vartheta'(G) \le \vartheta(G)$:

(U, c) optimal representation. Let $\tilde{Y} \in SYM_{n}$ be: $\tilde{y}_{ij} := \frac{u_i^T u_j}{(c^T u_i)(c^T u_j)} - 1, \ i \ne j$

$\tilde{y}_{ii} := \vartheta(G) - 1, \quad i = 1,\dots,n$

- If $\boxed{\tilde{Y} \succeq 0}$ $\Rightarrow (\tilde{Y}, \tilde{t} = \vartheta(G))$ is a feasible sol. of $(*)$.

$\Rightarrow \vartheta'(G) \le \vartheta(G)$

$\tilde{Y} = D + U^T U$, D diagonal ≥ 0 and

$U^i = c - u_i / c^T u_i$ $\rightarrow$ i-th column of U

Thus $\tilde{Y} \succeq 0$.

(B) $\vartheta'(G) \ge \vartheta(G)$ a FACT

Theorem $G = (V,E)$ $\omega(\overline{G}) \le \vartheta(G) \le \chi(\overline{G})$

We had last time: $\alpha(G) \le \vartheta(G)$

ω: size of a largest clique (complete subgraph) of G

α: size of a largest indep. set

$\chi(G)$: chromatic number.

$\vartheta(G) = 1 \Rightarrow G = K_n \Rightarrow \boxtimes$

$\vartheta(G) > 1 \Rightarrow \chi(\overline{G}) \ge 2$.

Rescale $(*)$

$$\begin{aligned} &\text{Min } t \\ &y_{ij} = -1/(t-1), \quad \{i,j\} \notin E \\ &y_{ii} = 1, \quad i = 1,\dots,n \\ &Y \succeq 0 \end{aligned}$$

Definition

$k \in \mathbb{R}$, vector k-coloring of G is $f: V \to S^{n-1}$: $f(v)^T f(w) = -\frac{1}{k-1}$, $\{v,w\} \in E$

Lemma. If G has k-coloring then it has vector k-coloring.

[FACT]

$k = 3$

Perfect Graphs G perfect if $\omega(G') = \chi(G')$ for each induced subgraph G'. 3

__Corollary.__ • G perfect $\Rightarrow$ $\omega(G) = \vartheta(\bar{G}) = \chi(G)$

• G perfect $\Rightarrow$ $\chi(G)$ polynomial !! by SDP:

* suffices to solve with accuracy $\varepsilon < \frac{1}{2}$ since value is integer.

The only known polynomial method