$\boxed{\text{Greedy Algorithm}}$ $(X, \mathcal{F})$; $X = \{1, \dots, n\}$; $w : X \to \mathbb{Q}$

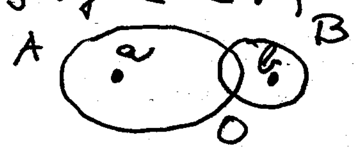
$\boxed{\text{GOAL}}$ Find $A \in \mathcal{F}$ s.t. $w(A) = \sum_{i \in X} w(i)$ is <u>maximized</u>.

<u>Notation</u>: $w(i) \equiv w_i$

- W.l.o.g. notation s.t. $w_1 \geq w_2 \geq \dots \geq w_n$ | m max s.t. $w_m > 0$
- $J := \emptyset$
- $\boxed{\text{For}}$ $i = 1, \dots, m$ $\boxed{\text{Do}}$ if $J \cup \{i\} \in \mathcal{F} \Rightarrow J := J \cup \{i\}$.

<u>Thm.</u> $(X, \mathcal{F})$ hereditary non-empty. Greedy alg. works for each w iff $(X, \mathcal{F})$ matroid.

$\boxed{\text{Proof}}$ <u>Necessary</u>: if $\mathcal{F}$ violates (3') then $A, B \in \mathcal{F}$ both max ($\subseteq$) indep. subsets of $Z \subseteq X$, $|A| > |B|$.



$b > a$ But $|A \setminus B| a > |B \setminus A| b$

$\boxed{\text{Suff.}}$ x' : char. vector of output of GA.

$T_i = \{1, \dots, i\}$

$\boxed{\text{Lemma}}$ $x'(T_i) \geq x(T_i)$ [Greedy Alg.]

x : char. vector of optimal set of $\mathcal{F}$ $\Rightarrow$

$$w(x) = \sum_{i=1}^{m} w_i x_i = \sum_{1}^{m} w_i (x(T_i) - x(T_{i-1})) =$$

$$\sum_{i=1}^{m-1} (w_i - w_{i-1}) x(T_i) + w_m x(T_m) \leq (\text{Lemma})$$

$$\sum^{m-1} (w_i - w_{i-1}) x'(T_i) + w_m x'(T_m).$$

$\boxtimes$

We only used : $x'(T_i) \geq r(T_i)$ and $x(T_i) \leq r(T_i)$.

Hence GA solves LP $\left(\max \sum_1^m w_i x_i \,;\, (\forall A \subseteq X) \sum_{i \in A} x_i \leq r(A);\ x \geq 0\right)$

$\boxed{\text{Corollary}}$ (Edmonds)

For any matroid, $\operatorname{conv}(\chi_A \,;\, A \in \mathcal{I}) = P_M =$

$= \{ x \geq 0 \,;\ \forall A \subset X,\ x(A) \leq r(A) \}$.

$\boxed{\text{Proof}}$ ① Each char. vector of independent set clearly belongs of P_M

$\Rightarrow \operatorname{conv}(\dots) \subseteq P_M$

② Each vertex of P_M is, by GA, char. vector of indep. set

$\Downarrow$

$P_M \subseteq \operatorname{conv}(\dots)$. ☒

$\boxed{\text{Matroid intersection}}$ $(X, \mathcal{I}_1), (X, \mathcal{I}_2)$.

$$\max |J|;\ J \in \mathcal{I}_1 \cap \mathcal{I}_2 = \min_{A \subseteq X} r_1(A) + r_2(X - A)$$

<u>Proof</u>. (1) $J \in \mathcal{I}_1 \cap \mathcal{I}_2 \Rightarrow$

$\forall A \subseteq X,\ J \cap A \in \mathcal{I}_1,\ J \cap (X \setminus A) \in \mathcal{I}_2 \Rightarrow \boxed{\max \le \min}$

(2) By induction on $|X|$.

let $k = \min_A r_1(A) + r_2(X \setminus A)$

let $\{x\} \in \mathcal{I}_1 \cap \mathcal{I}_2$ $\{ |\mathcal{I}_1 \cap \mathcal{I}_2| = 1$

$\Rightarrow$ let $A = \{x;\ r_1(\{x\}) = 0\} \Rightarrow$
$r_1(A) + r_2(X - A) = 0 \}$

let $X' = X - \{x\}$.

(a) $\min_{A \subseteq X'} r_1(A) + r_2(X' - A) = k \Rightarrow$
$\Rightarrow$ OK by induction.

(b) let $\mathcal{I}_i' = \mathcal{I}_i$ contracted to X'.

(i) $\min_{A \subseteq X'} r_1'(A) + r_2'(X - A) \geq k - 1 \Rightarrow$
by induction there is a common subset of $\mathcal{I}_1' \cap \mathcal{I}_2'$ of size $\geq k-1 \Rightarrow$ add $\{x\}$ OK

(ii) <u>Else</u> $\exists A, B \subseteq X' : r_1(A) + r_2(X' - A) \le k - 1$

- $r_1(B \cup \{x\}) - 1 + r_2((X' \setminus B) \cup \{x\}) - 1 \le k - 2$

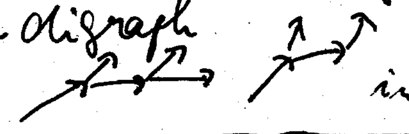
$\boxed{\text{Submodularity}}$

$$r_1(A \cup B \cup \{x\}) + r_1(A \cap B) + r_2(X - (A \cap B)) + r_2(X \setminus (A \cup B \cup \{x\})) \le 2k - 1$$

One of the pairs gives contr. with k. ☒

(1) max matching in bipartite graphs,

(2) max branching in a digraph



indeg ≤ 1

Min-Max Theorems

$\cup$|

Good Characterizations

$\cup$|

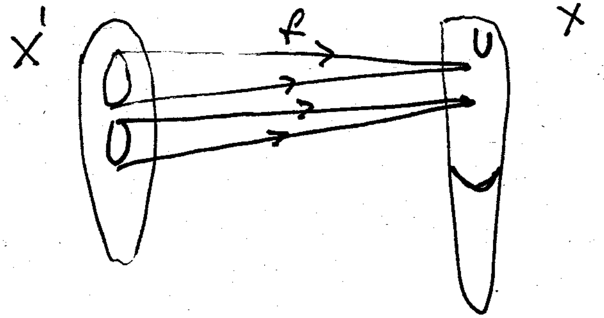
$NP \cap co\text{-}NP$

$\boxed{\text{Edmonds}}$

Image of a matroid (X', φ'); $f : X' \longrightarrow X$; $\varphi = \{f(I);\ I \in \varphi'\}$.

Then (X, φ) matroid, $U \subset X \Rightarrow r(U) = \min_{T \subseteq U} \{|U - T| + r'(f^{-1}(T))\}$.

__Proof.__ 👁 $r(U)$ = max size of a common independent set of φ' and the partition matroid (X', W) induced by the family $(f^{-1}(s);\ s \in U)$.



Definition $M_i = (X_i; \varphi_i)$, $i = 1, \dots, m$, $X = \bigcup_1^m X_i$.

The __union__ $\bigcup_{i=1}^m M_i = (X, \{I = I_1 \cup \dots \cup I_m;\ I_i \in \varphi_i\})$.

Theorem Union is matroid with rank fction

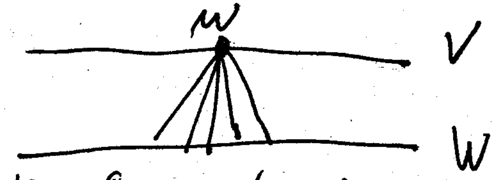
$$r(U) = \min_{T \subseteq U} \{|U - T| + r_1(T \cap X_1) + \dots + r_k(T \cap X_k)\}.$$

__Proof.__ Make X_i disjoint, use __image__ of matroids. $\boxtimes$

EXAMPLE

Transversal matroids

(V, W, E) bipartite graph



$\varphi_u = \{\{z\};\ \{u, z\} \in E\}$.

$M = \bigcup_{u \in V} \varphi_u$.

→ set of transversals of a set system.

max size of union of k independent sets

$$= \min_{T \subseteq X} \{ |X \setminus T| + k\, r(T) \}.$$

X can be covered by k independent sets iff

$$\forall U \subseteq X, \quad k\, r(U) \geq |U|$$

There are k disjoint bases iff

$$\forall U \subseteq X, \quad k\,(r(X) - r(U)) \leq |X - U|$$

A finite subset X of a vector space can be covered by k linearly independent sets iff

$$\forall U \subseteq X, \quad k \cdot r(U) \geq |U|$$