<u>Matroid</u> (X, φ), $\varphi \subseteq 2^X$ 1

(1) $\emptyset \in \varphi$ (2) $A \in \varphi$, $A' \subseteq A \Rightarrow A' \in \varphi$ (hereditary)

(3) exchange axiom $U, V \in \varphi$, $|U| > |V| \Rightarrow$
$$(\exists x \in U \setminus V)(V \cup \{x\} \in \varphi)$$

(3') $A \subseteq X \Rightarrow$ all maximal (w.r.t. $\subseteq$) independent subsets of A have the same size.

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(3) $\Leftrightarrow$ (3') ; <u>Examples</u>: I A matrix over field $\mathbb{F}$

X set of columns of A, $\varphi = \{Y \subseteq X ; Y$ linearly indep. over $\mathbb{F}\}$ $\boxed{\text{Terminology}}$ elmts of $\varphi \equiv$ independent sets

Example II $G = (V,E)$ graph, $X = E$, $\mathcal{Y} = \{ Y \subseteq X ;\ Y \text{ acyclic} \}$.

(*) 3' Matroids are exactly hereditary systems where **rank** can be defined **well**

Definition $(X, \mathcal{Y})$ matroid, $Y \subseteq X \Rightarrow$

!! $r(Y) \overset{\text{def}}{=} |A|$; A max (w.r.t. $\subseteq$) independent subset of Y.

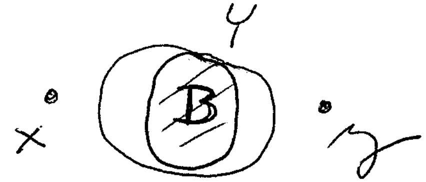
Definition Maximal ($\subseteq$) independent set is called a <u>basis</u>.

Thm $r: 2^X \longrightarrow \mathbb{N}$ is the rank of a matroid iff

(R1) $r(\emptyset)=0$ (R2) $r(Y) \leq r(Y\cup\{y\}) \leq r(Y)+1$

(R3) $r(Y\cup\{y\}) = r(Y\cup\{z\}) = r(Y) \Rightarrow r(Y) = r(Y\cup\{y,z\})$.

Proof (R3) holds for matroids:



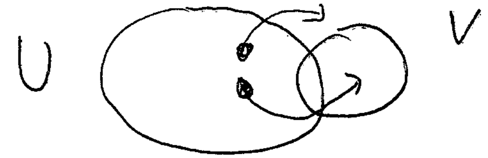
„$\Leftarrow$" Define $\mathcal{I}$ by: $A\in\mathcal{I}$ iff $r(A)=|A|$. $(X,\mathcal{I})$ matroid

(1) by (R1), (2) by (R2): $|A'|>r(A')$, $A'\subseteq A\in\mathcal{I} \Rightarrow$

(by R2) $r(A) \leq r(A') + |A\setminus A'| < r(A)$ [contradiction]

If (3) not true $\Rightarrow$ by repeated (R3), $r(\underbrace{V\cup(U\setminus V)}_{U}) = r(V)$

contradiction with $|U| = r(U)$.



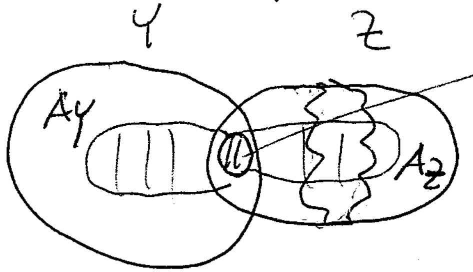
Thm $r : 2^X \to \mathbb{N}$ is the rank of a matroid iff

R1' $0 \le r(Y) \le |Y|$, $Y \subseteq X$ R2' $Z \subseteq Y \Rightarrow r(Z) \le r(Y)$

R3' $r(Y \cup Z) + r(Y \cap Z) \le r(Y) + r(Z)$ $\boxed{\text{submodularity}}$

Matroids are the systems where rank is monotone, submod.

Proof All simple but: why R3' holds for matroids:



A max indep. in $Y \cap Z$

A_Y max indep. in Y, $A \subseteq A_Y$

A_Z max indep. in Z, $A \subseteq A_Z$

$$r(Y) + r(Z) = |A_Y| + |A_Z| = |A_Y \cap A_Z| + |A_Y \cup A_Z| = r(Y \cap Z) + |A_Y \cup A_Z|$$

$r(Y \cup Z) \le |A_Y \cup A_Z|$: A_Y cannot be extended by more than $|A_Z \setminus Y|$ to indep. set in $Y \cup Z$.

Submodular fctions model well VALUE in combinatorial auctions: $f : 2^X \to \mathbb{R}$, $x \in X \Rightarrow \Delta f_x(T) \overset{\boxed{def}}{=} f(T \cup \{x\}) - f(T)$.

Theorem $f : X \to \mathbb{R}$ submodular iff $\boxed{\forall x \in X, \Delta f_x \text{ nonincreasing}}$

$$\forall \left(U \subseteq X,\ y, z \in X \setminus U\right)\left(f(U \cup \{y\}) + f(U \cup \{z\}) \geq f(U) + f(U \cup \{y, z\})\right)$$

"$\Rightarrow$" simple

"$\Leftarrow$" Want: $f(Y) + f(Z) \geq f(Y \cap Z) + f(Y \cup Z)$.

By induction on $|Y \triangle Z|$: ⓐ $|Y \triangle Z| \leq 2$ follows from the ass.

ⓑ $|Y \triangle Z| \geq 3 \Rightarrow$ w.l.o.g. $|Y \setminus Z| \geq 2$; let $t \in Y \setminus Z$.

$$f(Y \cup Z) - f(Y) \overset{1}{\leq} f(Y \setminus \{t\} \cup Z) - f(Y \setminus \{t\}) \overset{2}{\leq} f(Z) - f(Y \cap Z)$$

① $|Y \triangle (Y \setminus \{t\} \cup Z)| < |Y \triangle Z|$ ② $|Y \setminus \{t\} \triangle Z| < |Y \triangle Z|$.

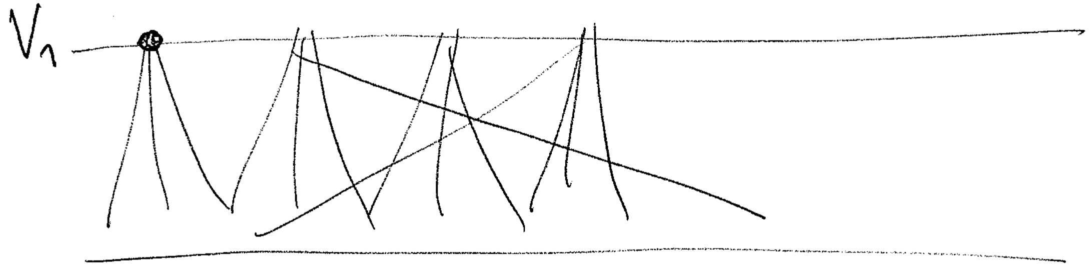
Deletion $M = (X, \varphi)$, $Y \subseteq X \Rightarrow M - Y = (X - Y; \{A - Y;\ A \in \varphi\})$

Direct Sum M_1, M_2 matroids, $X_1 \cap X_2 = \emptyset$.

$M_1 + M_2 = (X_1 \cup X_2, \{Y;\ Y \cap X_1 \in \varphi_1 \wedge Y \cap X_2 \in \varphi_2\})$.

Partition Matroid X_i, $i = 1, \dots, n$, disjoint, $\varphi_i = \{A \subseteq X_i;\ |A| \leq 1\}$.

$\sum_i (X_i, \varphi_i)$ is **Partition Matroid**



G bipartite graph

$X = E$

$\varphi = \{Y \subseteq E;\ v \in V_1 \Rightarrow \deg_Y(v) \leq 1\}$.

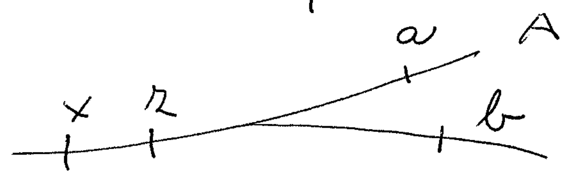
Example III Matroid *simple* if $r(A) = |A|$ whenever $|A| < 3$.

$A \subseteq X$ *closed* if $y \in X \setminus A \Rightarrow r(A \cup \{y\}) > r(A)$.

Each simple matroid of rank 3 ($r(X) = 3$) determined by

$L(M) = \{A \subseteq X \,;\, |A| > 2,\ r(A) = 2,\ A \text{ closed}\}$.

Lemma $A, B \in L(M) \Rightarrow |A \cap B| \leq 1$



(R3) contr.

Definition $\mathcal{C} \subset 2^X$ *configuration* if (1) $A \in \mathcal{C} \Rightarrow |A| \geq 3$

(2) $A, B \in \mathcal{C} \Rightarrow |A \cap B| \leq 1$.

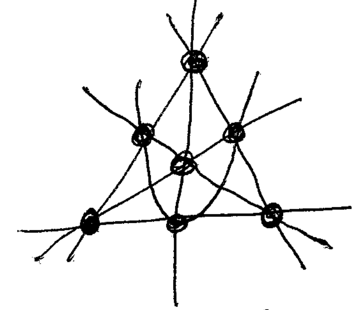
Theorem Each configuration is the set $L(M)$ of a simple matroid of rank 3 on X.

Define r:

- $r(A) = |A|$ for $|A| \leq 2$
- $A \subseteq$ elmt of $\mathcal{C} \Rightarrow r(A) = 2$
- otherwise $r(A) = 3$

satisfies R1, R2, R3

Fano matroid



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Contraction

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$\boxed{\text{Duality}}$