Back to Max-Cut Approximation Algorithms

Approximation Ratio of Goemans-Williamson Alg. : $\alpha_{GW} \approx 0.87856720..$

$\boxed{\text{Better}}$

(1) dense graphs (cn^2 edges, $c>0$ fixed) : Polynomial-time approximation scheme [use of regularity lemma]

(2) bounded max degree

But it is conjectured that no improvement possible for All graphs [Unique Games Conj.]

Theorem (Håstad) • Assuming $P \neq NP$, no polynomial-time algorithm can approximate Max Cut better than $16/17 \approx 0,94$.

- Given a 3-SAT formula φ, Håstad's construction yields graph $G = G(\varphi)$ s.t.
  - (1) Max Cut $\geq x$ if φ satisfiable (x eff. computed from φ)
  - (2) Max Cut $\leq 16/17\, x$ if φ is unsatisfiable.

Approximation Ratio and Integrality Gap

$$\alpha_{GW} = \inf_G \frac{\mathrm{Algo}(G)}{\mathrm{Opt}(G)} \; ; \quad \mathrm{Algo}(G)\text{: expected size of cut found by random hyperplane rounding in GW algorithm}$$

$$\mathrm{SDP}(G) = \max_{\cancel{\{i,j\}\in E}} \left\{ \sum_{\{i,j\}\in E} \frac{1 - v_i^T v_j}{2} \; ; \; \|v_1\| = \dots = \|v_n\| = 1 \right\} \quad \text{semidefinite relaxation}$$

$\boxed{\text{Integrality Gap}}$ $\boxed{\mathrm{Gap} := \sup_G \frac{\mathrm{SDP}(G)}{\mathrm{Opt}(G)}}$

$\boxed{\text{Metatheorem}}$ $\frac{1}{\mathrm{Gap}} \geq$ Approximation Ratio

[for maximisation algorithms based on semidefinite or linear progr.]

Theorem (Feige, Schechtman) The integrality gap of the Goemans–Williamson SDP relaxation of Max Cut satisfies

$$\text{Gap} \geq \frac{1}{\alpha_{GW}} \simeq 1.1382 \; ; \; \forall \varepsilon > 0 \; \exists G : \frac{SDP}{Opt} \geq \frac{1}{\alpha_{GW}} - \varepsilon$$

__We had__ : $\text{Algo}/_{SDP} \geq \alpha_{GW}$ and $\text{Opt} \geq \text{Algo} \implies$

$$GAP = \sup\left(SDP/Opt\right) \leq \frac{1}{\alpha_{GW}}$$

__Hence__ : $GAP = \dfrac{1}{\alpha_{GW}}$
