Critical Goods Distribution System ⋆

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Abstract. Distribution crises are manifested by a significant discrepancy between the demand and the supply of a critically important good, for a period of time. In this paper, we suggest a hybrid market mechanism for minimizing the negative consequences of sudden distribution crises.

Keywords: Fair distribution · Market mechanism · Auction.

1 Introduction

Distribution crises are extreme situations where the supply of a critically important good is so small that the actors realistically do not have the necessary minimum available to ensure function. Such situations have two basic characteristics from a distribution point of view:

1. The *ethical fairness* of the distribution of supply of critical goods is important. According to the rules of fairness, which are declared in advance, it is possible to hypothetically distribute the supply of a critical commodity fairly among organizations.

In other words, the buyers have *rights to purchase a fair amount* of the commodity in question. The total amount of rights to purchase the commodity is equal to the amount of the commodity available on the market.

2. The existence of a *market for a critical commodity* is the second essential aspect of the distribution. Lack of a critically important good leads to a significant increase in price, organizations try to obtain more than a fair amount to cover their needs, and this is always at the expense of other participants.

Since the flow of money and critical goods in a free market during a distribution crisis is mainly concentrated between sellers and active buyers, our aim is to extend the flow to passive buyers, in order to avoid situations where the distribution of scarce strategic material reaches only the most active.

Our strategy may be thought of as analogous to the basic strategy of [4] which however studies a different problem, namely maximizing the potential of the marketplace itself to serve as a re-distributive tool.

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1.1 Main idea

For simplicity, we assume that there is only one commodity to be distributed. Our system is formed by a sequence of markets happening sequentially over discrete time. Each market has two parts: 1. Sellers announce the supply for the critical commodity and buyers announce the demand for the commodity. On the basis of this input, the system assigns, according to pre-agreed rules of fairness, to each buyer a fair amount of *rights to purchase the commodity.* 2. In each market of the sequence, both the critical commodity and the rights are traded. There is only one limitation: at the end of each market, each buyer must have at least as big amount of rights as of the commodity. Uncoupled rights disappear in the end of each market.

The idea is that some (active) buyers want to buy more of the good right away, so according to the rules of the market, they can buy from hesitant (passive) buyers an amount of rights assigned to them. With this money saved, even these passive buyers can buy the commodity in next markets, because the rights are recalculated before each market, and thus renewed.

1.2 Frustration and Willingness to Pay

We need to introduce some measures indicating how successful the proposed hybrid mechanism is. We recall that our goal is to widen the flow of the critical commodity in the time of a distribution crisis. For a single market, we define the *frustration* of a buyer b as the ratio $0 \leq a/r \leq 1$ where r is the amount of the rights potentially assigned to b and a is the difference between r and the amount of the good purchased by b, if this is at least zero, and it is defined to be zero otherwise. We note that the notion of frustration extends to normal times without crises and to the unrestricted free market with the critical commodity. In normal times, the amount of assigned rights is equal to the demand and moreover b can buy all the goods it demands. Hence, its frustration is zero. Our concept of *frustration* is somewhat similar to the concept of *deprivation cost* introduced in [5].

Our aim is to show that in a sequence of markets of our system, *the frustration of each buyer becomes smaller than it would be in the free market.* In order for that to happen it is needed that the frustrated buyers increase their *willingness to pay* (formally introduced in section 3). Our proposed mechanism changes the willingness to pay by trading the assigned rights.

1.3 Main Contribution

We suggest a new distribution system for crises consisting of a sequence of markets and based on autonomous behavior of participants. For simplicity, we assume that there is only one *indivisible good* to be distributed and we call it *Good.* Indivisibility is a natural choice because of packaging. We initiate the study of properties of such systems in the middle of a crisis when the participants' individual utility is dominated by the individual utility in the current market, i.e., the participants do not perceive trends in the crisis.

We show that an auction-based mechanism approximates an equilibrium of each of the markets of the system in a polynomial number of steps and we observe how the frustration decreases over time, confirming the goal of the paper.

2 Fairness

In this section we study rules of the *ethical fairness* for distributing the available supply of one commodity called the *Good*. These rules need to be declared in the beginning of each crisis and may differ for different crises. According to declared fairness rules, it has to be possible to *hypothetically* distribute the available supply fairly among participants.

Let us denote by G the finite set representing the available supply of Good to be distributed. We denote the finite set of buyers by B.

Definition 1. *Let each buyer $b \in B$ have its* truthful demand *d_b for Good, and let $|G| \leq \sum_{b\in B} d_b$. A* distribution *of G is a function $f = f_{|G|,(d_b;b\in B)} : B \to \mathbf{N}$ satisfying $\sum_{b\in B} f(b) = |G|$ which, given the parameters $|G|$ and $(d_b; b \in B)$, determines for each $b \in B$, the amount $f(b)$ of G assigned to b.*

What are desirable properties of a distribution? Clearly, it should be *monotone* in the individual demands and *anonymized*, i.e., yield the same outputs if the buyers are permuted. Also, f should be *computable efficiently.* We also need to address the issue of the *truthfulness of the inputs* since there are situations when it may be advantageous for a buyer to declare a higher than actual need or for a seller to declare lower than actual availability of the good. We are not aware of a general mechanism ensuring truthfulness in our setting; see section 3.2 for a discussion.

A choice of a *fair* distribution depends on the actual crisis. Determining aspects of fair distributions is our work in progress. Algorithmic fairness is currently extensively studied, see e.g. [3], [6], [7], [8]. In this section, we describe only one candidate, the *contested garment distribution (CGD)*, defined by Aumann and Maschler [1] as a formalization of an ancient bancrupcy resolution method.

2.1 CGD for divisible good

Let us assume we are dividing an estate E instead of elements of G, and we assume E is *divisible*. Let $d_1 \leq \ldots \leq d_r$, $r = |B|$, be the demands of the buyers. In the description of CGD, we follow [1].

Let us think of the estate as gradually growing. When it is small, all buyers divide it equally. This continues until buyer 1 has received $d_1/2$; for the time being she then stops receiving supplies, and the remaining supply is divided equally between the remaining $r - 1$ buyers. This, in turn, continues until buyer 2 has received $d_2/2$, at which point she stops receiving supplies for the time being, and the remaining supply is divided equally between the remaining $r-2$ buyers. The process continues until each buyer has received half of her claim. This happens when $E = D/2$, where we let $D = \sum_{i=1}^{r} d_i$.

When $E = D/2$, the process is the mirror image of the above. Instead of thinking in terms of i's award x_i, one thinks in terms of her loss $d_i - x_i$, the amount by which her award falls short of her demand. When the total loss $D-E$ is small, it is shared equally between all buyers. The buyers continue sharing total loss equally, until buyer 1 has lost $d_1/2$ (which is the same as receiving $d_1/2$). For the time being she then stops losing, and an additional total loss is divided equally between the remaining $r-1$ buyers. The process continues until each buyer has lost half of her demand, which happens when $E = D/2$. This is precisely to where we got in describing the first part of the procedure.

Adaptation to the distribution of indivisible good. Distributions of divisible goods can be naturally adopted to distribution of indivisible good as follows: (1) Find the *fractional* distribution as if the good is divisible, (2) Round down the fractional distribution for the participants, (3) Distribute the surplus reflecting the societal preferences by a chosen optimization mechanism. A simple possibility is to linearly order the participants and round up the fractional solution for the participants in an initial segment. This may help the central authority to *enhance (its understanding of) the fairness* by treating participants in an unequal but socially beneficial and accepted way.

3 A Sequence of Markets for one indivisible good

In the previous section, we introduced a *fair distribution* of the limited available supply of Good to buyers, i.e., a fair distribution of *rights to buy Good.* Clearly, the number of these rights equals $|G|$. We call them *Rights* and we let R be the set of all the Rights.

In this section, each critical episode will consist of a finite sequence of trading events which we call *Markets*, happening regularly one after another in time. The participants of each Market of the sequence are the same and are divided into *buyers* and *sellers* depending on whether they produce or consume Good. We denote by B the set of the buyers and by S the set of the sellers.

We will assume that for each Market of the sequence, an allocation of Good to sellers and a fair allocation of Right to the buyers is defined and *is independent* on previous Markets. We are modeling a middle of a distribution crisis and we thus assume that the agents' utilities do not depend on time. We leave the study of the beginning and end of the crisis for future work.

Good and Right are traded freely except of two conditions: (1) after each Market, each buyer needs to have at least as many items of Right as the items of Good and (2) money obtained by selling items of Right can be used for buying Good in the future Markets only.

3.1 A view of a single Market

Let us describe how a Market could work. The first step towards a practical implementation is to create a Certified Public Portal (CEP) facilitating trading both in normal and critical times. For example by the Czech law, in various crisis settings, authorities in charge can require all dealings with strategic goods in a prescribed portal, and all the production and storage reported. The Certified Public Portal sends out a message to the central authority that a distribution crisis is happening for a commodity called Good. The central authority reacts by declaring a crisis and declares that CEP is the designed portal for the distribution of Good during this crisis. The central authority declares the schedule for trading in the portal:

a. Time period for declaring the demand and the offer in the portal in which trading is frozen. Mechanisms for truthful declarations need to be in place.

b. The portal freezes demands and offers and issues the items of Right according to the agreed fair distribution.

c. The portal opens trading with Good and Right.

d. End of trading: Central authority penalizes all the buyers who have more items of Good than items of Right. Unmatched items of Right disappear.

There needs to be a protocol for maintaining *money* obtained by selling items of Right. It is an ethical requirement that such money stays in the system and is used for buying items of Good or Right in future Markets. This motivates introducing the third commodity called Money in the formal definition of Market below. Another reason to have Money as a commodity is that the utility for Money differs among the participants and we will use this fact for studying the considered sequence of Markets of a critical episode. For similar treatment of Money, see [4].

Definition 2 (Single Market of a Critical Episode). *In each Market, three indivisible commodities will be traded:* Good, Right, Money*. We denote by G (R, M respectively) the set of all the items of Good (Right, Money respectively). The trading is facilitated by selling and buying items of commodities. In these acts, the price of Good (Right respectively) may differ. However, we require that the price of each unit of Money is always equal to* 1.

Individual utilities. *We denote by $u_p(H,x)$ the utility of x items of set H for a participant p. (1) All $u_p(H,x)$ in this paper will be* monotone*. (2) For each participant p, $u_p(M,x)$ depends* linearly *on x and sellers have a positive utility only for Money. (3) For each buyer b, $u_b(G,x) + u_b(G,y) \geq u_b(G,x+y)$ holds.*

Initial endowments. *Each seller s has an initial endowment g^s of Good and each buyer b has an initial endowment r^b of Right and m^b of Money and trades them according to its utility function. We assume that these initial endowments are known to all participants before Market starts.*

Definition 3 (Equilibrium). *If $X \subset G \cup R \cup M$ then we let $m(X)$ ($g(X), r(X)$ respectively) denote the number of items of Money (Good, Right respectively) in X. A* solution *of Market consists of (1) the prices π_R, π_G per item of Right and Good (we recall that the price per item of M is equal to* 1 *always), and (2) a* *partition of a subset of the union of all the initial endowments into* baskets B_p, $p \in S \cup B$.

Given a solution, we say that the assigned basket of a participant p is feasible *if it satisfies: (1) the total price of B_p is at most the total price of p's initial endowment, (2) the money obtained from selling Right are not used for buying Good, i.e., if $p \in B$ then $m(B_p) \geq \pi_R(r^p - r(B_p))$ and (3) if $p \in B$ then $r(B_p) \geq g(B_p)$.*

A solution is feasible *if all the baskets are feasible. A solution is an* equilibrium *if for each participant, (1) its basket is feasible and (2) its utility of its basket is maximum among all feasible sets of items.*

Remark 1. In a feasible solution, we require that the money obtained from selling Right are not used for buying Good in the same Market. The reason is to give an advantage to the active buyers, who can thus obtain Good earlier than the passive buyers.

The dynamics of the Market is determined by $u_b(M,x)$ and $u_b(G,x)$, b buyer. These individual utilities are crucial for studying the Market and its outcome and they are not disclosed. Below we introduce a related function called *willingness to pay* which, unlike the utility functions, can be observed in the Market.

Definition 4. *The* Willingness to pay *of a buyer b is a non-negative integer function that associates, with a non-negative integer x, the maximum number $w_b(x)$ of items of Money of the same utility as x items of Good, i.e., $w_b(x)$ is maximum such that $u_b(M, w_b(x)) = u_b(G,x)$.*

3.2 Strategy and coalition proofness

Our strategical aim is that the rights are sold as soon as possible so that passive buyers get funds without delay and can buy Good in subsequent Markets. We require that after each Market of the sequence, for each buyer there is a check whether it has at least as many items of Good as of Right. We list below features of the proposed system supporting the strategy and coalition proofness.

(1) The distribution system is implemented in a certified portal, the history of trading provides indirect mechanisms for (checking) truthful reporting of demand and supply, (2) The complete history of trading of the Right is known by using tailored data structures, (3) We suggest to have some special participants which help ensuring the strategy and coalition proofness, e.g., a 'central bank' in charge of the rights. (4) It is possible to implement Market using anonymous mechanisms. In this paper, we study one such *buyers-driven* mechanism, see Definition 5 and Corollary 1. Studying *sellers-driven* mechanism is initiated in the last section.

Definition 5. *The* Couple mechanism *is the following way to implement the Market which is driven by the activity of the buyers: (1) In the beginning, the central authority assigns the items of Good to buyers arbitrarily so that each buyer has the same number of items of Good and Right. Each buyer arbitrarily* *pairs the items of Good and Right into items of a new type of good called* Couple. *(2) The buyers subsequently sell and buy the items of Couple. (3) In the end of trading the prices are cleared, i.e., the price of each item of Couple is equally divided between Good and Right and the resulting money are cleared to sellers and buyers. (4) An* equilibrium *for this mechanism is defined as the market clearing for the trading part of the mechanism.*

Remark 2. In the above mechanism, the price of Good is equal to the price of Right. This is strategically correct, but restricts the autonomous behavior in the Market. The algorithm of the next section also aims at approximating a Market equilibrium in which the prices of Good and Right are equal. A study of less restrictive market mechanism is initiated in the last section.

The results of Theorem 1, Corollary 1 and Theorem 2 remain valid for any fixed ratio of the prices of Good and Right.

4 An Algorithm Approximating Market equilibrium

In this section we introduce an efficient auction-based algorithm for finding an approximate equilibrium of Market. For the purpose of the algorithm, we introduce one more commodity of indivisible items called the Couple. Each item of Couple is a pair (s,t) where s is an item of Good and t is an item of Right. We denote by $u_p(C,x)$ the utility of x items of Couple for a participant p, and we let $u_p(C,x) = u_p(G,x)$. The algorithm *auctions* items of Couple. We will denote the current price of one item of Good (Right, Money, Couple respectively) by π_G (π_R, π_M, π_C respectively).

The algorithm description. Let $0 < \epsilon < 1$. The algorithm is divided into *iterations*. During each iteration, some items of Couple are sold for π_C and some for $(1+\epsilon)\pi_C$, and analogously for Good and Right. The price π_M is always constant equal to 1. Each iteration is divided into *rounds*. An iteration ends when the price of Couple is raised from π_C to $(1+\epsilon)\pi_C$. Initially, we let $\pi_M = \pi_G = \pi_R := 1, \pi_C := 2$ and each buyer gets the surplus *cash* covering its initial endowment.

- Round: consider buyers one by one. Let buyer b be considered. Let us denote by o^b the number of items of Couple b currently has, and by o^b_+ the number of items of Couple b currently has of price $(1+\epsilon)\pi_C$.
- Let S^b be b'th optimal basket given the current price π_C, i.e., a set of items of Couple and of Money of max total utility which b can buy with its cash plus $\pi_C o^b$. Let $c(S^b)$ denote the number of items of Couple in S^b.
  1. If $c(S^b) < o^b$ then b does nothing, the algorithm moves to the next buyer. [if this happens then the current basket of b is optimal for the previous price $\pi_C/(1+\epsilon)$ and $o^b_+ = 0$.]
  2. If $c(S^b) \geq o^b$ then b buys items of Couple via the *Outbid*.

OUTBID:

- We keep as the invariant of the algorithm that the cash of each buyer b is always at least $\pi_R(r^b - c(B_b))$ where B_b is b's current basket.
- The system buys by cash one by one and at most $c(S^b) - o^b_+$ items of Couple for price π_C and sells them to b for cash price $(1+\epsilon)\pi_C$ per item, maintaining the invariant. First, it buys from b itself.
- An alternative for buying the items of Couple is to buy separately items of Good and Right and compose them into items of Couple. This happens when some items of Right and (necessarily the same amount of items of) Good are not yet coupled in previous tradings. We observe that this happens only if they are available for the initial price from the participants. In this situation, the system again buys items of Right first from the buyer b. However, the system pays nothing if it buys items of Right from an initial endowment of a buyer for the initial price since the payment is already in the surplus money.

– If no more Couple is available at price π_C after the Outbid then the current round and iteration terminate, $\pi_G := (1+\epsilon)\pi_G$, $\pi_R := (1+\epsilon)\pi_R$, $\pi_C := (1+\epsilon)\pi_C$ and the surplus money is updated: everybody who had Good or Right in its initial endowment gets extra surplus money, $\epsilon\pi_G$ per item of Good or $\epsilon\pi_R$ per item of Right.
– If a round went through all buyers, the algorithm proceeds with next round.
– When nobody wants to buy new items of Couple, the whole trading ends. The system takes all items of Money from the buyers, sells them to the buyers and sellers and keeps whatever items remain.
– The OUTPUT of the algorithm consists of (1) the collection of the final baskets of each participant and (2) the price-vector of the terminal prices π_G, π_R, π_C.

We note that the outcome of the algorithm has an interesting feature: the terminal price of Right is equal to the terminal price of Good.

Theorem 1. *Let $0 < \epsilon < 1$ be fixed and assume each act of a buyer in rounds needs at most a constant times $|B|$ steps. Let the initial endowments satisfy, for each buyer $b \in B$, the following properties:*
(0) $2/\epsilon < m^b$,
(1) $m^b \geq 4r^b$,
(2) for each $x \leq r^b$, $u_b(G,x) \geq 2u_b(M,x)$ *and*
(3) for each $x \geq 1/2m^b$, $u_b(M,m^b) \geq u_b(M,m^b - x) + u_b(G,x)$.

1. *The time-complexity of the auction-based algorithm is at most $|B|^3(1 + \log_{1+\epsilon} m)$; hence, the auction-based algorithm is polynomial in the input size.*
2. *For each participant, its basket assigned by the algorithm is feasible and its price plus 1 is bigger than the total price of its initial endowment.*
3. *Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1-2\epsilon)$ times the utility of its optimal feasible basket.*

Remark 3. The assumptions (0)-(3) of the theorem are needed for the proof. (0), (1) and (3) are not restrictive for applications since it is natural to assume that the buyers are institutions (hospitals) which have money and use their individual utility function for deciding how to spend them. (2) is scaling.

Remark 4. Concerning the assumption that each act of a buyer in each round needs at most a constant times $|B|$ steps. All utility functions are known and monotone, $u_b(M, x)$ is linear and $u_b(G, x)$ is concave. Let X be the current basket of b. Buyer b does not need to know S^b, it only needs to

(1) Find maximum k such that $u_b(G, g(X)+k) - u_b(G, g(X)) > u_b(M, m(X)) - u_b(M, m(X) - \pi_C k) = u_b(M, \pi_C k)$, thus needs to solve: given a constant K, find max k such that $u_b(G, g(X) + k) - u_b(G, g(X)) > Kk$; *we assume that this can be done in a constant time by buyers.*

(2) Buy at most k items of Couple from buyers, satisfying the invariant of the algorithm.

4.1 Proof of Theorem 1

We denote by m, r, g the total initial endowments of Money, Right and Good and recall that $r = g$. We prove the theorem in a sequence of claims.

Claim 0. At each stage of the algorithm, the total surplus is at most $2m$: it is true in the beginning by assumption (1), the surplus is gradually decreased during each iteration and at the end of each iteration, deleted funds are given back.

Claim 1. In the first iteration, all items of Good and Right are paired.

Proof. By assumptions (2) all the buyers prefer to buy at least the fair amount of items of Couple for the initial price; by assumption (1) there is enough cash in the initial surplus of each buyer to do it.

Claim 2. After the end of the first iteration: (1) a buyer owes to the system only cash for items of Money in its initial endowment and (2) total cash among participants is always at most m.

Proof. The first part follows from Claim 1 since all items of Good and Right are sold and bought at the end of the first interaction. For the second part we note that among sellers, the total cash is $g\pi_G$ since all items of Good were sold in the first iteration and among buyers, the total cash is at most $m - g\pi_G$ since the buyers payed for the items of Good and there is no cash left from the initial endowments of Right since all items of Right were sold and bought in the first iteration.

Claim 3. The number of rounds in an iteration is at most $2 + |B|$.

Proof. We observe that in each fully completed round, either none of the buyers buys items of Couple and the trading ends, or none of the buyers buys items of Couple in the next round and the trading ends or at least one buyer acts for the last time in this iteration: if in the current round every buyer buys items of

Couple only from itself then in the next round nobody buys since nobody got additional cash. Hence let a buyer b buy items of Couple from another buyer in the current round. It means that b gets no additional cash in this iteration since it has no items of Couple for π_C, otherwise it would have to buy these first by the rules of the outbid and the current round is the last active round for b.

Claim 4. The total number of iterations is at most $1 + \log_{1+\epsilon} m$.

Proof. Each iteration raises the price of Couple by the factor of $(1 + \epsilon)$ and the max price per unit of Couple cannot be bigger than the total surplus.

Claim 5. Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket.

Proof. (1) Buyers owe nothing to the system since after the end of the trading they keep only the items of Money they can buy with their remaining cash.

(2) After the end of the trading and buying items of Money, each participant is left with less than 1 dollar by the second part of Claim 2.

(3) The basket of each seller is optimal since all items of Good were sold.

(4) The only reason why the basket of buyer b is not optimal is: For some items of Couple, b payed $(1 + \epsilon)\pi_C$ where π_C is the terminal price of Couple. Let c (y respectively) denote the total number of items of Couple (Money respectively) in b's *optimal basket*. The utility of the *optimal basket* of b is thus $u_b(C, c)+u_b(M, y)$. By assumption (3), $y > c\pi_C$.

In b's *terminal basket*, there are c items of Couple and at least $y - \epsilon\pi_C c - 1$ items of Money. First let $\epsilon\pi_C c \geq 1$. The utility of b's *terminal basket* is thus, using the assumption on the linearity of the utility of Money, at least $u_b(C, c) + u_b(M, (1 - 2\epsilon)y) = u_b(C, c) + (1 - 2\epsilon)u_b(M, y)$.

Secondly let $\epsilon\pi_C c < 1$. The utility of b's *terminal basket* is thus at least $u_b(C, c) + u_b(M, (m_b - 2))$ and Claim 5 holds since we assume $1 > \epsilon > 2/m^b$.

Proof of Theorem 1

(1) It follows from Claims 3, 4 that the time complexity of the auction-based algorithm behaves asymptotically as $|B|^3(1 + \log_{1+\epsilon} m)$.

(2) follows from (2) of the proof of Claim 5. (3) is Claim 5.

Claim 1 of the above analysis of the auction-based algorithm means that the algorithm also provides an implementation of the Couple mechanism of Definition 5. Hence, we get the following

Corollary 1. *The auction-based algorithm is an implementation of the Couple mechanism which, under assumptions of Theorem 1, leads in at most* $|B|^3(1 + \log_{1+\epsilon} m)$ *steps to a feasible solution where each buyer or seller gets a basket of utility at least* $(1 - 2\epsilon)$ *times the utility of its optimal feasible basket for the terminal prices of the algorithm.*

5 Development of Willingness to Pay and Frustration in the Sequence of Markets

The Markets happen subsequently in a sequence. How do the parameters of subsequent Markets of the sequence change? We distinguish two regimes, (1) Deep in the crisis and (2) At the beginning or approaching the end of the crisis.

In this paper we study only the regime (1). This enables us to assume that *the individual utility of Good does not change in subsequent Markets*.

What changes is the individual utility of Money: buyers who sell items of Right enter the subsequent Market with more Money and the new items of Money can be used, by the rules, only to buy items of Good or Right. Hence, the utility function of Money changes for these buyers. We note that, since the individual utility of Good is unchanged, that the change of the individual utility of Money can equivalently be described as a change of the willingness to pay.

Definition 6 (potential willingness to pay). *Let buyer b sell, in the current Market, items of the Right for the total price z_b. If b buys Right then we let $z_b = 0$. Let w_b denote its willingness to pay in the current Market. Then for the next Market, its* potential willingness to pay*, denoted by w'_b, is $w'_b(x) := w_b(x) + z_b$, x arbitrary.*

Theorem 2. *Let us consider a sequence of Markets satisfying (1) the total supply, the individual demand and the individual utility of Good do not change in the sequence, (2) the Markets are implemented by the auction-based algorithm and (3) after a Market of the sequence ends, the willingness to pay in the next Market is equal to the potential willingness to pay for the next market. Then in all but possibly the first Market, each individual frustration is at most $1/2$.*

Proof. Let a Market $M(i), i \geq 1$ of the sequence ended and let us consider the next Market $M(i+1)$. By the assumptions of the theorem, the auction-based algorithm repeats the steps of Market $M(i)$. After the final step of the auction for $M(i)$, the willingness to pay of the buyers with zero frustration in $M(i)$ (let us call them *happy*) is saturated.

However, the buyers with positive frustration in $M(i)$ (let us call them *frustrated*) remain active since they acquired additional funds in $M(i)$. Let b be such a frustrated buyer. We recall that its willingness to pay, given the current price of the Couple (which is equal to the final price of Couple in $M(i)$), assures that b is willing to buy an additional number of items of the Couple which is equal to the number n_b of items of Right b currently sold (which is equal to the number of items of the Right b sold in $M(i)$). This is because b currently has cash for these n_b sold items of Right.

Let us denote by S the set of n_b items of Couple containing the items of Right buyer b sold so far in $M(i+1)$.

Buyer b buys the Couple of S at an increased price which in turn frees funds of active buyers who may buy back. During this process the frustration of b can only go down. Hence, we only need to consider the case that the frustration of b

is strictly bigger than 1/2. Let $0 < n'_b < n_b$ be such that $n_b - n'_b$ is equal to the half of the number of assigned rights to b.

Let us assume b doubled price of Couple in its first bid, and buys n'_b additional items of Couple of S. Buyer b only needs to buy items of Good by which b spends all z_b additional items of Money it got from the previous Market $M(i)$:

- $2n'_b z_b/n_b$ items of Money for buying n'_b items of Good from S, and
- $(n_b - 2n'_b)z_b/n_b$ items of Money needed to increase the price of $r^b - n_b$ items of Good b already has, since $n_b - n'_b = r^b/2$.

We observe that happy buyers from which b bought new items of Couple are not willing to buy back since:

They obtain in total $2n'_b z_b/n_b$ items of Money for the sold n'_b items of Couple of S but in order to increase the price further, $2(n_b - n'_b)z_b/n_b$ items of this obtained Money is needed to increase the price of the remaining $(n_b - n'_b)$ items of Couple in S. Clearly by the definition of n'_b and since $n_b \leq r^b$, $(n_b - n'_b) \geq n'_b$.

Summarising, the frustration of b after in $M(i+1)$ ends is at most 1/2 by spending in addition only Money obtained from selling Right in $M(i)$.

6 Seller-Driven Market: Preliminary Results

Next, we numerically study more realistic seller-driven Market mechanism which is a variant of the double-auction with two kinds of auctioned goods. A continuation of this study is a work in progress [2].

A sequence of the below described Markets can be thought of as a multi-agent game. We use a state of the art deep reinforcement learning algorithm to approximate optimal strategies of the players. The algorithm tries to find a strategy for each participant which maximizes the expected future sum of its utility. Each agent's strategy is parameterized by a deep neural network. The strategy (policy) is trained by Twin Delayed Deep Deterministic Policy Gradient (TD3) [9]. Let us begin by defining the Market mechanism.

Definition 7 (Sellers Market Mechanism). *In the beginning of each Market of the sequence, (1) each buyer b receives m_b additional units of Money, and each seller receives g additional units of Good. (2) Each seller declares the amount of Good for sale along with the selling price. (3) Each buyer b declares the demand d_b for Good and (4) the resulting items of Right are distributed according to the CGD, see section 2. Then, (5) each buyer declares the amount of Rights he is willing to sell along with the selling price.*

Each buyer, given the declared quantities and prices, orders a number of offered items of the Good and the Right by declaring its desired prices and volumes for the Good and the Right. The Market pairs the compatible offers and orders, starting with buyer with highest desired price of Good, and cheapest offers by other participants. In order to buy items of Good, the buyer needs a sufficient amount of items of Right. Therefore, a buyer first buys items of Good and pairs them with its items of Right until it has no Right left. Then, it buys equal amount of Good and Right. Finally, each buyer b consumes at most its demand d_b of purchased Good and keeps the possible surplus for the future.

6.1 Utilities

Next, we describe the utilities of participants in a sequence of Markets which models a period of a developed distribution crisis. For a better understanding, we measure the amount of Good and Money in *units* in this section, similarly as if they are divisible.

The utility of a seller s *in the end of each Market of the sequence* consists of (1) the amount of Money s received during that Market (denoted by ΔM_s) and (2) a small negative utility (denoted by c_{store}) per unit of Good the seller still keeps which models the cost of storing the Good and motivates sellers to sell the Good promptly. At the end of the whole period, the sellers also get some small utility (denoted by $c_{\text{end supply}}$) per unit of Good they are left with since they would be able to sell the remaining Good eventually.

Utility of the buyers reflects their need to keep a steady supply of Good throughout the period. At the end of each Market, their utility is the number of items of Good they have, up to their demand if it is saturated. At the end of the period, they also get some small utility for the Money they have. Formally,

Definition 8 (Utility). *Consider a sequence of Markets $X_1, \ldots, X_T$. For $t = 1, \ldots, T$ let (1) $d_b(t)$ denote the demand of buyer b in market $X(t)$, (2) let $M_p(t)$ and $G_p(t)$ be the amount of Money and Good participant p has at the end of Market X_t, (3) let $\Delta M_p(t)$ denote the amount of Money p earned in $X(t)$, and (4) we also let $D(T) = 1$ and $D(t) = 0$ for $t < T$.*

For a seller s,

$$u_s(t) = \Delta M_s(t) + c_{\text{store}} G_s(t) + D(t) c_{\text{end supply}} G_s(t), \tag{1}$$

For a buyer b,

$$u_b(t) = \min\{d_b(t), G_b(t)\} + D(t) c_{\text{money}} M_b(t). \tag{2}$$

6.2 Results

In this section, we present results of a particular instance of a sequence of the above described Markets. We aim to model a situation in which there is a large discrepancy between the active and passive buyers. Specifically, we consider a sequence of $T = 10$ Markets with four buyers and four sellers. The demands[3] and incomes of buyers are fixed to values given in Table 1. The sellers receive $g = 1/4$ units of Good per Market. At the start of the simulation, the participants have no Money or Goods.

We use the following values for the constants: $c_{\text{store}} = -1/8$, $c_{\text{end supply}} = 1/2$ and $c_{\text{money}} = 1$. These ensure that the utility is mainly influenced by the primary motivation of each participant.

[3] More precisely, d_b is decreased by the amount of Good b has at the beginning of a Market, if any.

Buyer	1	2	3	4
m_b	4/32	5/32	6/32	1/32
d_b	1/2	1/2	1/2	5/2

Table 1. Demands and earnings of the buyers.

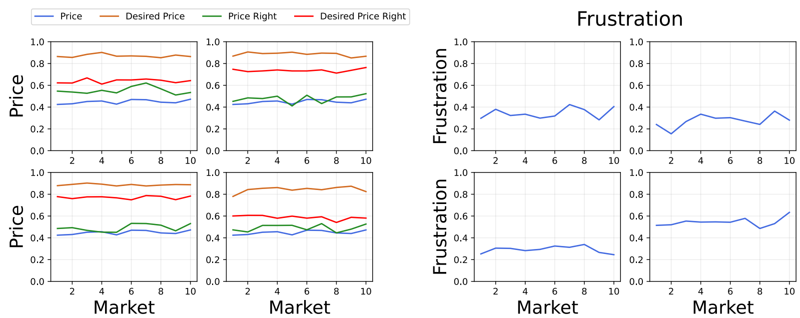

Fig. 1. Left: Selling and desired price of Good and Right; Right: Frustration of each buyer. The buyers are ordered from left to right, and top to bottom, i.e., the passive buyer is bottom right.

The first three buyers together have 93.75% of the Money and thus in the free market will receive 93.75% of the Goods. In contrast, the fair distribution following CGD is uniform if all Goods from the previous Markets are sold. If the sellers choose to offer more than the amount corresponding to uniform distribution, the CGD allocates the surplus Rights to the passive buyer first.

To train the strategy, the agents played 10 000 games, but the system stabilized after only $\approx$ 4 000 games. The final prices for each Market are shown in Figure 1. Since the participants strategies are not deterministic, we average all results over 100 sequences of Markets generated with final strategies.

Let us focus on the Goods first. The desired price for all buyers is larger than the selling price, thus the price set by sellers is acceptable for the buyers. The selling price is on average lower than the market clearing price, yet only $\approx$ 83% of the Good gets sold. The amount of Goods each buyer purchased during the sequence of Markets was $(1.94, 2.26, 2.31, 1.75)$. The passive buyer was thus able to obtain $\approx$ 21% of the total amount of Good, while in the free market he would have Money for only $\approx$ 6%.

The price of Right generally follows the price of Good and is acceptable for all buyers. This might seem counter intuitive, since the passive buyer aims to sell Right. However, in our Market mechanism, a buyer only buys Right in the second stage, if he has Money and used all Right assigned to him. In this situation, all buyers would buy Right for the mean selling price. The active buyers are willing to pay more for the Right than the passive buyer. The active buyers buy on average ≈ 0.036 units of Right per Market, which is $\approx$ 58% of the Right assigned to them by the uniform distribution.

Figure 1 shows the evolution of frustration during the sequence of Markets. As expected, the frustration is higher for buyers with lower income. However, it remains positive for all buyers, with the passive buyer's frustration being ≈ 0.54. A possible explanation: since the strategies of agents are stochastic, agents often take sub-optimal actions. This is supported by the fact that if we fix trained strategies to their means, the frustration of active buyers becomes zero and ≈ 0.46 for the passive buyer[4]. However, since the strategies were trained against stochastic strategies of the other agents, we might no longer be close to an equilibrium. We plan to return to these results in the future work.

7 Conclusion

In this paper, we define and study a new market mechanism for distribution crises based on autonomous behavior of participants. The main ingredients of the mechanism are the model of fairness and the definition of Market. We show that a simple auction-based Couple mechanism implementing the Market approximates an equilibrium of the Market in a polynomial number of steps. We also confirm that the frustration decreases in the sequence of Markets implemented by the Couple mechanism, confirming the goal of the paper.

Finally, in the last section we present some initial results on another (sellers-driven) market mechanism implementing the Market. More thorough study of this mechanism is work in progress.

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[4] Furthermore, the frustration decreases over the sequence of Markets.

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