Critical Goods Distribution System

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Distribution crises occur when there is a significant mismatch between the demand and supply of essential resources over an extended period. Without market regulations, these resources can be easily monopolized by the more affluent members of society, leaving the needs of the rest unfulfilled. While centralized planning can lead to fairer distribution, it also comes with economic and time-related inefficiencies. In this paper, we propose to model the crises as multi-round trading environments that balance fair allocation of buying rights with a market-based approach, taking advantage of the benefits of both via rationing. We examine two implementations of its single-round markets. The first market is a variation of the Arrow-Debreu model that incorporates buying rights, and we approximate its market-clearing solution through a polynomial-time auction algorithm. The second market involves non-myopic traders who optimize over multiple trading rounds. We determine the second market's equilibrium and demonstrate that it is coalition-proof. To assess the entire environment's equity, we use the concept of frustration, defined as the scaled difference between the amount of goods a buyer is entitled to according to their buying rights and the amount they can acquire through trading. We also introduce a concept of "Price of Anarchy" that serves as an analogy to the original definition in the context of frustration. Our main results show that with both markets, the inclusion of buying rights reduces the Price of Anarchy by at least half.

Under review.

1 INTRODUCTION

Crises of distribution are situations where the availability of a vital resource is so limited that individuals or organizations are unable to acquire the minimum amount necessary for functioning. These types of crises can occur due to a variety of reasons such as natural disasters, wars, or economic instabilities. From a distribution perspective, these situations possess two key characteristics that are important to consider.

Firstly, the ethical fairness of the distribution of the scarce resource is crucial. During a crisis, the distribution of resources is often not equitable, with some individuals or organizations receiving a disproportionate amount of resources while others receive little to none. This is where the principle of fairness comes in, as it aims to ensure that the distribution of resources is as equitable as possible. According to the rules of fairness, which are declared in advance, it is possible to hypothetically divide the resource fairly among different entities. In other words, buyers have the right to acquire a fair share of the commodity in question, and the total amount of rights to purchase the commodity is equal to the total amount available in the market. A central authority can then distribute the good in quantity equal to the right, which we consider an ideal benchmark allocation. This centralised approach however comes with economic and time-related inefficiencies [Moroney and Lovell, 1997].

On the other hand, markets, modeled as large single or double auctions with many sellers and buyers on each side, have the potential to distribute goods in a flexible and reliable manner [Cripps and Swinkels, 2006]. To improve the system's efficacy, the second important aspect of distribution crises is hence the presence of a market for the vital resource. The downside of this approach is that in the absence of stable supply, the scarcity of the resource leads to a significant rise in prices, with organizations attempting to acquire more than their fair share, always at the expense of others. This can lead to a situation where the most powerful or well-connected individuals or organizations are able to acquire the majority of resources, leaving the less fortunate with little to no access to the vital resource. To prevent such scenarios, *our goal is to increase the flow of the resource* to passive buyers, rather than allowing the distribution of scarce resources to be concentrated among the most active participants in the market. This can be achieved by creating a market for the resource, where buyers can purchase the resource at a fair price by implementing a rationing system where resources are traded based on pre-established rules of fairness.

In summary, distribution crises are extreme situations where the availability of a vital resource is limited, and it is important to consider the ethical fairness of the distribution in the presence of a market to ensure an equitable distribution of resources. Such approach may be thought of as analogous to the basic strategy of [Dworczak et al., 2021] which, however, studies a different problem, namely maximizing the potential of the marketplace to serve a re-distributive purpose.

1.1 Structure of the system

For simplicity, we assume that there is only one commodity to be distributed. Our system consists of a sequence of markets taking place sequentially over discrete time. Each market has two parts:

(1) Sellers announce their supply of the critical commodity, while buyers announce their demand for it. Based on this information, the system assigns buying rights in accordance with pre-agreed rules of fairness.
(2) In each market of the sequence, both the critical commodity and the buying rights are traded. However, there is one restriction: at the end of each market, each buyer must hold at least as many rights as they do of the commodity. Any uncoupled rights are eliminated at the end of each market.

The proposed hybrid mechanism aims to widen the flow of a critical commodity during a distribution crisis by allowing active buyers to purchase rights from hesitant, passive buyers. These saved funds can then be used by passive buyers to acquire the commodity in subsequent markets, as rights are recalculated and renewed before each market.

To improve the robustness of the system against strategic manipulation from the side of the traders, we require that after each market of the sequence, for each buyer there is a check whether their amount of the critical commodity is at most as large as their amount of right. We further require that the distribution system is implemented in a certified portal, where the history of trading provides indirect mechanisms for (checking) truthful reporting of demand and supply, the complete history of trading of right is known and archived, and the particular trading mechanisms are agnostic to the trader's identities.

In order to gauge the effectiveness of this mechanism, we introduce a measure of "frustration" for individual buyers. Specifically, for a single market, *frustration* is defined as the ratio $0 \leq a/r \leq 1$, where r represents the amount of rights potentially assigned to a buyer b, and a is the difference between r and the actual amount of the good purchased by b, assuming this difference is non-negative. If the difference is negative, the frustration is defined as zero. It is worth noting that this concept of frustration applies not only in times of crises, but also during normal market conditions and in unrestricted free markets for the critical commodity. In normal times, assigned rights are bigger or equal to demand, and buyers are able to purchase all of the goods they desire, thus their frustration is zero. This notion of frustration is similar to the concept of "deprivation cost" discussed in the work of Holguín-Veras et al.

Our ultimate goal is to demonstrate that, in a sequence of markets using this system, the frustration of each buyer is smaller than it would be in a free market. This can be achieved by increasing the "willingness to pay" of frustrated buyers through the trading of assigned rights. The *willingness to pay* of a buyer b associates, with a positive x, maximum amount $w_b(x)$ of money of the same utility as amount x of the critical commodity.

1.2 Main contributions

Our primary contribution is the development of a formal system that clearly distinguishes the allocation of buying rights and the trading of goods during the midst of a crisis characterized by a relatively stable but limited supply and an excessive demand. Our main goal is to investigate the impact of incorporating a fairness mechanism for allocating rights to purchase in a sequence of over-demanded markets, in order to enhance social good by improving fairness of the free market trading. Specifically, we formalize our hybrid trading environment, including the fairness mechanisms and the measure of equity, defined as frustration, in Section 2.1. We then use this frustration measure to analyze the evolution of the Price of Anarchy in this environment with two different implementations of the single-round markets. To the best of our knowledge, we are the first to rigorously analyze a system explicitly combining a market mechanism with a fairness mechanism allocating buying rights for more socially just redistribution of critical goods during the times of need. It is important to note that our findings are independent of the specific fairness mechanism used.

In Section 3, we delve into the analysis of solutions in systems where traders are myopic, in the form of market-clearing prices with rights. We formulate an auction-based algorithm that can approximate the solution in polynomial time while providing a guarantee on its quality.

Theorem 3.3 (informal). *There exists an algorithm with time complexity cubic in the number of traders that approximates the maximum clearing solutions of the system with myopic traders such that each trader receives a utility at least $(1 - 2\epsilon)$-times their optimum.*

Using this result, we are able to bound the overall Price of Anarchy in the system.

THEOREM 3.5 (INFORMAL). *Consider a sequence of markets with solutions computed using the algorithm from Theorem 3.3. Then up the first market, the Price of Anarchy is at most* $1/2$.

Section 4 then considers a more complex system with traders who optimize over the entire crisis. We examine properties of the arising equilibrium and design an algorithm computing it efficiently.

THEOREM 4.6 (INFORMAL). *There exists an algorithm with time complexity linear in the number of traders that computes a coalition-proof equilibrium of the system with non-myopic traders.*

Further analysis of the equilibrium reveals the equilibrium offers a similar asymptotic upper bound on the Price of Anarchy as in the system with myopic traders.

THEOREM 4.9 (INFORMAL). *Consider a sequence of markets where traders follow the equilibrial strategies from Theorem 4.6. Then the Price of Anarchy is at most* $1/2$ *as the number of markets tends to infinity.*

In the last part of the paper, we summarize the desired features of the trading system and provide several potential future directions.

1.3 Related work

Allocations a set of resources to a set of individuals in a fair manner have been extensively studied over the past decades. The objective of a fair distribution is to identify an allocation mechanism that satisfies certain criteria generally referred to as fairness criteria. In the literature, there is no clear definition of fairness, rather a wide variety of notions has been introduced; see for example [Thomson, 2011] for a survey. Many recent works considered the question of whether it is possible to simultaneously achieve fairness and efficiency, and studied Pareto optimality and fair allocations for various fairness notions. For example, [Caragiannis et al., 2019] presented inherent connection between Envy Freeness, Pareto optimality, and the notion of maximum Nash welfare.

Our work contributes to the field of redistributive mechanisms, particularly those aimed at reducing inequalities. A similar study in the literature examines a two-sided market trading goods of homogeneous quality, with the goal of optimizing the total utility of traders [Dworczak et al., 2021]. However, our approach differs in that we consider more general fairness measures, rather than just the social welfare, and assume that utilities are common knowledge. This work has been recently expanded to include a setting with heterogeneous quality of tradable objects, various measures of allocation optimality, and imperfect observations of traders [Akbarpour et al., 2020]. Another related work examines multiple market and non-market mechanisms for allocating a limited number of identical goods to multiple buyers [Condorelli, 2013]. The author argues that when buyers' willingness to pay aligns with the designer's allocation preferences, market mechanisms are optimal, and vice versa. In crisis situations, it is reasonable to assume that critical resources are highly valuable to all participants, yet some may lack the money to acquire them. Combined with the fact that it is in society's best interest to allocate goods fairly, these findings suggest that relying solely on unregulated markets may not be the best approach in a distribution crisis.

Emissions allowances and tradable allowance markets present to a certain extent a similar mechanism to our work. Historically, tradable property rights were allocated by regulators directly to firms. This led to a number of inefficiencies, such as misallocation, regulatory distortions or barriers to entry the markets. Contemporary market designs utilize auctions for the allocation of tradable property rights. As presented in [Dormady, 2014], tradable allowance markets play an invaluable role in ensuring the efficiency of carbon markets and in the prevention of market power by large and dominant agents. In [Wang et al., 2020], efficiency of multi-round trading auctions for allocation of carbon emission rights is studied.

[Alkaabneh et al., 2021] develops a dynamic programming model for optimizing resource allocation by food banks among the agencies they serve. The framework measures effectiveness based on the nutritional value of the allocation decisions, efficiency as the utility of the agencies served, and equity as fairness in the allocation of food among those agencies.

In our work, we differ from the existing related works by designing a trading-rationing mechanism that introduces rights to purchase a fair amount of the scarce commodity, allocates them in a fair manner and trades them together with the critical commodity in order to ensure that scarce resources can be allocated among agents in a fairer manner than by the free market. To the extent of our knowledge, this is the first work that introduces such hybrid trading-rationing mechanism, and also analytically solves such a complex rationing scheme.

2 PROBLEM DEFINITION

Formally, we model a crisis as a multi-round trading environment, which we call *Crisis*. We have one scarce resource, which we call *Good*. The other two commodities, representing funds and right to purchase Good, will be referred to as *Money* and *Right*. Crisis consists of a sequence of $\mathcal{T}$ rounds which we call Markets. The traders in this trading environment form two disjoint sets of *sellers* and *buyers*, depending on whether they trade Good or Money. Each single-round Market of this multi-round trading environment starts by the sellers declaring the amount of Good for sale and the buyers declaring their *Demands* for Good. A central *fairness mechanism* then assigns buyers with appropriate amount of Right. In the beginning of each Market, each seller and buyer also receive an amount of Good and Money, respectively.

The traders then trade these *initial endowments* of Good, Right and Money with the restriction that *at the end of each Market, each buyer has at least as much Right as Good.* This restriction forms the core of our idea. We further require that Money obtained by selling Right cannot be used in the current Market for buying Good (see Remark 1 for a justification). We refer to the single-round trading that happens after the Right is distributed as *Trading*. Hence, each Market consists of the fairness mechanism and the Trading.

At the end of each Market, the sellers consume the obtained Money and the unsold Good and the buyers consume all the obtained Right and Good up to their declared Demand. The buyers keep the Money they obtained for the next Market. Hence, the Markets of the sequence forming a Crisis are not interdependent.

Remark 1. *There needs to be a protocol for maintaining Money obtained by selling Right. It is an ethical requirement that such Money stays in the system and is used for buying Good or Right during the current Crisis. We also require that obtained Money is used in* a future, not the current *Market, in order to give an advantage to the active buyers, who are along with the sellers the "kings" of the Crisis. The advantage consists in obtaining Good earlier than the passive buyers.*

Another reason to have Money as a commodity is that the utility for Money differs among the participants and we will use this fact for studying our multi-round trading environment. For similar treatment of Money, see [Dworczak et al., 2021].

2.1 Trading environment

Market number $\tau \le \mathcal{T}$ is defined as $\mathbb{M}^\tau = \mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$, where S is a set of sellers and B is a set of buyers. We denote $T = S \cup B$ the set of all traders and assume $S \cap B = \emptyset$. The $G^\tau = (G^\tau_t | t \in T)$ and $M^\tau = (M^\tau_t | t \in B)$ denote the sets of Good and Money each trader and buyer has at the beginning of a Market τ, respectively. We let $V^\tau \le \sum_{s\in S} G^\tau_s$ be the offered volume of Good in the market $\mathbb{M}^\tau$. We also denote subsets of a set with a subscript, for example $G^\tau_A = (G^\tau_a | a \in A)$

for a set $A \subset T$. The set of Demand $D^\tau = (D_b^\tau | b \in B)$ give the amount of Good each buyer hopes to acquire in the Market. The function u_t^τ is the utility of trader t in Market $\mathbb{M}^\tau$, defined as follows.

Definition 2.1 (Utility function of $\mathbb{M}^\tau$). The *utility function of trader t in market $\mathbb{M}^\tau$ is $u_t^\tau : \mathbb{R}^2 \to \mathbb{R}$ where $u_t^\tau(x,y) = u_t^\tau(x,0) + u_t^\tau(0,y)$ denotes the t's utility of amount x of Good and amount y of Money. We require that the utility function (1) is* monotone *in each coordinate. Moreover, for each trader t, (2) $u_t^\tau(0,x)$ depends* linearly *on x, (3) sellers have a positive utility only for Money, and (4) For each buyer b, $u_b^\tau(x,0) + u_b^\tau(y,0) \geq u_b^\tau(x+y,0)$.*

Finally, ϕ^τ is the fairness mechanism of Market $\mathbb{M}^\tau$. It has the following form.

Definition 2.2. A fairness mechanism is a function $\phi : \mathbb{R}_0^{+,|B|+1} \to \mathbb{R}_0^{+,|B|}$ which, given offered volume of Good V and Demand D of buyers, assigns allocation of Right to the buyers which satisfies $\forall V, V' \in \mathbb{R}_0^+, D \in \mathbb{R}_0^{+,|B|}, \forall b \in B, \forall \tau \in \mathbb{N}$ and all permutations of $|B|$ elements α

(1) $\sum_{b\in B} \phi_b(V,D) = V$,
(2) $D_b = 0 \Rightarrow \phi_b(V,D) = 0$,
(3) $D_b \geq D_b' \Rightarrow \phi_b(V,D) \geq \phi_b(V,D')$,
(4) $\phi_b(V,\alpha(D)) = \phi_{\alpha^{-1}(b)}(V,D)$,
(5) $V \geq V' \Rightarrow \phi_b(V,D) \geq \phi_b(V',D)$.

To work in the system as intended, the fairness mechanism has to distribute Right for all offered Good, assign no Right to buyers without Demand for the scarce resource, be non-decreasing with Demand of each buyer, and treat buyers with equal Demand equally. These conditions imply that the fairness function has a special "distribution-induced" form

$$\phi_b\left(\sum_{s\in S} v_s^\tau, D\right) = \alpha_b(D,V) \sum_{s\in S} v_s^\tau,$$

where α is a distribution. Indeed, condition 1 implies $\sum_{b\in B} \alpha_b(D,V) = 1$ and conditions 2 & 3 together imply $0 \leq \alpha_b(D,V) \leq 1$. The last condition restricts functional dependence of α on the Demand and V. Our results do not depend on a particular choice of a fairness mechanism. We include two well known examples for illustration purpose only.

Example 2.3. One example of a fairness mechanism is the *proportional fairness mechanism*, which allocates Right to b proportionally to his Demand

$$\phi_b\left(\sum_{s\in S} v_s^\tau, D\right) = \frac{D_b \sum_{s\in S} v_s^\tau}{\sum_{b\in B} D_b}, \qquad \alpha_b^\phi(D,V) = \frac{D_b}{\sum_{b\in B} D_b},$$

where the distribution is not dependant on the total volume distributed. Another example of a fairness mechanism is the *contested garment distribution* [Aumann and Maschler, 1985].

Market number τ, $\mathbb{M}^\tau$, consists of two steps. First, each buyer b is assigned $\phi_b(V^\tau, D^\tau)$ amount of Right by the fairness mechanism ϕ. In the second step, traders trade assigned Good, Right and Money in the Trading. The Trading is a standard market with two restrictions: (1) the final basket of each buyer has the amount of Right at least as big as the amount of Good and (2) money obtained for selling Right cannot be used to buy Good in the current Trading.

Definition 2.4 (Solution of $\mathbb{M}^\tau$). If X is a set containing amounts of Good, Right and Money then we let $\mathcal{M}(X)$ ($\mathcal{G}(X), \mathcal{R}(X)$ respectively) denote the amount of Money (Good, Right respectively) in X. A *solution* of Market $\mathbb{M}^\tau$ consists of (1) the price q^τ (p^τ respectively) per unit of Right (Good respectively), and (2) a partition of a subset of the union of all the initial endowments into *baskets* B_t, $t \in T$. Given a solution, we say that the assigned basket B_t of a trader t is *t-feasible* if it satisfies,

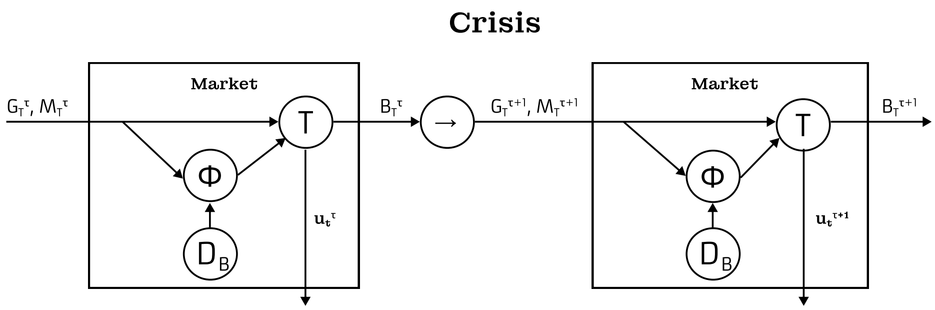

Fig. 1. An illustration of a Crisis consisting of a series of Markets. The fairness function ϕ is followed by the Trading phase, T. The baskets obtained by traders are transferred to the next Market.

assuming the price of a unit of Money is 1, that the total price of B_t is at most the total price of t's initial endowment, and further if t is a buyer then (1) $\mathcal{R}(B_t) \geq \mathcal{G}(B_t)$ and (2) B_t contains the amount of Money which t obtained from selling Right, i.e., $\mathcal{M}(B_t) = M_t^\tau - p^\tau \mathcal{G}(B_t) + q^\tau(\phi_t^\tau(V^\tau, D) - \mathcal{R}(B_t))$. A solution is *feasible* if for each trader t, B_t is t-feasible.

As remarked earlier, the subsequent Markets are not independent; a feasible solution of $\mathbb{M}^\tau$ influences initial endowments of Money in $\mathbb{M}^{\tau+1}$. The cascaded Markets formally define the *Crisis* as a sequence $\mathbb{C} = \mathbb{C}(S, B, \mathcal{T}, G, M, D, u, \phi, \mathcal{B})$ of $\mathcal{T}$ Markets and *their feasible solutions*, where S, B have the same meaning as in the previous subsection. Further, $G = (G^\tau : \tau \leq \mathcal{T})$, $M = (M^\tau : \tau \leq \mathcal{T})$, $D = (D^\tau : \tau \leq \mathcal{T})$, $u = (u^\tau : \tau \leq \mathcal{T})$ and $\phi = (\phi^\tau : \tau \leq \mathcal{T})$. Finally, $\mathcal{B} = (\mathcal{B}^\tau : \tau \leq \mathcal{T})$ where each $\mathcal{B}^\tau = (q^\tau, p^\tau, (B_t^\tau; t \in T))$ is a feasible solution of Market $\mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$. We further require that for each $\tau < \mathcal{T}$ and $t \in B$, $M_t^{\tau+1} \geq \mathcal{M}(B_t^\tau)$ (see Remark 1). An illustration of the structure of the Crisis is available in Figure 1.

2.2 Frustration and the Price of Anarchy

The Right can be thought of as the amount of Good a buyer is socially entitled to. In other words, if the distributional crisis is managed fully by a central authority, the buyer would receive that amount. A contrast between this “most fair" centralized solution, and whatever a buyer is able to acquire is captured by his *frustration*.

Formally, the frustration of a buyer is normalized difference between the Right he would be assigned and the amount of Good he acquired in a Market if that is at least zero, and zero otherwise. We note that this definition allows to measure frustration also in the free market, where no Right exists. However, a common notion of fair allocation may still exist.

Definition 2.5. Let $\mathbb{M}^\tau(S, B, G^\tau, V^\tau, M^\tau, D^\tau, u^\tau, \phi^\tau)$ be a Market. Then the *frustration* of buyer $b \in B$ of a non-zero demand in market $\mathbb{M}^\tau$, denoted by f_b^τ, is

$$f_b^\tau = \max\left\{0, \frac{\phi_t^\tau(V^\tau, D) - \mathcal{G}(B_t^\tau)}{\phi_t^\tau(V^\tau, D)}\right\},$$

Consider now the point of view of the central authority, such as government of a country. As a result of trading the Good, the final allocation differs from distributing it centrally. This difference measures the inefficiency of the trading to allocate Good fairly. This is conceptually similar to the Price of Anarchy (PoA), which measures the price the system pays for autonomous behaviour of the traders [Koutsoupias and Papadimitriou, 1999]. As such, we define the Price of Anarchy in our system as a scaled sum of frustrations of the buyers, i.e.,

$$PoA^{\tau} = \frac{\sum_{i=1}^{\tau} \sum_{b \in B} f_b^i}{\tau |B|}.$$

Note that $PoA^{\tau} \geq 0$ since $f_b^{\tau} \geq 0$ and $PoA^{\tau} = 0 \Rightarrow f_b^i = 0 \ \forall b \in B, \forall i \in \{1, \ldots \tau\}$. In the latter case, trading Good in the Market had the same social impact as distributing it centrally.

In this work, we aim to show that as a result of implementing Right, the Price of Anarchy decreases compared to the free market. Note that for the latter the PoA can be arbitrarily close to one if a buyer receives low amount of Money and has a high Demand. We consider two types of Markets based on whether the traders are myopic or not. In each case we provide an upper bound on the PoA therein. Due to space constraints, full versions of all proofs are in Appendix A.

3 MARKET WITH MYOPIC TRADERS

We begin by defining a concept of optimal solution of Market $\mathbb{M}^{\tau}$, which we study in this section.

Definition 3.1 (Optimal solution of $\mathbb{M}^{\tau}$ and $\mathbb{C}$). A solution of the Market is *optimal* if for each trader t, (1) B_t is t-feasible and (2) $u_t(\mathcal{G}(B_t), \mathcal{M}(B_t))$ is maximum among all $u_t(\mathcal{G}(Q), \mathcal{M}(Q))$, where Q is a t-feasible subset of the union of all the initial endowments. We further say a solution of a Crisis is optimal if for each τ, $\mathcal{B}^{\tau}$ is an optimal solution of $\mathbb{M}^{\tau}$.

Remark 2 (Justification of the definition of optimal solution). *As mentioned in the Introduction, we model the middle of multi-round distribution crises and thus we measure optimality of the feasible solution of a multi-round trading environment by how optimal are the feasible solutions of each round. In general, this does not need to maximize the sum of utilities of the $\mathcal{T}$ baskets of a trader t since* strategically *it may be advantageous to sacrifice optimality at time τ to gain higher individual utility at later times. We assume that in the middle of a distribution crisis such strategizing does not take place.*

3.1 Auction theoretic formulation

We introduce an efficient auction-based algorithm for finding an approximate optimal solution of Market $\mathbb{M}^{\tau}$. Since our discussion is valid for any Market, we will omit the upper index τ in this section to ease notation. We will assume in this section that Good (and thus also Right and Money) are *indivisible*. This is more practical for distribution crises since commodities come in packages. Analogous results with simpler proofs hold also for divisible commodities.

For the purpose of the algorithm, we introduce one more commodity of indivisible items called *Couple*. Each item of Couple is a pair (x_1, x_2) where x_1 is an item of Good and x_2 is an item of Right. For a trader t, we will assume their *utility of x items of Couple* is the same as of x items of Good, i.e., equal to $u_t(x, 0)$. The algorithm auctions items of Couple. We will denote the current price of one item of Good (Right, Couple respectively) by p (q, c respectively). We assume that the price of one item of Money is equal to 1. Finally, we recall that M_t denotes the initial endowment of Money of trader t and for ease of notation, we denote by R_t the initial endowment of Right of t.

The algorithm description. Let $0 < \epsilon < 1$. The algorithm is divided into *iterations*. During each iteration, some items of Couple are sold for c and some for $(1+\epsilon)c$, and analogously for Good and Right. Each iteration is divided into *rounds*. An iteration ends when the price of Couple is raised from c to $(1+\epsilon)c$. Initially, we let $p = q \leftarrow 1, c \leftarrow 2$ and each buyer gets the surplus *cash* covering its initial endowment of Money and Right. Cash is a dummy commodity representing the flow of Money in the system.

- Round: we fix an arbitrary order of buyers and consider them one by one in this order. Let buyer b be considered. Let us denote by o^b the number of items of Couple b currently has, and by o^b_+ the number of items of Couple b currently has of price $(1+\epsilon)c$.
- Let S^b be a set of items of Couple and of Money of max total utility which b can buy with its current cash plus co^b. Let $C(X)$ denote the number of items of Couple in set X.

(1) If $C(S^b) < o^b$ then b does nothing, the algorithm moves to the next buyer.
[if this happens then the current basket of b is optimal for the previous price $c/(1+\epsilon)$ and $o^b_+ = 0$.]

(2) If $C(S^b) \geq o^b$ then b buys items of Couple via the *Outbid*.

OUTBID:

- We keep as the invariant of the algorithm that the cash of each buyer b is always at least $q(R_b - C(B_b))$ where B_b is b's current basket.
- The system buys by cash one by one and at most $C(S^b) - o^b_+$ items of Couple for price c and sells them to b for cash price $(1+\epsilon)c$ per item, maintaining the invariant. First, it buys from b itself.
- An alternative for buying the items of Couple is to buy separately items of Good and Right and compose them into items of Couple. This happens when some items of Right and (necessarily the same amount of items of) Good are not yet coupled in previous tradings. We observe that this happens only if they are available for the initial price from the traders. In this situation, the system again buys items of Right first from the buyer b. However, the system pays nothing if it buys items of Right from an initial endowment of a buyer for the initial price since the payment is already in the surplus cash.

- If no more Couple is available at price c after the Outbid then the current round and iteration terminate, $p \leftarrow (1+\epsilon)p$, $q \leftarrow (1+\epsilon)q$, $c \leftarrow (1+\epsilon)c$ and the cash is updated: everybody who had Good or Right in its initial endowment gets extra cash, ϵp per item of Good or ϵq per item of Right.
- If a round went through all buyers, the algorithm proceeds with next round.
- When nobody wants to buy new items of Couple, the whole trading ends. The system takes all items of Money from the buyers, sells them to the buyers and sellers for cash and keeps whatever items remain.
- The OUTPUT of the algorithm consists of (1) the collection of the final baskets of each trader and (2) the terminal prices p, q, c.

We are able to analyze the algorithm only when initial endowments are feasible.

Definition 3.2. Let $0 < \epsilon < 1$. The initial endowments $(M_b, R_b); b \in B$ are *feasible for* ϵ if they satisfy, for each buyer $b \in B$, the following properties: (1) $M_b > \max(2/\epsilon, 4R_b)$, (2) for each $x \leq R_b$, $u_b(x, 0) \geq 2u_b(0, x)$ and (3) for each $x \geq \max(D_b + 1, 1/2M_b)$, $u_b(0, M_b) \geq u_b(0, M_b - x) + u_b(x, 0)$.
Moreover, the offered volume of Good is *feasible* if $V^\tau = \sum_{s \in S} G^\tau_s$.

The feasibility assumptions are not restrictive for applications where the buyers are institutions (hospitals) which have practically unlimited amount of money and use their individual utility function for deciding how to spend them.

The assumption that a buyer b has no utility from and thus does not buy more Good than is its declared Demand in the current Market makes sense for b since not buying more Good than they will currently consume makes sense for the middle of a distribution crisis and it is also favourable for a central authority who can check this property since it knows everything about trading Right.

Finally, the feasibility of the offered volume of Good makes sense for *myopic sellers.*

Theorem 3.3. *Let $0 < \epsilon < 1$. Assuming initial endowments of buyers are feasible for ϵ, the offered volume of Good is also feasible and each act of a buyer in a round needs at most a constant-time $|B|$ steps, the following holds.*

(1) *The time-complexity of the auction-based algorithm is at most $|B|^3(1 + \log_{1+\epsilon} m)$; hence, the auction-based algorithm is polynomial in the input size.*
(2) *For each participant, its basket assigned by the algorithm is feasible and its price plus 1 is bigger than the total price of its initial endowment.*
(3) *The terminal price of Right is equal to the terminal price of Good.*
(4) *Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket.*

Proof (Sketch). We denote, only in this sketch, the total initial endowments of Money, Right and Good by m, r, g and recall that $r = g$. We first observe that at each stage of the algorithm, the total surplus is at most $2m$ and that in the first iteration, all items of Good and Right are paired and after the first iteration, a buyer owes to the system cash only for Money of their initial endowment. (2) and (3) now follow directly from the description of the algorithm.

We further deduce that the number of rounds in an iteration is at most $2 + |B|$ and that the total number of iterations is at most $1 + \log_{1+\epsilon} m$. This proves (1).

Finally we show in Lemma A.6 that relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket. This proves (4).

□

Remark 3 (Complexity assumption of Theorem 3.3). *All utility functions are known and monotone, $u_b(0, x)$ is linear and $u_b(x, 0)$ is concave. Let X be the current basket of buyer b. In the algorithm, buyer b does not need to know S^b, it only needs to*

(1) *Find maximum k such that $u_b(\mathcal{G}(X) + k, 0) - u_b(\mathcal{G}(X), 0) > u_b(0, ck)$, thus needs to solve: given a constant K, find max k such that $u_b(\mathcal{G}(X) + k, 0) - u_b(\mathcal{G}(X), 0) > Kk$;* we assume that this can be done in a constant time by buyers.
(2) *Buy at most k items of Couple from buyers, satisfying the invariant of OUTBID.*

3.2 Development of willingness to pay and frustration over trading rounds

The Markets happen subsequently in a sequence. In this paper we study only the regime when the Markets happen deep in a distribution crisis. This enables us to assume from now on that *the individual utility of Good does not change in subsequent Markets.*

What changes is the individual utility of Money: buyers who sell items of Right enter the subsequent Market with more Money and the new items of Money can be used, by the rules, only to buy items of Good or Right. Hence, the utility function of Money changes for these buyers. We note that, since the individual utility of Good is unchanged, that the change of the individual utility of Money can equivalently be described as a change of the willingness to pay.

Definition 3.4 ((potential) willingness to pay). Let buyer b sell, in the current Market $\mathbb{M}^\tau$, items of the Right for the total price z_b. If b buys Right then we let $z_b = 0$. Their *willingness to pay*, denoted by w_b^τ associates, with a positive x, maximum amount $w_b(x)$ of Money of the same utility as amount x of Good, i.e., $w_b(x)$ satisfies $u_b^\tau(x, 0) = u_b^\tau(0, w_b(x))$. Their *potential willingness to pay*, denoted by $\omega_b^{\tau+1}$, is defined as $\omega_b^{\tau+1}(x) := w_b^\tau(x) + z_b$, x arbitrary.

Remark 4. *As defined, the potential willingness to pay of the hesitant traders increases in subsequent rounds due to selling their Right, but why it does not* decrease *for the "active" buyers who have less Money in subsequent rounds due to purchasing Good? The reasoning is analogous to comments after*

Definition 3.2. If buyers are institutions (e.g., hospitals) which have practically unlimited amount of money regularly supplied and their willingness to pay does not decrease in the middle of a crisis. We note that with non-myopic traders we adopt different assumptions and oscillations do occur.

Theorem 3.5. *Let us consider a Crisis satisfying (1) the total supply, the individual demand and the individual utility of Good do not change in the sequence, (2) the Markets are implemented by the auction-based algorithm and (3) after a Market of the sequence ends, the willingness to pay in the next Market is equal to the potential willingness to pay for the next market. Then in all but possibly the first Market, each individual frustration is at most* $1/2$.

Proof (Sketch). Let Market $\mathbb{M}^\tau, \tau \geq 1$ of the sequence ended and let us consider the next Market $\mathbb{M}^{\tau+1}$. By the assumptions of the theorem, the auction-based algorithm repeats the steps of Market $\mathbb{M}^\tau$. After the final step of the auction for $\mathbb{M}^\tau$, the willingness to pay of the buyers with zero frustration in $\mathbb{M}^\tau$ is saturated.

However, the buyers with positive frustration in $\mathbb{M}^\tau$ (let us call them *frustrated*) remain active since they acquired additional funds in $\mathbb{M}^\tau$. Let b be such a frustrated buyer. The willingness to pay assures that b is willing to buy an additional number of items of Couple which is equal to the number n_b of items of Right b currently sold, equivalently to the number of items of Right b sold in $\mathbb{M}^\tau$). Let S_b be the set of n_b items of Couple containing the items of Right buyer b sold so far in $\mathbb{M}^{\tau+1}$. Buyer b buys the Couple of S_b at an increased price which in turn frees funds of non-frustrated buyers who may buy back. During this process the frustration of b can only go down, and non-frustrated buyers stay with zero frustration. Hence, we only need to rule out the case that the final frustration of b is strictly bigger than $1/2$. □

4 MARKET WITH NON-MYOPIC TRADERS

While the previous section considered market-clearing as an optimal solution of a system with myopic traders, this section investigates solutions of multi-round trading systems with traders capable of optimizing over a long horizon. We derive an explicit formulation of the solution in a form of the interaction's equilibrium, examine its robustness with respect to coalitions, and formulate an upper bound on the arising Price of Anarchy.

4.1 Game theoretic formulation

Considering non-myopic traders substantially increases the complexity of the situation, making the Crisis a sequential game. Let us first introduce a specific extension of the single-round Market with such traders that we call the *Market game*.

Formally, the Market game is a tuple $(\mathbb{M}^\tau, \Pi, \mu)$, where $\mathbb{M}^\tau$ is a Market as defined previously, with particular utilities for sellers and buyers. The sellers are motivated by profit, so their utility is the amount of Money they acquired in a Market. We modify this simple model by subtracting the amount of Good they are left with after the Market. This represents the cost of storing, as well as the reputation damaged by not selling the scarce Good. Together,

$$u_s^\tau(\mathcal{G}(B_s^\tau), \mathcal{M}(B_s^\tau)) = \mathcal{M}(B_s^\tau) + c\mathcal{G}(B_s^\tau),$$

where $c < 0$ is a suitable constant. A buyer, on the other hand, wishes to keep a steady supply of Good during the Crisis. Here, we assume each buyer makes use only for the Good they consume at the end of the Market via

$$u_b^\tau(\mathcal{G}(B_b^\tau), \mathcal{M}(B_b^\tau)) = \min\left\{D_b, \mathcal{G}(B_b^\tau)\right\}.$$

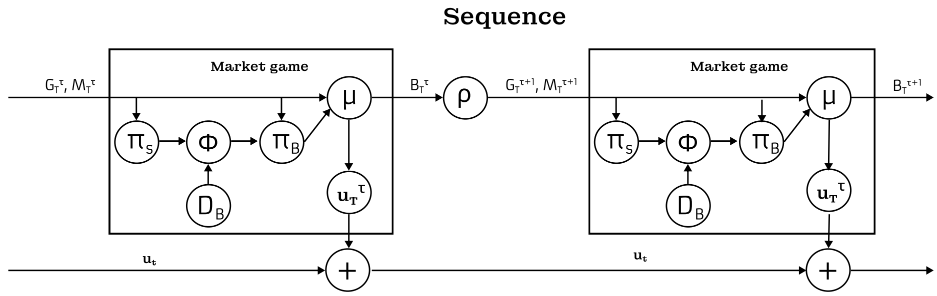

Fig. 2. An illustration of the Sequence of Market games, see main text.

Π is a set of strategies of the traders. We assume each seller has access to the amount of Good every seller has, as well as Money and Demand[1] each buyer has. On the other hand, each buyer has access only to the amount of Money and Good they already own, as well as their Demand. We consider this assumption sufficiently realistic as the sellers may invest into some market research, while availability of such information to the buyers remain rather limited. The strategy of each seller is thus a function $\pi_s : \mathbb{R}_0^{+,2|B|+|S|} \longrightarrow \mathbb{R}_0^{+,2}$, denoted as

$$\pi_s(G_B^\tau, M_B^\tau, G_S^\tau) = (v_s^\tau, p_s^\tau),$$

where $v_s^\tau \leq G_s^\tau$ is the offered volume of Good at price p_s^τ by seller s in Market τ. Based on the total offered volume of Good $V^\tau = \sum_{s \in S} v_s^\tau$ and Demand of buyers, a fairness mechanism ϕ allocates the Right. When offering Right for sale, a buyer is given the offers of sellers and the amount of Good, Money and Right they have. Their strategy for this task is hence a function $\hat{\pi}_b : \mathbb{R}_0^{+,2|S|+3} \longrightarrow \mathbb{R}_0^{+,2}$, denoted as

$$\hat{\pi}_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau) = (w_b^\tau, q_b^\tau),$$

where $w_b^\tau \leq R_b^\tau$ is the offered volume of Right at price q_b^τ. When declaring acceptable price and volume, a buyer is also given the offers of the other buyers. Summarising, the (complete) strategy of a buyer b is a function $\pi_b : \mathbb{R}_0^{+,2|S|+3+2(|B|-1)} \longrightarrow \mathbb{R}_0^{+,6}$, written as

$$\pi_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau, w_{-b}^\tau, q_{-b}^\tau) = (w_b^\tau, q_b^\tau, \overline{v}_b^\tau, \overline{p}_b^\tau, \overline{w}_b^\tau, \overline{q}_b^\tau),$$

where $\cdot_{-b}^\tau = \{\cdot_{b'}^\tau | b' \in B \setminus \{b\}\}$ and $\bar{\cdot}_b^\tau$ denote acceptable amounts of the corresponding quantities for buyer b. We will denote $\pi = \pi_S \times \pi_B$ the strategy profile of all traders. After all traders declare their bids, the Trading begins, which is done via our particular market mechanism.

Definition 4.1. The market mechanism is a function $\mu : \Pi \times \mathbb{R}_0^{+,2|T|+|B|} \longrightarrow \mathbb{R}_0^{+,2|T|}$, written as

$$\mu(v_S^\tau, p_S^\tau, w_B^\tau, q_B^\tau, \overline{v}_B^\tau, \overline{p}_B^\tau, \overline{w}_B^\tau, \overline{q}_B^\tau, G_T^\tau, M_T^\tau, R_B^\tau) = B_T^\tau,$$

where B_t^τ is basket containing the amount of Good and Money t gained during trading. The market mechanism we consider has two stages. In the first stage, the buyers use the Right they were assigned to buy as much Good as they desire. In the second stage, the buyers buy Good and Right in equal volume, until they buy their desired volume of either, or they have no Money left. In both stages, items offered at lower price are traded first. When more traders offer Good or Right at the same price, they are treated as a single trader until one runs out of items for sale. We describe the process in more detail using the notion of a *compatible buyer* for each trade. In the first stage, a buyer b is said to be compatible with offers of $S' \subset S$ if $\overline{p}_b^\tau \geq p_{s'}^\tau$, $R_b^\tau > 0$ and $v_{s'}^\tau > 0\ \forall s' \in S'$. We denote the set of buyers compatible with offers of sellers in S' during the first stage as $C_1(S')$.

[1]Since the fairness mechanism is assumed to be public knowledge, the traders also know (for some offered volume of Good) the amount of Right each buyer will be assigned.

ALGORITHM 1: Market mechanism

$\Delta G_t^\tau, \Delta M_t^\tau \leftarrow 0 \; \forall t \in T$
$g \leftarrow \text{sort}(\text{unique}(\{p_s | s \in S\}))$ // Sorted list of unique prices of Good
$r \leftarrow \text{sort}(\text{unique}(\{q_b | b \in B\}))$
/* First stage when buyers use their Right */
for p *in* g **do**
  $S' \leftarrow \{s | s \in S, p_s = p\}$ // Set of sellers offering Good at a given price
  while $|C_1(S')| > 0$ **do**
    $v^\tau \leftarrow \sum_{s \in S'} v_s^\tau$ // Total offered volume at this price
    $\overline{v}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')} (M_b^\tau / p, R_b^\tau - w_b^\tau, \overline{v}_b^\tau)$ // Total desired and affordable volume
    $V^\tau \leftarrow \min(v^\tau, \overline{v}^\tau)$
    $\Delta G_s^\tau \leftarrow \Delta G_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
    $v_s^\tau \leftarrow v_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
    $\Delta G_b^\tau \leftarrow \Delta G_b^\tau + V^\tau / |C_1(S')|, \; \forall s \in S'$
    $R_b^\tau \leftarrow R_b^\tau - V^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
    $M_b^\tau \leftarrow M_b^\tau - pV^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
    $\overline{v}_b^\tau \leftarrow \overline{v}_b^\tau - V^\tau / |C_1(S')|, \; \forall b \in C_1(S')$
  end
end
/* Second stage when buyers buy Right and Good in equal quantity */
for p *in* l **do**
  for q *in* r **do**
    $S' \leftarrow \{s | s \in S, p_s = p\}$
    $B' \leftarrow \{b | b \in B, q_b = q\}$
    while $|C_2(S', B')| > 0$ **do**
      $v^\tau \leftarrow \sum_{s \in S'} v_s^\tau$ // Total offered volume of Good at this price
      $w^\tau \leftarrow \sum_{b \in B'} w_b^\tau$ // Total offered volume of Right at this price
      $\overline{v}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')} (M_b^\tau / (p+q), \overline{v}_b^\tau)$
      $\overline{w}^\tau \leftarrow |C_1(S')| \; \min_{b \in C_1(S')} (M_b^\tau / (p+q), \overline{w}_b^\tau)$
      $V^\tau \leftarrow \min(v^\tau, \overline{v}^\tau, w^\tau, \overline{w}^\tau)$ // Total volume to be traded
      $\Delta G_s^\tau \leftarrow \Delta G_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
      $v_s^\tau \leftarrow v_s^\tau - V^\tau / |S'|, \; \forall s \in S'$
      $w_b^\tau \leftarrow w_b^\tau - V^\tau / |S'|, \; \forall b \in B'$
      $\Delta G_b^\tau \leftarrow \Delta G_b^\tau + V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $M_b^\tau \leftarrow M_b^\tau - (p+q) V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $\overline{v}_b^\tau \leftarrow \overline{v}_b^\tau - V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
      $\overline{w}_b^\tau \leftarrow \overline{w}_b^\tau - V^\tau / |C_2(S', B')|, \; \forall b \in C_2(S', B')$
    end
  end
end
return: $(G_T^\tau + \Delta G_T^\tau, M_T^\tau + \Delta M_T^\tau)$

Similarly, a buyer b is compatible in the second stage with offers of sellers in S' and buyers in $B' \subset B \setminus \{b\}$ if $\overline{p}_b^\tau \geq p_{s'}^\tau$, $\overline{q}_b^\tau \geq q_{b'}^\tau$ and $v_{s'}^\tau, w_{b'}^\tau > 0 \; \forall s' \in S', b' \in B'$. We denote the set of buyers compatible with offers of sellers in S' and buyers in B' as $C_2(S', B')$. The structure of the market mechanism is outlined in Algorithm 1, and the overall Market game can be found in Algorithm 2.

Similarly as in the previous section, we are interested in the performance of the system over the period of multiple rounds, each realized by a Market game. To distinguish this implementation of Crisis

ALGORITHM 2: Market

input: $G_t^\tau, M_t^\tau,\ \forall t \in T$
$(v_s^\tau, p_s^\tau) \leftarrow \pi_s(G_B^\tau, M_B^\tau, G_S^\tau),\ \forall s \in S$
$R_B^\tau \leftarrow \phi(\sum_{s\in S} v_s^\tau, D_B)$
$(w_b^\tau, p_b^\tau) \leftarrow \hat{\pi}_b(v_S^\tau, p_S^\tau, G_B^\tau, M_B^\tau, R_B^\tau),\ \forall b \in B$
$(w_b^\tau, p_b^\tau, \overline{v}_b^\tau, \overline{p}_b^\tau, \overline{w}_b^\tau, \overline{q}_b^\tau) \leftarrow \pi_b(v_S^\tau, p_S^\tau, G_B^\tau, M_B^\tau, R_B^\tau, w_{-b}^\tau, q_{-b}^\tau),\ \forall b \in B$
$B_T^\tau \leftarrow \mu(v_S^\tau, p_S^\tau, w_B^\tau, q_B^\tau, \overline{v}_B^\tau, \overline{p}_B^\tau, \overline{w}_B^\tau, \overline{q}_B^\tau, M_B^\tau, R_B^\tau)$
$u_T^\tau \leftarrow u(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau))$
return: $(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau), u_T^\tau)$

ALGORITHM 3: Sequence

$G_t^1, M_t^1, u_t \leftarrow 0,\ \forall t \in T$
for $\tau \in \{1, \ldots \mathcal{T}\}$ **do**
  $\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau), u_T^\tau \leftarrow \mathbb{M}(G_T^\tau, M_T^\tau)$
  $u_t \leftarrow u_t + u_t^\tau,\ \forall t \in T$
  $G_t^{\tau+1}, M_t^{\tau+1} \leftarrow \rho(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau)),\ \forall t \in T$
end
return: u_T

with non-myopic traders from its myopic alternative, we refer to the sequential game consisting of $\mathcal{T}$ Market games as the *Sequence*. Formally, the Sequence is a tuple $\mathbb{S}(\mathcal{T}) = (S, B, D, u, \mathcal{T}, \phi, \mu, \omega, \rho)$, where S, B, D, ϕ and μ have the same meaning as before, and ρ is a *transition* function. The purpose of the transition function is to consume resources[2] after each Market, and give traders additional Good and Money for the next Market.

Definition 4.2. The transition function $\rho : \mathbb{R}_0^{+,2} \to \mathbb{R}_0^{+,2}$ is given by

$$\rho(\mathcal{G}(B_t^\tau), \mathcal{M}(B_t^\tau)) = \begin{cases} (g_t + \mathcal{G}(B_t^\tau), 0) & t \in S, \\ (\max(0, \mathcal{G}(B_t^\tau) - D_t), m_t + \mathcal{M}(B_t^\tau)) & t \in B, \end{cases}$$

where g_s and m_b is the amount of Good and Money sellers and buyers gain after each Market game respectively. Our results rely on the assumption that the quantities g_s and m_b are stable during the Sequence, and without a loss of generality, we set $\sum_{s\in S} g_s = 1$ and $\sum_{b\in B} m_b = 1$. This assumption is not only crucial for our analysis but also practical in the midst of a crisis.

The structure of the entire Sequence is given in Algorithm 3 and visualized in Figure 2. We further assume the traders aggregate their utilities over the whole Sequence so that their resulting outcomes are $u_t = \sum_{\tau=1}^{\mathcal{T}} u_t^\tau$. As a solution of this interaction we employ the standard notion of equilibrial strategies, such that no deviation of a trader may increase their utility.

Definition 4.3. For a Sequence $\mathbb{S}(\mathcal{T})$, a strategy profile π is an equilibrium, if for any other strategy profile $\overline{\pi}_t$ of t

$$\sum_{\tau=1}^{\mathcal{T}} u_t^\tau(\pi) \geq \sum_{\tau=1}^{\mathcal{T}} u_t^\tau(\overline{\pi}_t, \pi_{-t}), \qquad \forall t \in T,$$

where $u_t^\tau(\pi)$ is the utility received by trader t in Market τ under strategy profile π.

At first glance, it may be unclear how the inclusion of Right in trading affects the strategies of rational traders. We will demonstrate that a particular equilibrium of the Sequence is a natural

[2]Formally, we also need to remove Money of sellers to prevent them from obtaining utility for it multiple times.

extension of the free market equilibrium, with a strictly lower Price of Anarchy. Informally, this proposed strategy is that the sellers post the highest price that the buyers can afford to pay, taking into account the cost of Right. As a result, buyers will purchase all available Good, which also means some will purchase Right. Those who sell Right will do so for the same price as for the Good. Buyers will not accept prices higher than this for either Right or the Good. Additionally, sellers will not offer more Good than they can sell in each Market. We refer to such strategies as *Greedy*, and they will be of our focus further on.

Definition 4.4. The Greedy strategies of the sellers are defined as

$$\pi_s(G_B^\tau, M_B^\tau, G_S^\tau) = (g_s, p^\tau)\,, \tag{1}$$

where p^τ is the solution of

$$\sum_{b\in B} M_b^\tau - \max(0, p^\tau R_b^\tau - M_b^\tau) = p^\tau \sum_{b\in B} R_b^\tau. \tag{2}$$

For buyers, the Greedy strategies are

$$\pi_b(v_S^\tau, p_S^\tau, G_b^\tau, M_b^\tau, R_b^\tau, w_{-b}^\tau, q_{-b}^\tau) = \left(\max(0, R_b^\tau - M_b^\tau/P^\tau), P^\tau, R_b^\tau, P^\tau, \max(0, M_b^\tau/P^\tau - R_b^\tau), P^\tau\right), \tag{3}$$

where $P^\tau = \frac{\sum_{s\in S} p_s^\tau}{|S|}$ is the average of the selling price of Good.

We justify the exact form of Eq. (2) in the next section.

Remark 5. *The Greedy strategy is for sellers to offer at most the amount of Good they receive at the beginning of each Market, as if the Good is perishable. At first this seems to force sellers to store Good, but that is not the case as if all traders follow the Greedy strategy as all Good is sold. However, it prevents certain deviations of buyers from Greedy to be beneficial. See Theorem 4.6 for more details.*

Remark 6 (Correctness of Definition 4.4). *It may not be immediately clear how a seller can derive p^τ from Eq. (2), as it requires knowledge of R_B^τ. However, if all sellers adopt the Greedy strategy, the offered volume is $\sum_{s\in S} g_s = 1$, and the fairness function is publicly available. In contrast, b does not know M_{-b}^τ, and must therefore estimate the price of the Right by averaging the selling price of the Good.*

Our primary focus is on a shortage of Good, which we formalize by demanding that each buyer can never purchase more than their Demand, even if they try to buy as much as possible in every Market of the Sequence. This is analogous to point (3) of Definition 3.2. To be more specific,

Definition 4.5. Let $\mathbb{M}$ and $\mathbb{S}$ be a Market and a Sequence respectively. We say $\mathbb{M}$ has feasible initial endowment, if when traders utilize the Greedy strategy outlined in Definition 4.4, then $D_b > \mathcal{G}(B_b^\tau)$, $\forall b \in B$. We say $\mathbb{S}$ is feasible if every Market of $\mathbb{S}$ is feasible.

4.2 Greedy equilibrial strategies

We will now demonstrate how the Market operates when all traders follow the Greedy strategy. All sellers and buyers post the same selling prices, p^τ and q^τ, respectively, for the Good and Right. The amount of Good that all sellers can sell in the first stage is $\sum_{b\in B}\min\{R_b^\tau, M_b^\tau/p^\tau\}$. We will call a buyer *rich* if $p^\tau R_b^\tau < M_b^\tau$, and *poor* otherwise. In the second stage, poor buyers will offer all the Right that they cannot use in the first stage, as they know the prices set by sellers before offering Right. The amount of offered Good is $\sum_{b\in B} R_b^\tau$, which means the amount of Good sold in the second stage is

$$\sum_{b\in B} R_b^\tau - \min(R_b^\tau, M_b^\tau/p^\tau) = \sum_{b\in B}\max(0, R_b^\tau - M_b^\tau/p^\tau).$$

The amount of Money sellers get from the first and second stage is

$$\Delta M_S^\tau = \sum_{b \in B} p^\tau R_b^\tau,$$

since all Good is sold, see Remark 5.

Note that this is less or equal than the amount of Money the buyers posses in total because the Money used for selling Right may only be used in the following Market. That Money is the amount of Right sold in the second stage times the price

$$\Delta M_B^\tau = \sum_{b \in B} q^\tau \max(0, R_b^\tau - M_b^\tau / p^\tau) \geq 0, \tag{4}$$

where the equality holds only when $p^\tau R_b^\tau = M_b^\tau, \forall b \in B$, so they can all buy everything in the first stage. The inequality of the buyers thus dictates how much of the Money can be used to buy Good. We will refer to ΔM_S as *useful Money* in a Market and to ΔM_B as *useless Money*. The total Money in the system is the sum of both, i.e.,

$$\sum_{b \in B} M_b^\tau = \Delta M_S^\tau + \Delta M_B^\tau = \sum_{b \in B} p^\tau R_b^\tau + q^\tau \max(0, R_b^\tau - M_b^\tau / p^\tau).$$

We can also derive this equation by taking into account that the cost of the Good is p^τ in the initial stage and $p^\tau + q^\tau$ in the subsequent stage. This implies that

$$\sum_{b \in B} M_b^\tau - \frac{q^\tau}{p^\tau} \max(0, p^\tau R_b^\tau - M_b^\tau) = p^\tau \sum_{b \in B} R_b^\tau, \tag{5}$$

which offers a nice intuition: the Money each buyer has is effectively decreased by the (scaled) frustration which p^τ would induce. This is because the rich buyers will buy the amount of Right exactly equal to frustration of the poor, hence the scaling by the relative price. If the traders follow Greedy, then $q^\tau = p^\tau$ and Eq. (5) reduces to Eq. (2).

Remark 7. *In the following Market, the useless Money is allocated to buyers who were poor in the previous Market. Specifically, in the following Market a buyer will have*

$$M_b^{\tau+1} = m_b + q^\tau R_b^\tau f_b^\tau = m_b + \frac{q^\tau}{p^\tau} \max(0, p^\tau R_b^\tau - M_b^\tau), \tag{6}$$

or, in other words, they earn a portion of the Money proportional to their frustration f_b^τ (see Derfinition 2.5) in this Market.

Theorem 4.6. *The Greedy strategies in Eqs.* (1) *and* (3) *form a coalition-proof equilibrium of a feasible Sequence of any length. The equilibrium can be computed efficiently.*

Proof (Sketch). Let us address the three claims separately.

(1) *Greedy strategy is an equilibrium:* Seemingly, the sellers can simply choose the (free) market-clearing price, which would imply all Money would need to be spend on Good. However, a poor buyer would not sell his Right for free, since increasing the price gives them more Money for the next Market, increasing their utility[3]. This would imply that not all Good would be sold, decreasing the sellers' utility. Therefore, in non-trivial cases, $p^\tau < \frac{\sum_{b \in B} M_b^\tau}{\sum_{b \in B} R_b^\tau}$, leaving some useless Money which the poor buyers split among themselves.

We focus on deviations in a single Market, followed by Greedy strategy. Any deviation from the Greedy strategy by any trader will either have no impact on the Market or result in less Good being sold. The main focus is on how the price will change in the following Market.

[3]Furthermore, decreasing the amount of Good sold would decrease the price in the following Market.

When sellers offer less Good in a Market and sells it in the next, we show the price in the following Market either decreases compared to if they followed the Greedy strategy or is lower than in the current Market. This means it is not beneficial to offer less Good. Next, consider a deviation of a buyer, which results in less Good being sold in the current Market. The buyer hopes this decreases in the next Market the price since more Good will be available. However, following the Greedy strategy, sellers will not offer the addition Good, leading to the same distribution of Rights as in the current Market. Additionally, since some Good is not sold, the total amount of Money in the next Market increases, and so does the price. Furthermore, if a deviating buyer was poor in the current Market, they will receive less Money from selling Rights, limiting the amount of Good they can buy in the next Market.

When considering a deviation in the last Market of the Sequence, it is clear that if less Good is sold, no trader can increase their utility. Therefore, it is not beneficial to deviate from the Greedy strategy in the last Market of the Sequence. Since we showed that deviating and following the Greedy strategy in the next Market is not beneficial for any trader, they are incentivized to follow the Greedy strategy in the second-to-last Market. The conclusion is reached through induction.

(2) *The equilibrium is coalition-proof:* The introduction of the Right negatively impacts rich buyers. Even if they all choose to deviate and not purchase the Right, their payoff will not increase. This is because the sellers will continue to offer the same amount of Good, resulting in a continuously increasing price. Additionally, sellers do not want to join a coalition that only sells to rich buyers, as it would require them to hold back Good for future Markets, reducing their utility.
(3) *The equilibrium can be computed efficiently:* The Greedy strategy can be computed efficiently, since it only requires solving Eq. (5), which involves a computation of a fixed point. This can be done efficiently in this case, since Eq. (2) can be partitioned into $|B| + 1$ price intervals, where in each the computation reduces to solving a simple linear equation.

□

When players follow the Greedy strategy, all Goods are sold, and the amount of Goods offered remains the same in every Market. Therefore, we will no longer include the upper index of allocated Right in the rest of this section and will use R_b to denote Right allocated to b in any Market. It is also worth noting that $\sum_{b\in B} R_b = 1$.

The existence of an analytic solution for a Sequence with an arbitrary fairness function remains uncertain. However, in certain specific cases, a solution can be found. One such example is a system with only one buyer, which is akin to a free market. Additionally, if the ratio R_b/m_b is the same for all buyers, the system again reduces to a free market, as can be observed from Eq. (4). Notably, an analytic solution also exists if all the rights are allocated to a single buyer, which implies a more general result.

Proposition 4.7. *Let all traders follow the Greedy strategy. Then the price p^τ in Market $\mathbb{M}^\tau$ of a Sequence with fairness function ϕ satisfies*

$$p^\tau \geq \frac{\sum_{b\in B}(\alpha_b + 1)M_b^\tau}{2}, \qquad \textit{where} \qquad \alpha_b = \phi_b(V, D).$$

Proof (Sketch). The fairness function ϕ can be expressed as a linear combination of distributions in which all Rights are allocated to a single buyer. The left hand side of Eq. (2) is concave in p^τ, which after simple algebraic manipulation leads to the statement of the proposition. □

We present a more thorough study of such fairness functions in Appendix B.

4.3 Development of frustration over trading rounds

One may observe certain similarities between Eqs. (5) and (6): the frustration decreases useful Money a buyer has, and increases the amount they will have in the following Market. Now we will show that the price oscillates around a fixed value and tends to it over time. Moreover, that fixed value is the (free-)market clearing price.

Proposition 4.8. *Let all traders follow the Greedy strategy. Then the mapping of the current price to the next one is a non-expansive mapping on $\mathbb{R}$ with the L1 norm, resulting in the limiting price being one.*

This suggests that the Sequence ultimately reaches a stable state. Additionally, the price is the same as it would be in a free market $\frac{\sum_{b\in B} m_b}{\sum_{s\in S} g_s} = 1$. This means the fairness mechanism primarily impacts the income of buyers.

Proof (Sketch). Similar to the free market, the price is an increasing function of the Money each buyer has, and therefore also of the total amount. This implies, after some algebraic manipulation, that if the price is smaller than one, it increases in the following Market, and vice versa. The remaining task is to demonstrate that $|p^\tau - 1| \geq |p^{\tau+1} - 1|$, which can be done through a simple case study.

□

Proposition 4.8 illustrates that if all traders employ Greedy strategies, the Sequence eventually stabilizes. In this scenario, the amount of Money and Goods entering and leaving the system are equal, since $\lim_{\tau\to\infty} p^\tau = 1$. The amount of Money buyers start a Market with hence stabilizes to

$$M_b = m_b + \max(0, R_b - M_b).$$

This leads to the Price of Anarchy being twice as high in the Sequence with free market than in a one with a fairness mechanism.

Theorem 4.9. *Consider a Sequence where traders follow the Greedy strategy. Then the Price of Anarchy in the Market with fairness is at most 1/2 of the free market's PoA as $\mathcal{T} \to \infty$.*

Proof. When investigating frustration, we can focus only on the poor buyers, for whom $R_b > M_b$. Asymptotically, Eq. (6) thus becomes

$$M_b = \frac{m_b + R_b}{2},$$

corresponding to the same amount of Good they can buy, since the price is equal to one. This means their frustration is

$$f_b = \frac{R_b - \frac{m_b+R_b}{2}}{R_b} = \frac{1}{2}\left(1 - \frac{m_b}{R_b}\right) \leq \frac{1}{2}.$$

In contrast, since in the free market $M_b = m_b$, the frustration of poor buyers is twice as high, i.e.,

$$f_b = \frac{R_b - m_b}{R_b} = 1 - \frac{m_b}{R_b}.$$

For each poor buyer, the frustration is asymptotically half of what it would be in the free market. Since the frustration of the rich buyers is zero, the Price of Anarchy is asymptotically half of what it would be in the free market, once the initial portion of the Sequence becomes insignificant. □

5 CONCLUSION

In this paper, we present a system that combines market and fairness mechanisms to more equitably redistribute critical goods during times of need. The fairness mechanism assigns the buyers their buying rights, representing the amount of goods they are entitled to. We define a multi-round trading environment in which the buyers' rights and goods offered by the sellers are traded. To evaluate the fairness of the system, we use a concept of "frustration", which measures the difference between the amount of right allocated to a buyer and the amount of good they are able to acquire in the market. We then define an analog of the Price of Anarchy in our system as the sum of individual frustrations. Our primary focus is to demonstrate that by using buying rights, the fairness of the system improves compared to an unregulated market.

To understand the performance of our system, we study two different implementations of the single-round market. If the buyers are myopic, we formulate the solution in the form of market-clearing prices and design an algorithm that computes an approximate solution in polynomial time. We also show that the Price of Anarchy in the system is upper bounded by 1/2. In the case of traders who optimize over multiple rounds, the solution takes the form of an equilibrium in a sequential game. We provide an explicit formulation of the solution and demonstrate that it is also computable in polynomial time. As the number of trading rounds increases, the upper bound on the Price of Anarchy again approaches 1/2. However, it remains an open question whether there exists a system that can achieve a zero Price of Anarchy in the limit.

Future work. We believe that our work has two major limitations. Firstly, we have focused on the most severe crises scenarios and assumed a steady, albeit small, resupply of goods and money over multiple trading rounds. We would like to further explore the impact of more complicated system dynamics, such as the bullwhip effect that occurs when there is a sudden surge in demand at the onset of a crisis. Secondly, our results are not specific to any particular fairness mechanism. Examining specific mechanisms may alter the system dynamics and potentially improve the bounds on the Price of Anarchy. Additionally, it may be possible to design practical optimal fairness rules using similar approaches as those employed for voting mechanisms, as seen in [Koster et al., 2022].

ACKNOWLEDGMENTS

This research was supported by the CRISDIS project of the Czech Ministry of the Interior no.VI04000107 and SIMTech-NTU Joint Laboratory on Complex Systems.

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A PROOFS

Theorem 3.3. *Let $0 < \epsilon < 1$. Assuming initial endowments of buyers are feasible for ϵ, the offered volume of Good is also feasible and each act of a buyer in a round needs at most a constant-time $|B|$ steps, the following holds.*

1. *The time-complexity of the auction-based algorithm is at most $|B|^3(1 + \log_{1+\epsilon} m)$; hence, the auction-based algorithm is polynomial in the input size.*
2. *For each participant, its basket assigned by the algorithm is feasible and its price plus 1 is bigger than the total price of its initial endowment.*
3. *The terminal price of Right is equal to the terminal price of Good.*
4. *Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket.*

We denote by m, r, g the total initial endowments of Money, Right and Good and recall that $r = g$. We prove the theorem in a sequence of lemmata.

Lemma A.1. *At each stage of the algorithm, the total surplus is at most $2m$.*

Proof. It is true in the beginning by assumption (1) of Definition 3.2, the surplus is gradually decreased during each iteration and at the end of each iteration, deleted funds are given back. □

Lemma A.2. *In the first iteration, all items of Good and Right are paired.*

Proof. By assumptions (2) of Definition 3.2, all buyers prefer to buy at least the fair amount of items of Couple for the initial price; by assumption (1) there is enough cash in the initial surplus of each buyer to do it. □

Lemma A.3. *After the end of the first iteration: (1) a buyer owes to the system only cash for items of Money in its initial endowment and (2) total cash among participants is always at most m.*

Proof. The first part follows from Lemma A.2 since all items of Good and Right are sold and bought at the end of the first interaction. For the second part we note that among sellers, the total cash is pg since all items of Good were sold in the first iteration and among buyers, the total cash is at most $m - pg$ since the buyers payed for the items of Good and there is no cash left from the initial endowments of Right since all items of Right were sold and bought in the first iteration. □

Lemma A.4. *The number of rounds in an iteration is at most $2 + |B|$.*

Proof. We observe that in each fully completed round, either none of the buyers buys items of Couple and the trading ends, or none of the buyers buys items of Couple in the next round and the trading ends or at least one buyer acts for the last time in this iteration: if in the current round every buyer buys items of Couple only from itself then in the next round nobody buys since nobody got additional cash. Hence let a buyer b buy items of Couple from another buyer in the current round. It means that b gets no additional cash in this iteration since it has no items of Couple for c, otherwise it would have to buy these first by the rules of the outbid and the current round is the last active round for b. □

Lemma A.5. *The total number of iterations is at most $1 + \log_{1+\epsilon} m$.*

Proof. Each iteration raises the price of Couple by the factor of $(1 + \epsilon)$ and the max price per unit of Couple cannot be bigger than the total surplus. □

Lemma A.6. *Relative to terminating prices, each buyer or seller gets a basket of utility at least $(1 - 2\epsilon)$ times the utility of its optimal feasible basket.*

Proof. (1) Buyers owe nothing to the system since after the end of the trading they keep only the items of Money they can buy with their remaining cash.

(2) After the end of the trading and buying items of Money, each participant is left with less than 1 dollar by the second part of Lemma A.3.

(3) The basket of each seller is optimal since all items of Good were sold.

(4) The only reason why the basket of buyer b is not optimal is: For some items of Couple, b payed $(1+\epsilon)c$ where c is the terminal price of Couple. Let x (y respectively) denote the total number of items of Couple (Money respectively) in b's *optimal basket*. The utility of the *optimal basket* of b is thus $u_b(x, 0) + u_b(0, y)$. By assumption (3) of Definition 3.2, $y > cx$.

In b's *terminal basket*, there are x items of Couple and at least $y - \epsilon cx - 1$ items of Money. First let $\epsilon cx \geq 1$. The utility of b's *terminal basket* is thus, using the assumption on the linearity of the utility of Money, at least $u_b(x, 0) + u_b(0, (1 - 2\epsilon)y) = u_b(x, 0) + (1 - 2\epsilon)u_b(0, y)$.

Secondly let $\epsilon cx < 1$. The utility of b's *terminal basket* is thus at least $u_b(x, 0) + u_b(0, (y - 2))$ and Claim 5 holds since we assume $1 > \epsilon > 2/M_b$. □

Now we are ready to prove Theorem 3.3.

Proof. (1) It follows from Lemmata A.4, A.5 that the time complexity of the auction-based algorithm behaves asymptotically as $|B|^3(1 + \log_{1+\epsilon} m)$.

(2) follows from (1) of Lemma A.3., (3) follows from the description of the algorithm and (4) is Lemma A.5. □

Theorem 3.5. *Let us consider a sequence of Markets satisfying (1) the total supply, the individual demand and the individual utility of Good do not change in the sequence, (2) the Markets are implemented by the auction-based algorithm and (3) after a Market of the sequence ends, the willingness to pay in the next Market is equal to the potential willingness to pay for the next market. Then in all but possibly the first Market, each individual frustration is at most* $1/2$.

Proof. Let Market Market $\mathbb{M}^\tau$, $\tau \geq 1$ of the sequence ended and let us consider the next Market $\mathbb{M}^{\tau+1}$. By the assumptions of the theorem, the auction-based algorithm repeats the steps of Market $\mathbb{M}^\tau$. After the final step of the auction for $\mathbb{M}^\tau$, the willingness to pay of the buyers with zero frustration in $\mathbb{M}^\tau$ (let us call them *happy*) is saturated.

However, the buyers with positive frustration in $\mathbb{M}^\tau$ (let us call them *frustrated*) remain active since they acquired additional funds in $\mathbb{M}^\tau$. Let b be such a frustrated buyer. We recall that its willingness to pay, given the current price of the Couple (which is equal to the final price of Couple in $\mathbb{M}^\tau$), assures that b is willing to buy an additional number of items of Couple which is equal to the number n_b of items of Right b currently sold (which is equal to the number of items of the Right b sold in $\mathbb{M}^\tau$). This is because b currently has cash for these n_b sold items of Right.

Let S_b be the set of n_b items of Couple containing the items of Right buyer b sold so far in $\mathbb{M}^{\tau+1}$.

Buyer b buys the Couple of S_b at an increased price which in turn frees funds of active buyers who may buy back. During this process the frustration of b can only go down, and active buyers stay with zero frustration. Hence, we only need to rule out the case that the final frustration of b is strictly bigger than $1/2$. Let $0 < n'_b < n_b$ be such that $n_b - n'_b = R_b/2$ where we denote by R_b the number of assigned rights to b in $\mathbb{M}^\tau$ (and thus also in $\mathbb{M}^{\tau+1}$).

Let us assume b doubled price of Couple in its first bid, and buys n'_b additional items of Couple of S. Buyer b only needs to buy items of Good by which b spends all z_b additional items of Money it got from the previous Market $\mathbb{M}^\tau$:

- $2n'_b z_b/n_b$ items of Money for buying n'_b items of Good from S_b, and

- $(n_b - 2n'_b)z_b/n_b$ items of Money needed to increase the price of $R_b - n_b$ items of Good b already has, since $n_b - n'_b = R_b/2$.

Happy buyers from which b bought new items of Couple are not willing to buy back since:

They obtain in total $2n'_b z_b/n_b$ items of Money for the sold n'_b items of Couple of S_b, but in order to increase the price further, $2(n_b - n'_b)z_b/n_b$ items of these obtained Money is needed to increase the price of the remaining $(n_b - n'_b)$ items of Couple in S_b. Clearly by the definition of n'_b and since $n_b \leq R_b$, $(n_b - n'_b) = R_b/2 \geq n'_b$.

Summarising, the final frustration of b in $\mathbb{M}^{\tau+1}$ is at most $1/2$ by spending in addition only Money obtained from selling Right in $\mathbb{M}^{\tau}$.

□

THEOREM 4.6. *The Greedy strategies* (1) *and* (3) *form a coalition-proof equilibrium of a feasible Sequence of any length. The equilibrium can be computed efficiently.*

PROOF. We proof the theorem as a sequence of lemmas. We begin with two useful lemmas, showing how the price changes with Money, and throughout the Sequence.

LEMMA A.7. *Let the traders follow the Greedy strategy. Then p^τ, given as a solution of Eq. (2) is increasing function of M_b^τ $\forall b \in B, \tau \in \{1, \dots \mathcal{T}\}$.*

PROOF. Taking the derivative of both sides yields

$$\frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} = \frac{\mathrm{d}}{\mathrm{d}M^\tau_{b'}} \sum_{b\in B} M_b^\tau - \max(0, p^\tau R_b - M_b^\tau) = 1 - \sum_{b\in B} \operatorname{sign}(p^\tau R_b - M_b^\tau)\left(\frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} R_b - 1\right) =$$

$$1 + \tilde{N} - \frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} \sum_{b\in B} \operatorname{sign}(p^\tau R_b - M_b^\tau) R_b, \qquad \Rightarrow \qquad \frac{\mathrm{d}p^\tau}{\mathrm{d}M^\tau_{b'}} = \frac{1+\tilde{N}}{1+\tilde{R}} \geq 1,$$

where $\tilde{N} > 0$ is the number of poor buyers and $0 < \tilde{R} < \tilde{N}$ is the sum of their Right. □

LEMMA A.8. *Let the traders follow the Greedy strategy. Then $p^{\tau-1} < 1 \Rightarrow p^\tau > p^{\tau-1}$, resp. $p^{\tau-1} > 1 \Rightarrow p^\tau < p^{\tau-1}$.*

PROOF. Substituting Eq. (6) into (5) gives

$$\sum_{b\in B} m_b + \max(0, p^{\tau-1}R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau) = p^\tau,$$

$$1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau) = p^\tau, \tag{7}$$

If $p^{\tau-1} < 1$, then the useful Money also is

$$\sum_{b\in B} M_b^{\tau-1} - \max(0, p^{\tau-1}R_b - M_b^{\tau-1}) < 1,$$

or in other words

$$\sum_{b\in B} M_b^{\tau-1} < 1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M_b^{\tau-1}).$$

Going back to Eq. (7), we see that

$$1 + \sum_{b\in B} \max(0, p^{\tau-1}R_b - M_b^{\tau-1}) = p^\tau + \sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau),$$

$$\sum_{b\in B} M_b^{\tau-1} < p^\tau + \sum_{b\in B} \max(0, p^\tau R_b - M_b^\tau) = \sum_{b\in B} M_b^\tau.$$

Using Lemma A.7 we get $p^\tau > p^{\tau-1}$. Similarly we can show $p^{\tau-1} > 1 \Rightarrow p^\tau < p^{\tau-1}$. □

To demonstrate that the Greedy strategy is an equilibrium, we show that no deviation from it is beneficial for any trader. We consider changes in the offered volume of Good and Right first.

Lemma A.9. *Let all traders follow Greedy except for a seller who offers less Good in a Market and sells it in the following Market. Then his utility decreases as a consequence.*

Proof. Note that the scenario in which the seller deviates is denoted with a hat. Since the fairness function is non-decreasing in the total offered volume, we get $\hat{R}_b^{\tau+1} \geq R_b^{\tau+1}$ for all buyers. For contradiction we also assume the deviation is beneficial for the seller, i.e. $\hat{p}^{\tau+1} \geq p^{\tau+1}$. Since at least one buyer didn't get the Money for selling Right in τ, we have

$$\sum_{b \in B} \max(0, \hat{p}^{\tau+1}\hat{R}_b^{\tau+1} - \hat{M}_b^{\tau+1}) > \sum_{b \in B} \max(0, p^{\tau+1}R_b^{\tau+1} - M_b^{\tau+1}). \tag{8}$$

Using Eq. (2) in both scenarios gives

$$\sum_{b \in B} M_b^{\tau+1} - \max(0, p^{\tau+1}R_b^{\tau+1} - M_b^{\tau+1}) = p^{\tau+1}, \tag{9}$$

$$p^\tau \mathcal{V} + \sum_{b \in B} M_b^{\tau+1} - \max(0, \hat{p}^{\tau+1}\hat{R}_b^{\tau+1} - \hat{M}_b^{\tau+1}) = \hat{p}^{\tau+1}(1 + \mathcal{V}), \tag{10}$$

or in combination with Eq. (8)

$$p^\tau \mathcal{V} + p^{\tau+1} > \hat{p}^{\tau+1}(1 + \mathcal{V}).$$

The price if s deviates is thus upper bounded by the weighted average of prices under Greedy. We can consider two cases.

(1) $p^\tau < 1$: Then by Lemma A.8, $p^{\tau+1} \geq p^\tau \Rightarrow p^{\tau+1} > \hat{p}^{\tau+1}$, which is a contradiction.
(2) $p^\tau > 1$: Again by Lemma A.8, $p^{\tau+1} \leq p^\tau \Rightarrow p^\tau > \hat{p}^{\tau+1}$. This means the seller could have sold Good in τ and increase his payoff.

□

Lemma A.10. *Let all traders follow Greedy except for a rich, resp. poor buyer who buys, resp. sells less Right in a Market. Then his utility decreases as a consequence.*

Proof. If a buyer deviates in this way, it leaves sellers with more Good for the next Market, and the buyers with more Money. But when following Greedy, the sellers will offer the same volume of Good in the next, leading to the same distribution of Right as in the last Market. And, since the buyers now have more Money then by Lemma A.7, the price increases. Moreover, if b was poor in τ, then he will receive less Money from selling Right, limiting the amount of Good he can buy in the next Market

$$\hat{M}_b^{\tau+1} = m_b + \max(0, p^\tau(R_b - \mathcal{V}) - M_b^\tau) < M_b^{\tau+1}.$$

□

Deviating from Greedy by changing of the price has a similar effect.

Lemma A.11. *Let all traders follow the Greedy strategy. Then no trader can increase his utility by changing the selling price of Good, resp. Right in a single Market.*

Proof. Let us start with $s \in S$ changing p_s^τ. There are two cases

(1) $p_s^\tau < p^\tau$: In this situation, s will get less Money in τ, which will stay in the system. However, in the following Market, the Money will be split proportionally to g_s, decreasing the utility of s.

(2) $p_s^\tau > p^\tau$: The acceptable price of Good is the average of the selling prices. This means s will not sell anything, decreasing his utility as was shown in Lemma A.9.

Similarly for $b \in B$

(1) $q_b^\tau < q^\tau$: When selling at a lower price, b will get less Money in the next Market, decreasing his utility. Furthermore, some rich buyers will be left with more Money, increasing the price in the following Market.
(2) $q_b^\tau > q^\tau$: The acceptable price of Right is $p^\tau = q^\tau$, so b will not sell any Right. This will again get him less Money and increase the price in the next Market.

□

Now, let us focus on coalition-proofness.

Lemma A.12. *There does not exist any coalition of traders that could increase their individual utilities by deviating from the Greedy strategy.*

Proof. Let us split the proof into three parts, depending on the composition of the coalition C.

(1) $C \subset B$: To increase the utility of buyers, they need to acquire more Good, meaning $T \setminus C$ will have less. Since $\mathcal{G}(B_s) = 0\ \forall s \in S$ when following Greedy, $B \setminus C$ need to acquire less. The only way C can accomplish that is if they don't buy Right from $B \setminus C$. But, similar to proof of Lemma A.10, this would only lead to an increase in price, lowering the utility of $b \in C$.
(2) $C \subset S$: The sellers have utility for Money, and since following Greedy they obtain all Money the buyers have, $S \setminus C$ needs to get less Money. The proportion in which the sellers split the Money of buyers is given by p_s^τ and g_s, of which they can influence p_s^τ. But any change in price will not decrease Money $S \setminus C$ get. Decreasing will leave some Money for next Market, of which $S \setminus C$ gets a portion. Increasing the price will leave C with extra Good, decreasing their utility.
(3) $C \subset T$: Again, $S \cap C$ can only increase their utility if $S \setminus C$ get less Money. This can only be accomplished if $B \cap C$ don't buy from $S \setminus C$, so they accept a lower price, which is still larger than selling price of $S \cap C$. But if the selling price is lower, and they are selling only to a subset of buyers, they cannot get more Money.

□

Finally, let us discuss the computational complexity of a Market utilizing the Greedy strategy.

Lemma A.13. *The Greedy strategy can be computed efficiently.*

Proof. The computation of the Greedy strategy given the price of Good can be done in constant time. What remains to show is that Eq. (2) can be solved efficiently. But that is the case since the price $p^\tau \in \mathbb{R}_0^+$, allowing us to split the interval into at most $|B| + 1$ intervals separated by points $\left\{\frac{M_b^\tau}{R_b^\tau}\right\}_{b \in B}$. In each interval, Eq. (2) can be partitioned into $|B| + 1$ price intervals, where in each the computation reduces to solving a simple linear equation

$$\sum_{b \in B} M_b^\tau - (p^\tau R_b^\tau - M_b^\tau)\mathrm{sign}(p^\tau R_b^\tau - M_b^\tau) = p^\tau \sum_{b \in B} R_b^\tau.$$

which can be solved efficiently. If it's solution lies in the corresponding interval, it is a solution of Eq. (2). By Brouwer fixed-point theorem the solution is guaranteed to exist, since the price is upper bounded by the free market clearing price. The complexity is thus linear in the number of buyers. □

This concludes the proof of Theorem 4.6 □

Proposition 4.8. *Let all traders follow the Greedy strategy. Then the mapping of the current price to the next one is a non-expansive mapping on $\mathbb{R}$ with the L1 norm, resulting in the limiting price being one.*

Proof. In the first part of the proof, we eliminate simple cases when the price can be equal to one. In other cases, the mapping is a contraction, which we will show in the second part of this proof.

Let us begin with a statement about uniqueness.

Lemma A.14. *For any $M_b^\tau \geq m_b, R_b \in [0, 1]$, Eq. (2) has a unique solution.*

Proof. For fixed Money and Right of buyers, the left hand side is a concave, decreasing piecewise linear function of the price. Since the right hand side is linear, they cross at at most one point. For $p^\tau = 0$, the left hand side is $\sum_{b\in B} M_b^\tau \geq \sum_{b\in B} m_b = 1$, while the right is zero, so a solution exists. □

It may happen that the price is one in a Market. But if that happens, the price stays for the remainder of the Sequence.

Lemma A.15. *Let all traders follow the Greedy strategy. Then $p^\tau = 1 \Rightarrow p^{\tau+1} = 1$.*

Proof. If $p^\tau = 1$, then

$$\sum_{b\in B} M_b^\tau - \max(0, R_b - M_b^\tau) = 1,$$

and in the next Market

$$\sum_{b\in B} M_b^{\tau+1} - \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) = p^{\tau+1},$$

but

$$\sum_{b\in B} M_b^{\tau+1} = \sum_{b\in B} m_b + \max(0, R_b - M_b^\tau) = 1 + \sum_{b\in B} \max(0, R_b - M_b^\tau) = \sum_{b\in B} M_b^\tau.$$

However, this implies that $p^{\tau+1} = p^\tau$. For contradiction, let B' be a set of poor buyers at τ. The price can only be influenced by the Money poor buyers have, since R_b is fixed. There are two options

(1) $p^{\tau+1} > p^\tau$: Then the poor buyers have more Money $\sum_{b'\in B'} M_{b'}^{\tau+1} > \sum_{b'\in B'} M_{b'}^\tau$. But that is not possible without increasing the amount of Money all buyers have, since for the rich $M_b^{\tau+1} = m_b$.

(2) $p^{\tau+1} < p^\tau$: In this case, the amount of Money the poor buyers have decreases, meaning for some $b \in B \setminus B'$, $M_b^{\tau+1} > m_b$. But for rich buyers $M_b^{\tau+1} = m_b$.

□

This result shows that the mapping is non-expansive if at some τ, $p^\tau = 1$. In other cases, we show the mapping is a contraction, i.e.

$$1 > \frac{|p^{\tau+1} - 1|}{|p^\tau - 1|}.$$

The rest of the proof of Proposition 4.8 is dedicated to proving this. Combining Eq. (2) and (6) we get

$$p^\tau - 1 = \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^\tau R_b - M_b^\tau),$$

so the contraction condition is

$$
\begin{aligned}
1 > \frac{|p^{\tau+1} - 1|}{|p^{\tau} - 1|} &= \left| \frac{\sum_{b\in B} \max(0, p^{\tau} R_b - M_b^{\tau}) - \max(0, p^{\tau+1} R_b - M_b^{\tau+1})}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^{\tau} R_b - M_b^{\tau})} \right| = \\
&\left| \frac{\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) - \max(0, p^{\tau} R_b - M_b^{\tau})}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^{\tau} R_b - M_b^{\tau})} \right| = \\
&\left| 1 - \frac{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^{\tau+1} R_b - M_b^{\tau+1})}{\sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}) - \max(0, p^{\tau} R_b - M_b^{\tau})} \right|. \quad (11)
\end{aligned}
$$

We split the remainder of the proof into two parts

(1) The first option to satisfy Eq. (11) is

$$
\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) < \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}), \quad (12a)
$$

$$
\sum_{b\in B} \max(0, p^{\tau} R_b - M_b^{\tau}) < \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}). \quad (12b)
$$

Eq. (12b) gives

$$
\begin{aligned}
\sum_{b\in B} \max(0, p^{\tau} R_b - M_b^{\tau}) &< \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}), \\
\sum_{b\in B} \max(0, p^{\tau} R_b - M_b^{\tau}) &< \sum_{b\in B} M_b^{\tau} - m_b, \\
-p^{\tau} + \sum_{b\in B} M_b^{\tau} &< -1 + \sum_{b\in B} M_b^{\tau}, \\
1 &< p^{\tau}. \quad (13)
\end{aligned}
$$

Then the Eq. (12a)

$$
\begin{aligned}
\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) &< \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}), \\
\sum_{b\in B} \max(0, p^{\tau+1} R_b - m_b - \max(0, p^{\tau} R_b - M_b^{\tau})) &< \sum_{b\in B} M_b^{\tau} - m_b.
\end{aligned}
$$

Let us show that the inequality holds for all buyers, not just their sum. If b is poor in τ, then the inequality becomes

$$
\max(m_b - M_b^{\tau}, (p^{\tau+1} - p^{\tau}) R_b) < 0,
$$

however $M_b^{\tau} > m_b$ for a poor buyer, and using Eq. (13) and Lemma A.8 yields $p^{\tau+1} < p^{\tau}$. Otherwise $M_b^{\tau} = m_b$ and $p^{\tau} R_b - M_b^{\tau} \leq 0$. Using Lemma A.8 we get $p^{\tau} R_b - M_b^{\tau} > p^{\tau+1} R_b - m_b$, so

$$
\max(0, p^{\tau+1} R_b - m_b) < 0,
$$

or b is rich in the next Market as well.

(2) Similarly the second option is

$$
\sum_{b\in B} \max(0, p^{\tau+1} R_b - M_b^{\tau+1}) > \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}),
$$

$$
\sum_{b\in B} \max(0, p^{\tau} R_b - M_b^{\tau}) > \sum_{b\in B} \max(0, p^{\tau-1} R_b - M_b^{\tau-1}),
$$

where the second condition similarly reduces to $1 > p^\tau$. Focusing on the first one gives

$$\sum_{b\in B} \max(0, p^{\tau+1}R_b - m_b - \max(0, p^\tau R_b - M_b^\tau) > \sum_{b\in B} M_b^\tau - m_b. \tag{14}$$

Again, for a poor buyer in τ we get

$$\max(m_b - M_b^\tau, (p^{\tau+1} - p^\tau)R_b) > 0$$

which is again true. For a rich buyer we get $\max(0, p^{\tau+1}R_b - m_b)$, which can be zero. However, since for the poor buyers the inequality was strict, Eq. (14) is satisfied.

□

B CANONICAL SOLUTIONS

In this section, we study a solution of the Sequence with perhaps the simplest class of fairness function. Specifically, this fairness function assigns all Right to a buyer with the n^{th} largest Demand[4]

$$\varphi_{b,n}\left(\sum_{s\in S} v_s^\tau, D\right) = \begin{cases} \sum_{s\in S} v_s^\tau & \text{if } D_b \text{ is the } n^{\text{th}} \text{ highest Demand,} \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

We call it the *n-canonical fairness function.* Let b_n be the buyer receiving all Right. In $\tau = 1$ we get

$$\sum_{b\in B} M_b^1 - \max(0, p^1 R_b - M_b^1) = p^1 \sum_{b\in B} R_b,$$

$$\sum_{b\in B} m_b - \max(0, p^1 R_b - m_b) = p^1,$$

$$1 - p^1 + m_{b_n} = p^1, \qquad \Rightarrow \qquad p^1 = \frac{1 + m_{b_n}}{2}. \tag{16}$$

Therefore, in the second Market, $M_{b_n}^2 = m_{b_n} + \frac{1+m_{b_n}}{2} - m_{b_n} = \frac{1+m_{b_n}}{2}$ and the price is

$$\sum_{b\in B} M_b^2 - \max(0, p^2 R_b - M_b^2) = p^2 \sum_{b\in B} R_b,$$

$$1 + \frac{1 + m_{b_n}}{2} - m_{b_n} - p^2 + m_{b_n} + \frac{1 + m_{b_n}}{2} - m_{b_n} = p^2$$

$$\Rightarrow p^2 = \frac{2 + 2m_{b_n}}{2} - m_{b_n} = 1.$$

This means b will have $M_b^3 = m_b + 1 - \frac{1+m_b}{2} = \frac{1+m_b}{2} = M_b^2$ and the cycle repeats.

Any fairness function ϕ can be constructed as a sum of weighted sum of canonical mechanisms

$$\phi(V, D) = \sum_{n=1}^{|B|} \alpha_n \varphi_{b,n}(V, D). \tag{17}$$

Since Eq. (2) is quasi-linear in this composition of Right

$$\sum_{b\in B} M_b^\tau - \max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - M_b^\tau) = p^\tau \sum_{b\in B} \alpha R_b^\tau + \beta \overline{R}_b^\tau,$$

$$\sum_{b\in B} (\alpha + \beta) M_b^\tau - \max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - (\alpha + \beta) M_b^\tau) = p^\tau \sum_{b\in B} \alpha R_b^\tau + \beta \overline{R}_b^\tau.$$

We want to split the maximum into two maxima, but

$$\max(0, p^\tau(\alpha R_b^\tau + \beta \overline{R}_b^\tau) - (\alpha + \beta) M_b^\tau) \leq \alpha \max(0, p^\tau R_b^\tau - M_b^\tau) + \beta \max(0, p^\tau \overline{R}_b^\tau - M_b^\tau),$$

[4]Ties are broken arbitrarily, but consistently.

so computing each price separately gives a lower bound on the price in the composed system, which can be obtained by solving

$$\alpha \sum_{b \in B} M_b^\tau - \max(0, p^\tau R_b - M_b^\tau) + \beta \sum_{b \in B} M_b^\tau - \max(0, p^\tau \overline{R}_b - M_b^\tau) = p^\tau \alpha \sum_{b \in B} R_b + p^\tau \beta \sum_{b \in B} \overline{R}_b$$

This approach can be further generalized to an arbitrary decomposition of the fairness function, leading to a lower bound

$$\sum_n \alpha_n \sum_{b \in B} M_b^\tau - \max(0, p^\tau R_{b,n} - M_b^\tau) = \sum_n \alpha_n p^\tau.$$

This construction is especially useful, since we can solve the n problems separately. This can be done for the canonical fairness function, leading to a lower bound on price in a Sequence with and arbitrary fairness function.

Proposition 4.7. *Let all traders follow the Greedy strategy. Then the price p^τ in Market $\mathbb{M}^\tau$ of a Sequence with fairness function ϕ satisfies*

$$p^\tau \geq \frac{\sum_{b \in B} (\alpha_b + 1) M_b^\tau}{2}, \qquad \textit{where} \qquad \alpha_b = \phi_b(V, D).$$

Proof. Let M_b^τ be the amount of Money buyers have at the start of Market τ. We can write the distribution ϕ as

$$\phi(V, D) = \sum_{n=1}^{|B|} \alpha_n \varphi_{b,n}(V, D).$$

where $\varphi_{b,n}(V, D)$ is the n-canonical fairness function defined in Eq. (15), and $\alpha_n \in [0, 1]$.

In the system with canonical fairness function we can obtain an explicit solution in a similar way to Eq. (16)

$$\sum_{b \in B} M_b^\tau - \max(0, p^\tau R_{b_n} - M_b^\tau) = p^\tau,$$

$$M_{b_n}^\tau - p^\tau + \sum_{b \in B} M_b^\tau = p^\tau, \qquad \Rightarrow \qquad p^\tau = \frac{M_{b_n}^\tau + \sum_{b \in B} M_b^\tau}{2}.$$

Weighing the prices according to the decomposition of the fairness function, we obtain the statement of the proposition. □