Bijective Proof of Kasteleyn's Toroidal Perfect Matching Cancellation Theorem

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Abstract

We give a constructive bijective proof for an identity that generalizes an observation of Kasteleyn: Let m, n be positive even numbers and let $T_{m,n}$ be the toroidal square grid which consists of m horizontal cycles $C_0^H, \ldots, C_{m-1}^H$ and n vertical cycles $C_0^V, \ldots, C_{n-1}^V$. Let A be one layer of the horizontal edges of $T_{m,n}$ and let B be one layer of the vertical edges of $T_{m,n}$. We say that a perfect matching is *even* if it has an even number of elements of each of A, B. We call a perfect matching *odd* if it is not even. If $h = (h_0, \ldots, h_{m-1})$ and $v = (v_0, \ldots, v_{n-1})$ are positive integer vectors then we let $\mathcal{E}_{m,n}(h, v)$ denote the set of the *even perfect matchings* of $T_{m,n}$ with exactly h_i edges from cycle C_i^H and exactly v_j edges from cycle C_j^V, for $0 \leq i < m$ and $0 \leq j < n$. Analogously let $\mathcal{O}_{m,n}(h, v)$ denote the set of the *odd perfect matchings* of $T_{m,n}$ with exactly h_i edges from cycle C_i^H and exactly v_j edges from cycle C_j^V, for $0 \leq i < m$ and $0 \leq j < n$. We show (combinatorially) that $|\mathcal{E}_{m,n}(h, v)| = |\mathcal{O}_{m,n}(h, v)|$, for all positive integer vectors h and v.

1 Introduction

A perfect matching of a graph $G = (V, E)$ is a collection of edges with the property that each vertex is adjacent to exactly one of these edges. In statistical physics a Perfect matching is

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called *dimer arrangement*. Very often graph G comes with an edge-weight function $w : E \to \mathbb{Q}$. The statistics of the weighted perfect matchings is then given by the generating function

$$P(G,x) = \sum_{M \text{ perfect matching of } G} x^{\sum_{e\in M} w(e)},$$

also known as the *dimer partition function* on G. We note that if G is bipartite, with bipartition classes V_1 and V_2, then $P(G,x)$ equals the permanent of $A(G)$, where $A = A(G)$ is the $|V_1| \times |V_2|$ matrix given by $A_{uv} = x^{w(uv)}$.

Dimer partition functions are among the most studied in statistical mechanics. One of their remarkable properties is that, if g denotes the minimum genus of an orientable surface in which G embeds, then the partition function of a dimer model on a graph G can be written as a linear combination of 2^{2g} Pfaffians of *Kasteleyn matrices*. These 2^{2g} Kasteleyn matrices are skew-symmetric $|V| \times |V|$ matrices determined by 2^{2g} orientations of G, called Kasteleyn (or Pfaffian) orientations. P. W. Kasteleyn himself proved this Pfaffian formula in the planar case and for the toroidal square grids where all the horizontal edge-weights and all the vertical edge-weights are the same [Kas61], and stated the general fact [Kas67]. A complete combinatorial proof of this statement was first obtained much later by Gallucio and Loebl [GL99], and independently by Tesler [Tes00]. In his seminal paper [Kas61], Kasteleyn found an analytic expression for the four Pfaffians in the case of the toroidal square grid where all the horizontal edge-weights and all the vertical edge-weights are the same, and noticed that:

> *One of these four Pfaffians, which corresponds to a particular Kasteleyn orientation, vanishes.*

This observation can be restated as follows: Let T be a toroidal square grid where we fix one layer of the horizontal edges (and denote it by A) and one layer of the vertical edges (and denote it by B). We will not do technical work with the Pfaffians and we thus only state the relevant facts regarding them (for details see [Kas61]). The Pfaffian is a signed generating function of the perfect matchings. In the particular Pfaffian which vanishes, a perfect matching of T has sign $+1$ if and only if it contains an even number of edges from each of A, B; we call these perfect matchings *even* and the remaining ones *odd*. Hence, the above observation of Kasteleyn may be stated as follows:

Theorem 1 *[Kas61] Let T be a toroidal square grid, where A (respectively, B) is one layer of the horizontal (respectively, vertical) edges. For each k, l, the number of odd perfect matchings of T with exactly k vertical edges and l horizontal edges is equal to the number of even perfect matchings of T with exactly k vertical edges and l horizontal edges.*

The main result of this paper is a bijective combinatorial proof of a generalization of the preceding theorem. Let m, n be positive even numbers and let $T_{m,n}$ be a toroidal square grid which consists of m horizontal cycles $C_0^H, \ldots, C_{m-1}^H$ and n vertical cycles $C_0^V, \ldots, C_{n-1}^V$. For $T = T_{m,n}$, let A and B be as in Theorem 1. If $h = (h_0, \ldots, h_{m-1})$ and $v = (v_0, \ldots, v_{n-1})$ are positive integer vectors then we let $\mathcal{E}_{m,n}(h, v)$ denote the set of the even perfect matchings of $T_{m,n}$ with exactly h_i edges from cycle C_i^H and exactly v_j edges from cycle C_j^V, for $0 \leq i < m$ and $0 \leq i < n$. Analogously, let $\mathcal{O}_{m,n}(h, v)$ be defined as $\mathcal{E}_{m,n}(h, v)$ but with respect to the set of odd perfect matchings. We show:

Theorem 2 *For all positive integer vectors h and v, there is a polynomial time computable bijection between the sets $\mathcal{E}_{m,n}(h, v)$ and $\mathcal{O}_{m,n}(h, v)$.*

The partition function of free fermions on a closed Riemann surface of genus g is also a linear combination of 2^{2g} determinants of Dirac operators [AGMV86]. It is widely assumed in theoretical physics that dimer models are discrete analogues of free fermions and thus a relation between Kasteleyn matrices and Dirac operators is expected. This is well understood in the planar case due to the work of Kenyon [Ken02], and in the genus one case by the work of Ferdinand [Fer67] that extend results of Kasteleyn [Kas61]. The higher genus $g > 1$ case remains mysterious. The only results available are numerical evidence for one graph embedded on the double torus [CSM02]. In particular, $2^{g-1}(2^g - 1)$ determinants of the Dirac operators vanish, and a natural open problem is whether a discrete analogue of this fact holds for the Pfaffians of Kasteleyn matrices. The case $g = 1$ is affirmative by Theorem 1. However, the algebraic techniques used to prove this fact are not suitable even for addressing the genus 2 case, where 6 out of 16 Pfaffians are required to vanish (or vanish 'in a limit').

The motivation for this paper has been to give a bijective combinatorial proof of Theorem 1. We hope that this may help understanding the genus 2 case.

2 Preliminaries

Throughout this work, let m and n be two positive integers. We solely consider the case where both m and n are even. We will introduce a couple of distinct graphs and/or digraphs with node set $V_{m,n} = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ all of them satisfying the following property: the endpoints (i, j) and (i', j') of any of their edges differ in at most one coordinate and exactly by 1,[1] i.e.,

$$(i - i' = \pm 1 \wedge j = j') \vee (i = i' \wedge j - j' = \pm 1).$$

Edges between nodes (i, j) and (i', j') such that $i = i'$ (respectively, $j = j'$) will be referred to as *horizontal* (respectively, *vertical*) edges.

The collection of nodes $(i, j) \in V_{m,n}$ for which $0 \leq j < n$ (respectively, $0 \leq i < m$) will be called the *i-th row* (respectively, *j-th column*), and i (respectively, j) will be called its index. Nodes that belong to a row (respectively, column) of even index will be called *even row nodes* (respectively, *even column nodes*). If a node is both an even (respectively, odd) row and column node, then it will be simply referred to as an *even node* (respectively, *odd node*).

[1]Here and throughout this work, arithmetic involving node labels is always modulo m and n over the first and second coordinates, respectively

The collection of nodes $(i,j) \in V_{m,n}$ such that $i+j$ is even (respectively, odd) will henceforth be referred to as *black* (respectively, *white*) nodes, denoted $B_{m,n}$ (respectively, $W_{m,n}$). From our ongoing discussion it follows that the graphs we will work with are all bipartite. In fact, $\{B_{m,n}, W_{m,n}\}$ will always be a feasible color class bipartition of $V_{m,n}$.

Paths and cycles will always be considered as subgraphs, say S, of whatever the graph or digraps, we are dealing with. We thus rely on the standard notation $V(S)$ and $E(S)$ to denote S's nodes and edges, respectively.

For a simple path or cycle S in whatever graph we consider, we say that node $v \in V(S)$ is a *corner* of S if one of the edges of S incident on v is a horizontal edge and the other one is a vertical edge.

We now establish a simple, yet key property, of certain type of cycles of $T_{m,n}$. The property concerns the number of nodes encircled by such cycles.

Lemma 3 *If C is a contractible cycle of $T_{m,n}$ all of whose corners have the same parity, then the disk encircled by C contains a single connected component of $T_{m,n} \setminus C$.*

Proof For the sake of contradiction, assume the disk encircled by C contains two connected components of $T_{m,n} \setminus C$. The only way in which this could happen is if the intersection of C with a face of $T_{m,n}$ is not connected. Since the corners of C all have the same parity, one of the following two cases holds for some $(i,j) \in V_{m,n}$: (1) there is a horizontal edge of C between (i,j) and $(i,j+1)$, and also between $(i+1,j)$ and $(i+1,j+1)$, or (2) there is a vertical edge of C between (i,j) and $(i+1,j)$, and also between $(i,j+1)$ and $(i+1,j+1)$. Because of the grid like structure of $T_{m,n}$, for case (1) to occurr, one of the corners of C must belong to the row of index i and another corner to the row of index $i+1$, contradicting the parity assumption. Case (2) can be dealt with the same way as case (1). □

Lemma 4 *If C is a contractible cycle of $T_{m,n}$ all of whose corners have the same parity, then the disk encircled by C contains an odd number of nodes of $T_{m,n} \setminus C$.*

Proof By Lemma 3 we know that C encircles one connected component of $T_{m,n} \setminus C$. Let P be the shortest segment of C (ties broken arbitrarily), whose endpoints are corners of C. Clearly, by minimality of P, all nodes of P must belong to either the same column or the same row. Without loss of generality, assume that the former case holds, and that the nodes of P are $(i_1,j),(i_1+1,j),\ldots,(i_2,j)$. Let R_1 (respectively, R_2) be the rectangle (cycle with four corners) of $T_{m,n}$ that contains P as a segment and whose opposite corners are (i_1,j) and $(i_2,j+2)$ (respectively, $(i_2,j-2)$). Because the parity hipothesis concerning C's corners and the grid structure of $T_{m,n}$, either R_1 or R_2 must encircle an area of the torus also encircled by C. Without loss of generality, assume R_1 is such a rectantgle. Thus, either $C = R_1$ or by minimality of P the situation is one of the three depicted in Figure 1. To conclude, observe that any path fully contained in a single row or column whose endpoints have the same parity has an odd number of nodes. The desired conclussion follows inmediately by induction and the fact that any rectangle whose corners have the same parity encircles a set of nodes of size a product of two odd numbers, i.e., an odd number of nodes. □

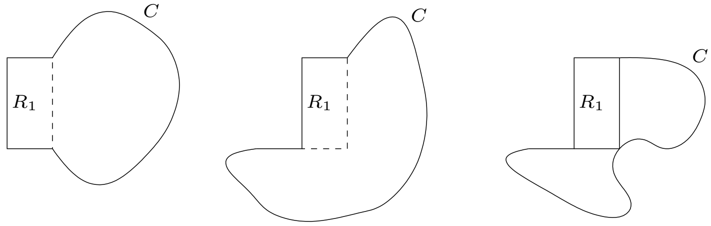


Figure 1: Possible relations between R_1 and C when $C \neq R_1$.

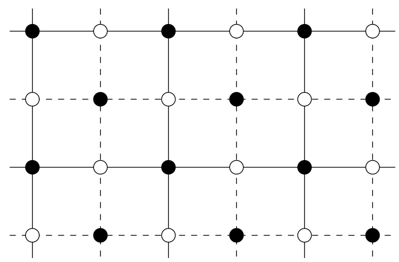


Figure 2: Depiction of $T_{4,6}$. Black/white nodes shown in black/white. Even/odd rows shown as solid/dashed lines.

Consider now a perfect matching M of $T_{m,n}$ (in contrast to our view of paths and cycles as subgraphs, we consider matchings as subsets of edges). We will associate to M a di-graph $D_{m,n} = D_{m,n}(M) = (U, D)$ over the same set of nodes of $T_{m,n}$, that is $U = V_{m,n}$, and such that (see Figure 3):

- If v is black, then $vw \in D$ if and only if vw is an edge of M.
- If v is white, then $vw \in D$ provided: (1) vw is not an edge of M, and (2) if uv is the edge of the matching M incident to v, then u, v, and w all lie in the same row or the same column of $T_{m,n}$.

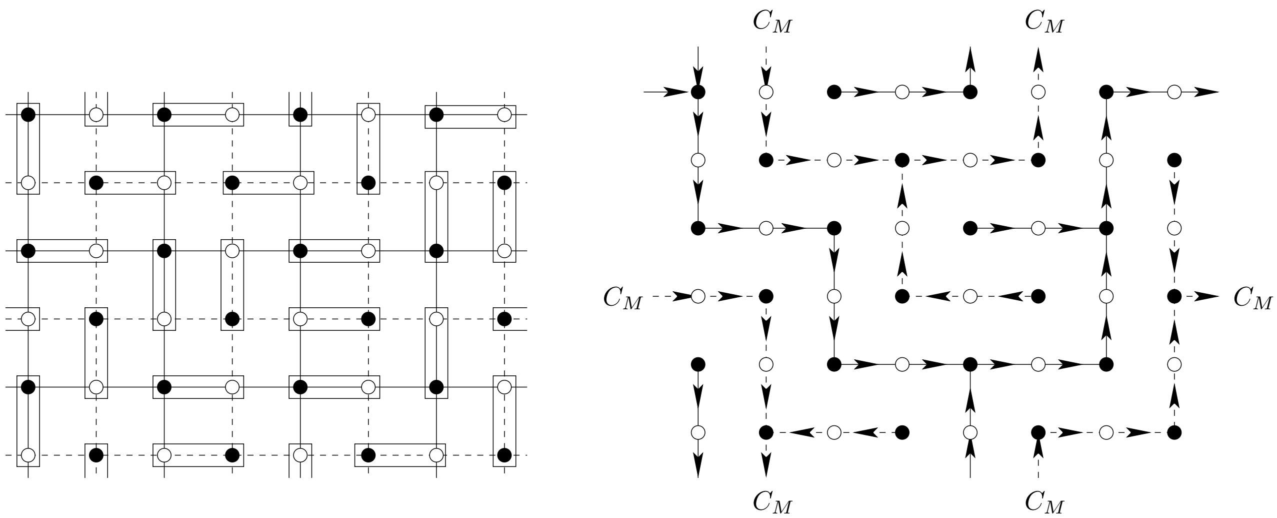
The association of M to $D_{m,n}(M)$ is inspired on one due to Thiant [Thi06, Ch. 5] which maps domino tilings of rectangles (equivalently, perfect matching of grids) to rooted trees. However, neither our context, nor the properties of the derived objects are related to [Thi06]. In particular, we consider toroidal grids in contrast to square grids, and hence $D_{m,n}(M)$ is not necessarily even a forest.

The digraph $D_{m,n}(M)$ has a fair amount of interesting structure. Our next result states the most obvious one.

Lemma 5 *If M is a perfect matching of $T_{m,n}$ and $D_{m,n} = D_{m,n}(M)$, then*

(i).- Every node of $D_{m,n}$ has out-degree one. every node of $D_{m,n}$ has out-degree one.

(ii).- If two black nodes belong to the same dipath of $D_{m,n}$, then either they both lie in even rows and columns, or they both lie in odd rows and columns.



(a) Matching M represented as rectangles.

(b) Digraph $D_{6,8}(M)$.

Figure 3: Perfect matching M of $T_{6,8}$ and associated $D_{6,8}(M)$. Even (respectively, odd) edges shown as solid (respectively, dashed) lines.

(iii).- Any dicycle of $D_{m,n}$ has all of its corners of the same parity.

Proof To establish Claim (i) it suffices to observe that the edges outgoing from a node are completely determined by the edges of M incident to that node, and observing that there is exactly one such matching edge per node.

To establish Claim (ii) just oberseve that if the outgoing edge from a black node is horizontal (respectively, vertical), then it is incident to a white node whose unique outgoing edge will also be horizontal (respectively, vertical) edge. Claim (iii) follow immediately from Claim (ii). □

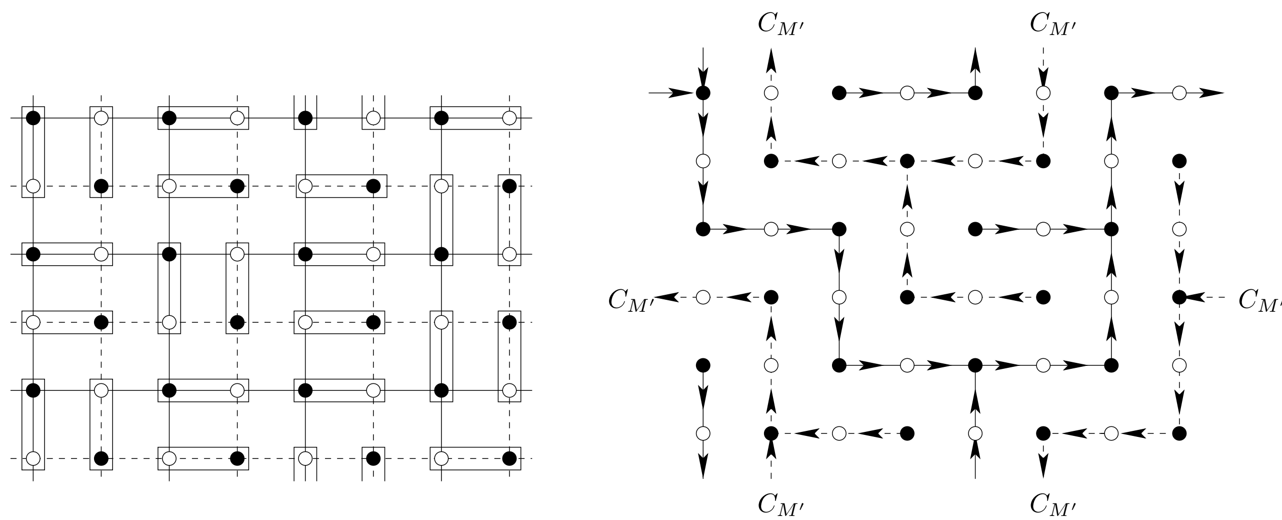
Note that for a perfect matching M of $T_{m,n}$, the components of digraph $D_{m,n}(M)$ can be partition into two classes depending on the parity of their corners – we will not directly use this property of $D_{m,n}$.

3 Bijection

Henceforth, let A (respectively, B), be the set of vertical (respectively, horizontal) edges whose endpoints are in the rows (respectively, columns) of index 0 and $m-1$ (respectively, $n-1$). Whether or not we consider the collection of edges in A and B are undirected or directed will be clear from context.

Let $\mathcal{M}_{m,n}$ be the collection of perfect matchings of $T_{m,n}$. We say that $M \in \mathcal{M}_{m,n}$ is of:

- Type EE: If $|M \cap A|$ and $|M \cap B|$ are both even.



(a) $M' = \Phi_{6,8}(M)$. (b) $D_{6,8}(M')$.

Figure 4: Illustration of $\Phi_{m,n}(\cdot)$ and associated $D_{m,n}(\cdot)$.

- Type *EO*: If $|M \cap A|$ is even and $|M \cap B|$ is odd.
- Type *OE*: If $|M \cap A|$ is odd and $|M \cap B|$ is even.
- Type *OO*: If $|M \cap A|$ and $|M \cap B|$ are both odd.

We henceforth let $\mathcal{M}^{(EE)}_{m,n}$, $\mathcal{M}^{(EO)}_{m,n}$, $\mathcal{M}^{(OE)}_{m,n}$, and $\mathcal{M}^{(OO)}_{m,n}$ be the collection of matchings of $T_{m,n}$ of type EE, EO, OE, and OO, respectively.

We now explicitly construct a mapping $\Phi_{m,n} : \mathcal{M}_{m,n} \to \mathcal{M}_{m,n}$. In order to define it, we fix a total ordering of the dicycles of digraphs over the node set $V_{m,n}$ which satisfies the *orientation independence property*, i.e. the ordering only depends on the node set of the dicycles (in particular, it ignores the orientation of its edges). Also, for a subgraph H of a digraph D, we denote by $U(H)$ the collection of edges of H ignoring orientations. Now, consider a perfect matching M of $T_{m,n}$. Let C_M be the first (according to the aforementioned fixed total ordering) dicycle of $D_{m,n}(M)$ and let

$$\Phi_{m,n}(M) = M \triangle U(C_M).$$

Since C_M alternates between edges and non-edges of M, it follows that $\Phi_{m,n}(M) \in \mathcal{M}_{m,n}$. (For the matching M of Figure 3(a), the matching $M' = \Phi_{m,n}(M)$ is illustrated in Figure 4(a)).

Proposition 6 *If a perfect matching $M \in \mathcal{M}_{m,n}$ has exactly h_i edges in row i and exactly v_j edges in column j, $0 \le i < m$, $0 \le j < n$, then so does $\Phi_{m,n}(M)$.*

Proof Let $e_0, e_1, \ldots, e_\ell = e_0$ be the sequence of (undirected) edges of $T_{m,n}$ traversed by C_M. Since $D_{m,n}$ is bipartite, we have that ℓ must be odd. Without loss of generality, we can assume $e_0 \in M$. The way in which $D_{m,n}$ was constructed immediately guarantees that the following two claims hold for all $0 \leq i < \ell$: (1) $e_i \in M$ if and only if i is even, and (2) if i is even, then e_i and e_{i+1} are either both horizontal or both vertical edges. By definition of $\Phi_{m,n}(\cdot)$, the desired conclusion follows. $\square$

Now, consider a perfect matching M of $T_{m,n}$ and let $D_{m,n} = D_{m,n}(M)$. We say that a dicycle C of $D_{m,n}$ is of:

- Type *ee*: If $|U(C) \cap A|$ and $|U(C) \cap B|$ are both even.
- Type *eo*: If $|U(C) \cap A|$ is even and $|U(C) \cap B|$ is odd.
- Type *oe*: If $|U(C) \cap A|$ is odd and $|U(C) \cap B|$ is even.
- Type *oo*: If $|U(C) \cap A|$ and $|U(C) \cap B|$ are both odd.

The following result establishes that the structure of $D_{m,n}(\cdot)$ is such that it forbids the appearance of cycles of type *ee*.

Lemma 7 *For every perfect matching M of $T_{m,n}$, the digraph $D_{m,n} = D_{m,n}(M)$ does not contain dicycles of type ee.*

Proof For the sake of contradiction, assume C is a dicycle in $D_{m,n}$ of type *ee*. It follows that C must be the symmetric difference of boundaries of faces $F_1, \ldots, F_k$ of $T_{m,n}$. Since C is symmetric difference of faces and connected, it follows that C is contractible. By Lemma 4 and Part (iii) of Lemma 5, the interior of the disk encircled by C must contain and odd number of vertices of $T_{m,n}$, contradicting the fact that C is the symmetric difference of two perfect matchings of $T_{m,n}$, and hence must encircle an even number of vertices of $T_{m,n}$. $\square$

Next, we show that for each perfect matching M, only vertex disjoint cycles of equal type can arise in $D_{m,n}(M)$.

Lemma 8 *Let M be a perfect matching of $T_{m,n}$ and let $D_{m,n} = D_{m,n}(M)$. If C and C' are two distinct dicycles of $D_{m,n}$, then they are vertex disjoint.*

Proof Direct consequence of the fact that every node of $D_{m,n}$ has out-degree 1 (see Lemma 5). $\square$

We are now ready to establish that $\Phi_{m,n}(\cdot)$ is one-to-one. In fact, we can prove something stronger.

Corollary 9 *The mapping $\Phi_{m,n}$ is an involution of $\mathcal{M}_{m,n}$.*

Proof Consider $M \in \mathcal{M}_{m,n}$ and let $M' = \Phi_{m,n}(M)$. Note that the only difference between $D_{m,n}(M)$ and $D_{m,n}(M')$ is that C_M and $C_{M'}$ have opposite orientations. Hence, because of the orientation independence property, $U(C_{M'}) = U(C_M)$, which implies that

$$\Phi_{m,n}(M') = M' \triangle U(C_{M'}) = \Big(M \triangle U(C_M)\Big) \triangle U(C_{M'}) = M.$$

$\square$

Lemma 10 *For every perfect matching M of $T_{m,n}$, the parity of $|\Phi_{m,n}(M) \cap A|$ and $|M \cap A| + |U(C_M) \cap A|$ is the same.*

Proof Simply observe that,

$$
\begin{aligned}
|\Phi_{m,n}(M) \cap A| &= |(M \triangle U(C_M)) \cap A| = |(M \cap A) \triangle (U(C_M)) \cap A)| \\
&= |M \cap A| + |U(C_M) \cap A| - 2|M \cap A \cap U(C_M)|.
\end{aligned}
$$

□

We still need to show that $\Phi_{m,n}(\cdot)$ maps matchings of type EE onto matchings of type distinct than EE, and viceversa.

Proposition 11 *The image of $\mathcal{M}_{m,n}^{(EE)}$ via $\Phi_{m,n}(\cdot)$ is $\mathcal{M}_{m,n}^{(EO)} \cup \mathcal{M}_{m,n}^{(OE)} \cup \mathcal{M}_{m,n}^{(OO)}$.*

Proof Let M be a perfect matching of $\mathcal{M}_{m,n}^{(EE)}$. Since M is of type EE, we have that $|M \cap A|$ is even. Hence, by Lemma 10, the parity of $|\Phi_{m,n}(M) \cap A|$ equals the parity of $|U(C_M) \cap A|$. Similarly, the parity of $|\Phi_{m,n}(M) \cap B|$ equals the parity of $|U(C_M) \cap B|$. Because of Lemma 7, dicycle C_M must be of type *eo*, *oe*, or *oo*, in which case, by the ongoing discussion, $\Phi_{m,n}(M)$ would be of type EO, OE, or OO, respectively. □

We now claim:

Proposition 12 *The following hold:*

(i).- If M is a perfect matching of $T_{m,n}$ of type EO, OE, OO, then C_M is of type eo, oe, oo, respectively.

(ii).- The image of $\mathcal{M}_{m,n}^{(EO)} \cup \mathcal{M}_{m,n}^{(OE)} \cup \mathcal{M}_{m,n}^{(OO)}$ via $\Phi_{m,n}(\cdot)$ is $\mathcal{M}_{m,n}^{(EE)}$.

Theorem 2 is an immediate consequence of Proposition 6, Corollary 9 and Propositions 11 and 12.

To prove Proposition 12, note that Lemma 10 and Part (i) immediately imply Part (ii). The next section is devoted to showing Part (i) of Proposition 12. The argument is not straightforward and relies on abstracting characteristics of the objects we are dealing with and then manipulating them in this new higher level setting. We develop the argument in the next section.

4 Consistent Toroidal Systems

Before precisely describing the abstract setting we will work in, we establish the following result, which essentially says that we can restrict our attention to certain "better behaved" perfect matchings. Say that a perfect matching M of $T_{m,n}$ is *well behaved*, if C_M neither contains a horizontal (respectively, vertical) edge both of whose endnodes are in row 0 nor both in row $m-1$ (respectively, neither both endnodes in column 0 nor both in column $n-1$).

Lemma 13 *If M is a perfect matching of $T_{m,n}$, then there is a well behaved perfect matching M' of $T_{m+4,n+4}$ for which the types of M and M' are the same, and the types of C_M and $C_{M'}$ are the same.*

Proof We explicitly build M' with the desired properties. First, for each edge in $M \setminus (A \cup B)$, say between nodes (i,j) and (i',j') add an edge into M' between $(i+2, j+2)$ and $(i'+2, j'+2)$.

Now, if there is a vertical edge in $M \cap A$ between $(0,j)$ and $(m-1,j)$, then include in M' the vertical edge between $(0,j)$ and $(m+3,j)$. Note that for each column j of $T_{m+4,n+4}$, $2 \leq j \leq n+1$, in which uncovered nodes remain (there are exactly four such nodes in each column) they can be covered in a (uniquely determined) way by two vertical edges.

Similarly, if there is an edge in $M \cap B$ between $(i,0)$ and $(i,n-1)$, include in M' an horizontal edge between $(i,0)$ and $(i,n+3)$, and proceed as in the previous paragraph but considering rows instead of columns and horizontal edges instead of vertical ones.

Still, the nodes located in rows 0, 1, $m+2$, and $m+3$, which also belong to one of the columns 0, 1, $n+2$, and $n+3$, form 4 groups of 4 uncovered nodes. Cover each such group of 4 nodes by two vertical edges and include them in M'.

It is not hard to see that the described construction is sound, i.e., M' is a perfect matching of $T_{m+4,n+4}$. Moreover, C_M can be naturally identified with $C_{M'}$, the latter of which can be easily seen to be well behaved.

Finally, let A' and B' be defined as A and B but with respect to $T_{m+4,n+4}$ instead of $T_{m,n}$. It is straightforward to verify that $|M \cap A| = |M' \cap A'|$, $|M \cap B| = |M' \cap B'|$, $|C_M \cap A| = |C_{M'} \cap A'|$, and $|C_M \cap B| = |C_{M'} \cap B'|$, thus establishing the lemma's statement. $\square$

We now introduce the abstract setting in which we will henceforth work. Let T denote the surface of a torus. We say that $(R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ is a *toroidal system* if the following holds (see Figure 5):

- **Rectangle:** R is a rectangle of the torus T. The *sides* of R are denoted clockwise H_t, V_l, H_b, V_r.[2] We call H_t, H_b the *horizontal sides* and V_l, V_r the *vertical sides* of R.

- **Nodes:** N is a set of nodes. An even number of *nodes* are located in each of the four boundary sides of R. We denote by A_t, A_b (respectively, B_l, B_r) the nodes located in H_t, H_b (respectively, V_l, V_r). It must hold that $|A_t| = |A_b|$ and $|B_l| = |B_r|$. Moreover, one node is located in each corner of R. Hence, e.g., $|A_t \cap B_l| = 1$. These nodes are called *corner nodes*.

  Nodes are bi-colored black and white alternately along the boundary of R. We henceforth assume that the corner node located at the intersection of V_l and H_t is colored white and hence completely fix the coloring of A_t, B_l, A_b, B_r. Traversing H_t, H_b starting from V_l, (respectively, V_l, V_r starting from H_t) one can naturally pair the nodes of A_t, A_b (respectively, B_l, B_r) in order of their appearance. We call *opposite* nodes each such pair and note that opposite nodes have different color.

[2] Here t, l, b, r stand for top, left, bottom, right, respectively.

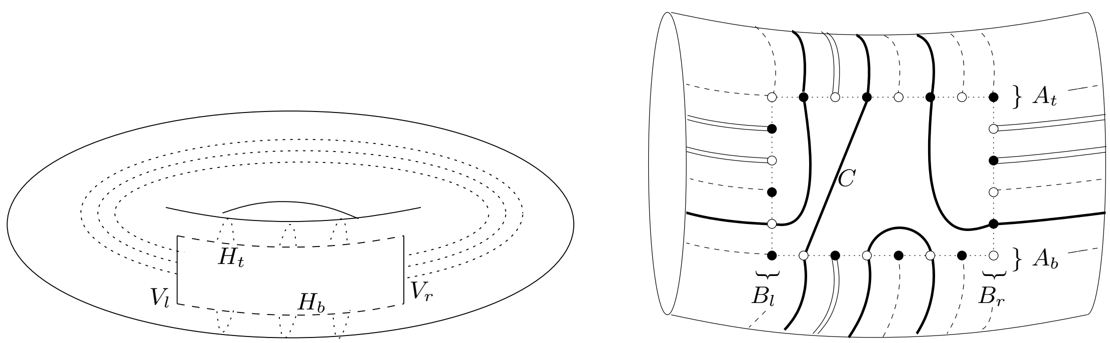
- **Edges:** Sets A and B are collections of disjoint closed curves on T. Elements of $A \cup B$ are called *edges*. Each edge in A (respectively, B) connects two opposite nodes of A_t, A_b (respectively, B_r, B_l) and is otherwise disjoint with R. Each edge of $C \cap A$ (respectively, $C \cap B$) has its black node in A_t and its white node in A_b (respectively, black node in B_r and its white node in B_l).

- **Outer Matching:** $M_{A,B}$ is a collection of edges of $A \cup B$ referred to as *matching edges*.

- **Toroidal cycle**: C is a non-intersecting non-contractible toroidal loop which is a subset of $R \cup A \cup B$ and avoids every corner node of R.

- **Cycle Matching**: $O(C)$ is a collection of disjoint closed intervals of $C \cap R$ such that each element of $O(C)$ is disjoint with each edge of $M_{A,B}$, has a node at one of its endpoints and the other endpoint is a point of the interior of R. Moreover, each node of C is covered by $M_{A,B} \cup O(C)$.

- **Inner Matching**: $O(R)$ is a collection of disjoint closed curves contained in R which cover each node uncovered by $M_{A,B} \cup C$ and such that each curve o of $O(R)$ satisfies: (1) o is disjoint with $M_{A,B} \cup C$, (2) o is the segment of a side of R between two neighboring nodes, or o intersects the boundary of R in exactly one point, which is then a node.



(a) Depiction in the torus T of rectangle R (horizontal sides H_t and H_b shown as dashed lines, and vertical sides V_l, V_r shown as solid lines). Dotted lines correspond to edges $A \cup B$.

(b) Depiction of a slice of the torus. Bi-colored node sets A_t, B_r, A_b, B_l located in the dotted line representing the sides of R. Dashed lines represent edges $A \cup B$. Double solid lines correspond to edges in $M_{A,B}$. Cycle C is represented as a thick solid line.

Figure 5:

- **Segments**: Connected components of $C \setminus (A \cup B)$ (respectively, $C \setminus A$, $C \setminus B$) are called *segments* (respectively, A-segments, B-segments) of C (see Figure 6). The endpoints of a segment are the only one of its points that can belong to the boundary of R (it follows that segments are disjoint). We will say that a *segment is bi-colored* when located at its endpoints there are nodes of distinct color, otherwise we say the segment is mono-chromatic.

We require that the nodes located at the endpoints of A-segments (respectively, B-segments) belong to A_t if they are black and to A_b if they are white (respectively, to B_r if they are black and to B_l if they are white).

We define $\mathcal{S}(M) = (R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$, for a a well behaved perfect matching M of $T_{m,n}$, as follows. We let R equal the rectangle of the torus with sides the edges in rows 0 and $m-1$ of $T_{m,n}$, and columns 0 and $n-1$ of $T_{m,n}$. We let N be the set of vertices of $T_{m,n}$ located in the boundary of R. The set of A (respectively, B) correspond to the set of edges of $T_{m,n}$ between nodes in row 0 and $m-1$ (respectively, 0 and $n-1$). The matching $M_{A,B}$ will be $M \cap (A \cup B)$ and the toroidal cycle C is set to C_M. The set $O(C)$ will consist of vertical edges of $C_M \setminus A$ incident to nodes in rows 0 or $m-1$ that are not covered by edges of $M_{A,B}$ as well as horizontal edges of $C_M \setminus B$ incident to nodes in columns 0 or $n-1$ also not covered by $M_{A,B}$. Finally, the set of $O(R)$ will consist of all edges $M \setminus (A \cup B)$ not in $O(C)$ which are incident to nodes in rows 0 or $m-1$, or to nodes in columns 0 and $n-1$. The following result may be easily verified.

Proposition 14 *If M is a well behaved perfect matching of $T_{m,n}$, then it is possible to rotate $T_{m,n}$ clockwise or counterclockwise by ninety degrees so that $\mathcal{S}(M)$ is a toroidal system.*

In fact, we will soon see that the toroidal system guaranteed by the preceding proposition in fact satisfies additional properties. Below, we first introduce some additional terminology and then the properties of toroidal systems that we will be concerned with.

An interval I of the loop H'_t (respectively, H'_b, V'_l, V'_r) consisting of H_t (respectively, H_b, V_l, V_r) and the edge between the endnodes of H_t (respectively, H_b, V_l, V_r) will be called *loop interval*.

For a segment we distinguish two situations (see again Figure 6):

- **Ears:** if the endpoints of the segment belong to the same side of R (H_t, V_r, H_b, or V_l).
- **Lines:** if the endpoints of the segment belong to opposite sides of R (H_t and H_b, or V_l and V_r).

We analogously define A-ears, B-ears and A-lines, B-lines. Let D be a disc of T bounded by exactly two A-lines and two loop intervals belonging to opposite loops H'_t and H'_b. We call D *A-regular* and its boundary loop intervals are called *ends* of D. A toroidal cycle C is called *A-regular toroidal cycle* if all its A-segments are A-lines. We define *B-regular disc* and *B-regular toroidal cycle* analogously. For instance, in Figure 6, the cycle is B-regular but not A-regular (because there are two A-segments which are ears). The cycle C is called *regular*, if it is both A-regular and B-regular.

A toroidal system $(R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ is called *consistent*, if the following properties hold:

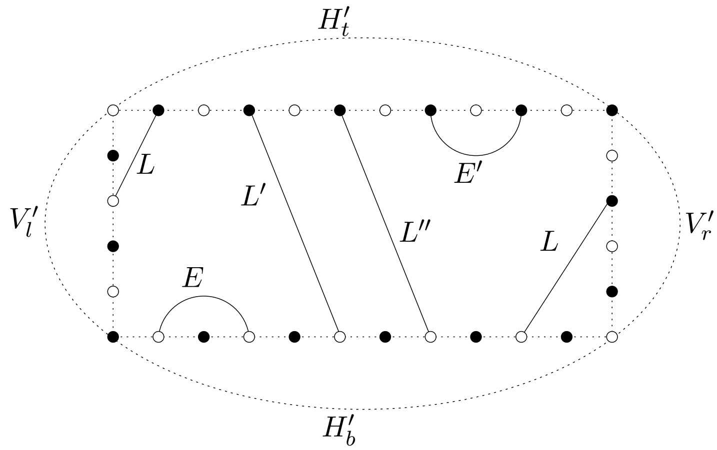


Figure 6: Illustration of a toroidal cycle C in rectangle R (the boundary of R is depicted as straight dotted lines): C has 6 segments, in fact 5 are A-segments (lines L, L' and L'' and ears E and E') and the whole cycle is a B-segment. The curved dotted lines are the completions of H_t, V_r, H_b, V_l that form H'_t, V'_r, H'_b, V'_l, respectively.

Segment consistency (S1): Let S be a segment of C and let e and e' be the edges whose endpoints are incident to the endnodes of S. Then, S is bi-colored if and only if $|\{e, e'\} \cap M_{A,B}|$ is even.

Line consistency (L1): If L is an A-line (respectively, B-line), then it has its black endnode in A_t and its white endnode in A_b (respectively, black endnode in B_r and its white endnode in B_l).

Ear consistency: Every ear E is mono-chromatic and satisfies:

Condition (E1): If E's endnodes are in A_t (respectively, B_r), then their color is black, otherwise their color is white.

Condition (E2): If D is the open disc of T bounded by the A-ear (respectively, B-ear) E and the loop-interval between E's endnodes, then E's endnodes are black if and only if D contains an odd number of endpoints of elements of $O(R) \cup O(C) \cup M_{A,B}$.

Region consistency:

Condition (R1): If D is an open A-regular (respectively, B-regular) disc, then D contains an even number of endpoints of elements of $O(R) \cup O(C) \cup M_{A,B}$.

Condition (R2): Let L be an A-line containing $k \geq 0$ edges of B. Let R' be obtained by concatenation of $k+1$ copies of R by its vertical boundaries. This completely determines $N', A', B', M'_{A',B'}, O(R'), O(C')$. Let L' denote the cover of L drawn in R': we note that L' has no edge of B'. If D is an open disc bounded by L', an interval of H'_b (respectively, an interval of V'_l), the vertical side V'_r (respectively, horizontal side H'_t), and an interval of H'_t (respectively, V'_r), then D contains and even number of endpoints of elements of $O(R') \cup O(C') \cup M'_{A',B'}$. We also require the analogous condition to hold but for B-lines.

It is not hard to see that the following strengthening of Proposition 14 holds.

Proposition 15 *If M is a well behaved perfect matching of $T_{m,n}$, then $\mathcal{S}(M)$ is a consistent toroidal system.*

We say that C has *type eo* if C has an even number of edges of A and an odd number of edges of B. Analogously, we define types ee, oe, oo. As usual, a type is also associated to the set $M_{A,B}$ according to the parity of its intersection with A and B, respectively.

Our goal for the remaining part of this article is to prove the following result (which together with Proposition 15 immediately implies the remaining unproved part of Proposition 12):

Theorem 16 *Let $(R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ be a consistent toroidal system. If neither the type of C nor the type of $M_{A,B}$ is ee, then the type of C is the same as the type of $M_{A,B}$.*

Henceforth, we say that two toroidal systems $\mathcal{S} = (R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ and $\mathcal{S}' = (R', N', \{A', B'\}, M'_{A',B'}, C', O(C'), O(R'))$ are *equivalent* if the type of C is the same as the type of C' and the type of $M_{A,B}$ is the same as the type of $M'_{A,B}$.

Theorem 16 will be a direct consequence of the the next two results.

Proposition 17 *If $\mathcal{S} = (R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ is a consistent toroidal system where neither the type of C nor the type of $M_{A,B}$ is ee, then either Theorem 16 holds for $\mathcal{S}$ or there is a regular consistent toroidal system $\mathcal{S}'$ which is equivalent to $\mathcal{S}$.*

Proof We describe how to iteratively construct a new A-regular consistent toroidal system, say $\mathcal{S}' = (R', N', \{A', B'\}, M'_{A',B'}, C', O(C'), O(R'))$, equivalent to $\mathcal{S}$. The system $\mathcal{S}'$ is then analogously adjusted (arguing with respect to B instead of A) to yield the claimed consistent regular system.

Let E be a black-endnodes-A-ear which is *minimal*, i.e., C does not intersect the loop interval between the endnodes of E. Let e_1 and e_2 be the edges of $A \cap C$ which are adjacent to E. Let the black-endnode of e_i be denoted by x_i, and the white-endnode by y_i, $i = 1, 2$. Denote by D the open disk with boundary E and the loop interval between x_1 and x_2. By the choice of E, the loop intervals between x_1, x_2 and between y_1, y_2 do not intersect C. Because of consistency conditions (S1) and (E1), there is exactly one edge of $M_{A,B} \cap C$ incident with an endnode of E: without loss of generality let it be e_1. Let us denote by S_1 the segment of C prolonging e_1 and by S_2 the segment of C extending e_2.

Next, we establish the following claim; the set of edges of $M_{A,B}$ that are incident with the interior of the loop interval between x_1 and x_2, henceforth denoted $M_{A,B}(E)$, has an even cardinality. Indeed, because (E2) holds for E, there is an odd number of endpoints of elements of $O(R) \cup O(C) \cup M_{A,B}$ contained in D. By minimality of E, none of the elements of $O(C)$ is contained in D and each element of $M_{A,B}$ is either contained in D or disjoint with it.

Hence, the interior of the loop interval between x_1 and x_2 is incident with an odd number of curves of $O(R)$; let us denote by $R(E)$ the set consisting of these curves. All (the odd number of) nodes in the interior of the loop interval between x_1 and x_2 not covered by elements of $R(E)$ must be covered by elements of $M_{A,B}(E)$. We conclude that $|M_{A,B}(E)| + |R(E)|$ and $|R(E)|$ are both odd, and hence $|M_{A,B}(E)|$ is even, thus proving the claim.

There are three possibilities for the types of S_1 and S_2: (1) S_1 is an A-line and S_2 is a white-endnodes-A-ear, (2) both S_1 and S_2 are A-lines, or (3) both S_1 and S_2 are white-endnodes-A-ears. Each possible case is illustrated in Figure 7(a), 7(b), and 7(c): note that for the last two cases there are two subcases depending on the relative positions of S_1 and S_2.

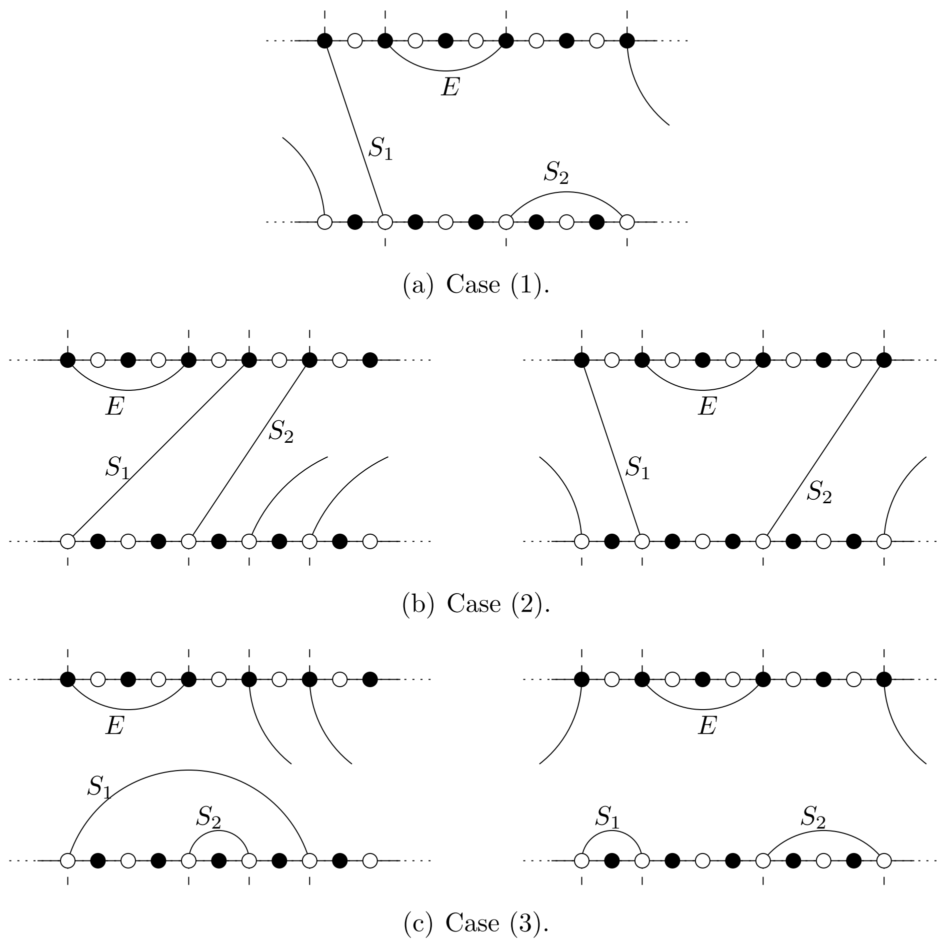
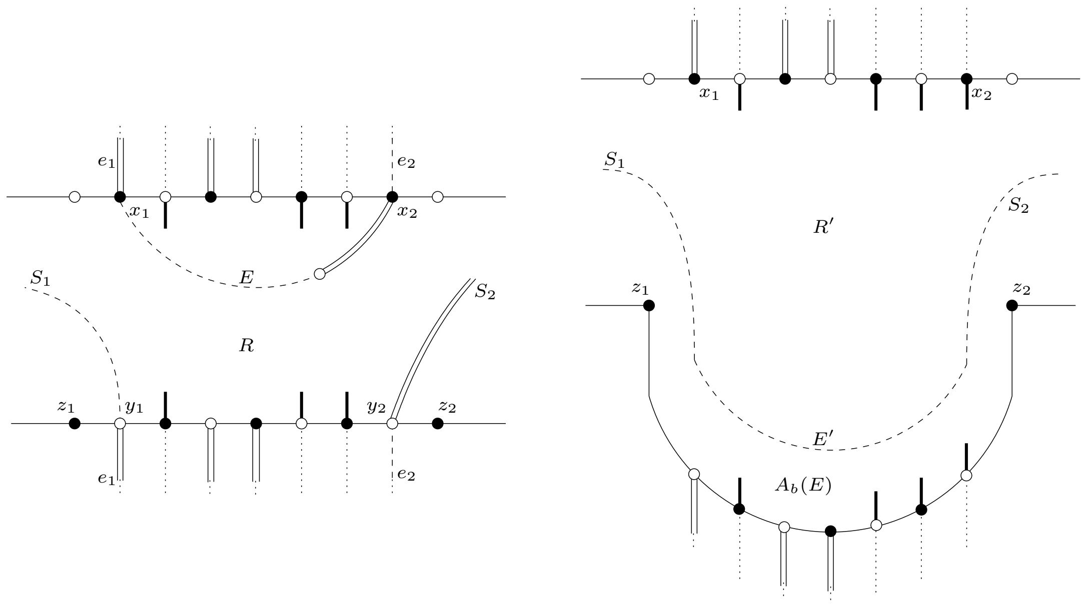


Figure 7:

By consistency conditions (L1) and (E1), the nodes located at the endpoints of $S_1 e_1 E e_2 S_2$ are colored black and white, respectively, in case (1); are both black, in case (2); and, are both white, in case (3).

Let z_i be the neighbor of y_i along H_b (see Figure 8(a)). We construct R' by attaching a new disc $A_b(E)$ to the loop interval between z_1 and z_2; in the interior of the outer boundary



(a) Illustration of a portion of a toroidal system $\mathcal{S}$ containing a black-endnodes-A-ear labeled E. Cycle C shown as dashed line, edges in A shown as dotted lines, elements of $O(R)$ shown as thick lines, elements of $M_{A,B}$ depicted as double lines outside R, and elements of $O(C)$ depicted also as double lines but inside R.

(b) Illustration of the portion of the toroidal system $\mathcal{S}'$ obtained by applying (to the system $\mathcal{S}$ illustrated in Figure 8(a)) the procedure described in the proof of Proposition 17. The components C', $M'_{A',B'}$, and the elements of $O(R')$ and $O(C')$ are depicted following the same conventions as in Figure 8(a).

Figure 8:

of $A_b(E)$ we place the same number of nodes as there are in the loop interval of R between z_1 and z_2 (see again Figure 8(a)). We then copy e_1Ee_2 to the just added disc and obtain C' from C as shown in Figure 8(b). Next, we remove nodes that end up in the interior of R'. This completely determines the new node set N'. Every A-edge e incident to nodes x_i and y_i, x_i between x_1 and x_2 (including them) and y_i between y_1 and y_2 (also including them), is replaced by an edge e' with endnodes x_i and the copy of y_i that was placed in the outer boundary of $A_b(E)$. All other edges are preserved. This completely determines the new set of edges A' and B'. Note that there is a one-to-one correspondence between old A/B-edges and new A'/B'-edges. The set of new edges associated to edges in $M_{A,B}$ determine $M'_{A',B'}$. The curves of $O(C')$ are those elements of $O(C)$ contained in C' which are incident to the boundary of R' (see Figure 8(b)). Finally, we add to $O(R')$ all elements in $O(R)$ that are incident to the boundary of R' as well as curves that match all the not-yet-covered new nodes (see Figures 8(a) and 8(b)).

We need to show that the new system $\mathcal{S}'$ is consistent. First, note that $S = S_1e_1Ee_2S_2$ is the only new segment created. In case (1), S is a line whose endnode in A_t is the corresponding node of S_1 (hence, by consistency condition (L1) of S_1, it is colored black) and whose other endnode in A_b is the corresponding one of S_2 (hence, by (E1) of S_2, it is colored white). It follows that the line S satisfies condition (L1). In case (2), S is a black-endnodes-A-ear with endnodes in A_t; and, in case (3), S is a white-endnodes-A-ear with endnodes in A_b. Thus, for these latter two cases, condition (E1) is satisfied.

We show next that S satisfies consistency condition (S1). Let e'_1 be the edge incident to the endnode of S_1 not covered by e_1. Similarly, let e'_2 be the edge incident to the endnode of S_2 not covered by e_2. Because consistent lines are bi-colored and consistent ears are monochromatic, we have that S is bicolored if and only if case (1) holds. Thus, to establish that (S1) holds in $\mathcal{S}'$ we need to show that $|\{e'_1, e'_2\} \cap M'_{A',B'}|$ is even if and only if case (1) holds. Since by condition (E1), the term $|\{e_1, e_2\} \cap M_{A,B}|$ is odd, we get that the parity of $|\{e'_1, e'_2\} \cap M'_{A',B'}|$ differs from the parity of $|\{e'_1, e_1\} \cap M_{A,B}| + |\{e_2, e'_2\} \cap M_{A,B}|$, which by consistency conditions (L1) and (E1) of $\mathcal{S}$ is odd if and only if case (1) occurs. This finishes the proof that $\mathcal{S}'$ satisfies (S1).

It is important to note that regarding B-ears and B-regular discs, we did not change the system. Hence, B-ears and open B-regular discs stay consistent.

Next, we show that A-ears and regions of $\mathcal{S}'$ are also consistent. First, let $\widetilde{E}$ be an A-ear of $\mathcal{S}'$ which does not involve the new segment S. Clearly, $\widetilde{E}$ is an A-ear in $\mathcal{S}$ as well. Observe that unless the disc $\widetilde{D}$ of the consistency condition (E2) of $\widetilde{E}$ intersects $A_b(E)$, condition (E2) is clearly satisfied. Thus, assume $\widetilde{D}$ intersects $A_b(E)$. Note that $|R(E)| + 1$ endpoints are removed from the interior of $\widetilde{D} \setminus A_b(E)$ and exactly the same number is added to the interior of $A_b(E)$. More precisely, $|R(E)|$ of the removed points are endpoints of elements of $O(R)$ incident to nodes between y_1 and y_2 and the remaining removed point is the endpoint of the element of $O(C)$ covering y_2. Similarly, $|R(E)| + 1$ of the added points belong to elements of $O(R')$ covering the nodes of the outer boundary of $A_b(E)$. The same reasoning establishes conditions (R1) and (R2) in $\mathcal{S}'$ when the region's boundary does not contain the new segment S.

We next argue that if S forms an A-ear, then (E2) holds for S in $\mathcal{S}'$. In case (1), no new ear is created. In case (3), the new segment S is a white-endnodes-A-ear and the interior of its open disc $D(S)$ consists of the interior of the open disc $D(S_1)$ of S_1, the interior of the open disc $D(S_2)$ of S_2, and the points on elements of $O(R')$ covering the nodes of the outer boundary of $A_b(E)$; we argued above that their number is even. Hence, by consistency of S_1 and S_2, segment S satisfies (E2). In case (2), the new segment S is a black-endnodes-A-ear. Because (L1) and (E2) hold for $\mathcal{S}$, the difference between the number of endpoints of elements of $O(R) \cup O(C) \cup M_{A,B}$ in its interior and in the open A-regular disc of $\mathcal{S}$ defined by S_1 and S_2 is exactly $|R(E)|$, which is odd. Hence, the new ear S also satisfies (E2).

It remains to check (R1) and (R2) for the regions which contain the new segment S as a line in their boundary. This situation only arises in case (1) when S_1 is a line and S_2 is a white-endnodes-A-ear in $\mathcal{S}$. Next, we determine the difference of the number of the inner points on each side of S with respect to S_1. On one side we add the even number of endpoints of elements of $O(R')$ covering the nodes of the outer boundary of $A_b(E)$. To the same side we add the points in the open disc $D(S_2)$ of S_2, whose number is even by (E2). On the other side we delete $|R(E)|$ (odd number) endpoints covered by elements of $O(R)$

incident with the interior of the loop interval between y_1 and y_2. We further delete an even number of endpoints in the interior of $D(S_2)$, and finally we delete exactly one endpoint in the interior of S_2 (the endpoint in the interior of R of the element of $O(C)$ incident to y_2). Summarizing, the aforementioned difference on each side of S with respect to S_1 is even and thus both conditions (R1) are (R2) are established. □

Summing up, if there is a counterexample to Theorem 16, then there is a regular counterexample $\mathcal{S} = (R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$. Assume $\mathcal{S}$ is such that $|A \cap C| = k$ and $|B \cap C| = l$. Let $\mathcal{S}' = (R', N', \{A', B'\}, M'_{A',B'}, C', O(C'), O(R'))$ be the system obtained by first concatenating l copies of $\mathcal{S}$ by the vertical boundaries of R, and then concatenating k copies of the resulting system by the horizontal boundaries of R. We claim that $\mathcal{S}'$ is a regular consistent toroidal system. Indeed, the regularity is clear since C' has exactly one edge of A' and exactly one edge of B'. Thus, conditions (E1) and (E2) vacuously hold for $\mathcal{S}'$. Moreover, conditions (S1) and (L1) are easily seen to be inhereted by $\mathcal{S}'$ from $\mathcal{S}$. Hence, the region consistency conditions (R1) and (R2) need to be checked and easily seen to follow from the consistency of $\mathcal{S}$.

If both k and l are odd, then the type of $M'_{A',B'}$ is the same as the type of $M_{A,B}$. Let exactly one of k, l, say k, be even. Furthermore, let $k > 0$. Since $\mathcal{S}$ is a counterexample to Theorem 16, it must be that $M_{A,B}$ has an odd number of edges of A and thence $M'_{A',B'}$ has an odd number of edges of A'. Clearly, $M'_{A',B'}$ will have an even number of edges of B'. Since C' has exactly one edge in each of A' and B', it follows that $\mathcal{S}'$ would also be a counterexample to Theorem 16. Summarizing, if there is a counterexample to Theorem 16, then there is a counterexample $\mathcal{S}$ with both $A \cap C$ and $B \cap C$ having at most one edge.

Proposition 18 *If $\mathcal{S} = (R, N, \{A, B\}, M_{A,B}, C, O(C), O(R))$ is a regular consistent toroidal system with both $A \cap C$ and $B \cap C$ consisting of at most one edge, then $\mathcal{S}$ satisfies Theorem 16.*

Proof First, we consider the case where $C \cap B = \emptyset$. Because C is not of type ee, it must be that $|C \cap A| = 1$. Let L be the only A-line. Let D be the open disc bounded by L, an interval of H_b, the vertical side of V_r, and an interval of H_t. By condition (R2), we have that D contains an even number of endpoints of elements of $O(R) \cup O(C) \cup M_{A,B}$. However, only endpoints of $O(R)$ belong to D, and therefore $O(R)$ covers an even number of nodes of the boundary of D. The remaining (even number) of nodes must be covered by $M_{A,B}$. Since each edge of $M_{A,B} \cap A$ covers an even number of nodes of the boundary of D, it follows that $|M_{A,B} \cap B|$ is even. Since $M_{A,B}$ can not be of type ee, we get that $M_{A,B}$ is of type oe, thus completing the analysis of the case being considered. The case that $C \cap B = \emptyset$ is analogous.

To conclude, let C have exactly one edge from both A and B. We note that $C \cap M_{A,B}$ may consist of zero, one or two edges. We consider in detail the case where $|C \cap M_{A,B}| = 2$. The remaining cases can be dealt with similarly. Let L be the only A-line of C. As in condition (R2) for L, we concatenate two copies of R by its vertical side. Let L' be the cover of L in the concatenated rectangle and let D be one of the two open discs obtained by subtracting L' from the interior of the concatenated copies of R. Let D' be the interior of the connected component of $R \setminus C$ containing opposite corners of R. By condition (R2) the disc D has an even number of endpoints of elements of $O(R)$. It follows that D' also has an even number of endpoints of elements of $O(R)$. Since D' has an even number of nodes in its boundary, we get that $M_{A,B}$ covers an even number of nodes of D''s boundary. However, each edge of $M_{A,B}$ is incident to a node in the boundary of D', so it follows that the cardinality of $M_{A,B}$ is even. Since $M_{A,B}$ is not of type *ee*, it must be of type *oo*. $\square$

5 Final comments

The proof argument for establishing Theorem 2 seems specially fit to handle the case where both m and n are even. If both were odd, the result is trivially true, since then $T_{m,n}$ does not have perfect matchings. It is somewhat puzzling that we could not find a simple adaptation of our proof argument for the case when exactly one of the integers m or n is even.

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