Recommended exercise problems for Set Theory, 2024/2025

1. lecture, 19. 2. 2025

1. Write "the set x is empty" (more precisely, "the set x has no elements") by a formula of the basic language of set theory.

2. lecture, 26. 2. 2025

2. Prove $((x \subseteq y) \wedge (y \subset z)) \to x \subset z$.

3. Write "the set x has exactly one element" by a formula of the basic language of set theory.

4. Prove that the "set of all sets" does not exist (using the separation schema).

5. Prove $(\forall z)(x \in z \leftrightarrow y \in z) \to x = y$ (using the pairing axiom).

6. Rewrite the triple (a, b, c) by unpacking the definition, using only the curly brackets.

7. Let a be an arbitrary set. What can be said about the sets
   (a) $\bigcup \mathcal{P}(a)$,
   (b) $\mathcal{P}(\bigcup a)$?

3. lecture, 5. 3. 2025

8. Show that the axiom of foundation (together with other axioms) forbids the existence of sets that are elements of themselves, and in general the existence of finite cycles in the membership relation (that is, sets $y_1, y_2, \ldots, y_n$ such that $y_1 \in y_2 \wedge y_2 \in y_3 \wedge \cdots \wedge y_{n-1} \in y_n \wedge y_n \in y_1$).

9. Assume that A, B are class variables representing class terms $\{x; \varphi(x)\}, \{y; \psi(y)\}$ where $\varphi(x), \psi(x)$ are formulas of the basic language of set theory. Rewrite the formula $A \subset B$ in the basic language of set theory.

10. Assume that a is a nonempty set. Is $\bigcap a$ necessarily a set?

11. Determine whether the universal class V satisfies $\mathcal{P}(\mathrm{V}) = \mathrm{V}$.

4. lecture, 12. 3. 2025

12. Show that for a class X it is not true in general that $X \times X^2 = X^3$ (choose for example $X = \{\emptyset\}$).

13. Verify that for an arbitrary relation R and the identity relation Id we have $R \circ \mathrm{Id} = \mathrm{Id} \circ R = R$.

14. Verify that for arbitrary sets x, y and the membership relation E we have

$$(x, y) \in \mathrm{E} \circ \mathrm{E} \leftrightarrow x \in \bigcup y.$$

15. Verify that for every nonempty set x and a nonempty class Y we have

$$\bigcup\bigcup\bigcup {}^{x}Y = x \cup Y.$$

5. lecture, 19. 3. 2025

16. Write the definition "an ordering R is a *well-ordering* on a class A" as a formula (using defined shortcuts and extensions of the language).

17. Find some sets x on which the membership relation is a strict ordering, and moreover the ordering $\mathrm{E} \cup \mathrm{Id}$ is a well-ordering on x.

6. lecture, 26. 3. 2025

18. Assume that the power set $\mathcal{P}(x)$ of x is ordered by the inclusion relation $\subseteq$. Verify that for every subset $A \subseteq \mathcal{P}(x)$ we have $\sup_{\subseteq} A = \bigcup A$.

19. Try to think about a "graph-theoretic" proof of the statement that the existence of a map f of a set x onto y and a map g of a set y onto x implies the existence of a bijection between x and y. You may assume that you have natural numbers with arithmetic available. Why can't the desired bijection be expressed as a set in general?

20. Prove that every nondecreasing function from $[0, 1]$ to $[0, 1]$ (not necessarily continuous) has a fixed point. You can follow the proof of the fixed-point lemma about monotone mappings on a power set.

21. Verify that the function $g : \omega \times \omega \to \omega$ defined as $g((m, n)) = 2^m \cdot 3^n$ is injective, and that the function $h : \omega \times \omega \to \omega$ defined as $h((m, n)) = 2^m \cdot (2n + 1) - 1$ is a bijection between $\omega \times \omega$ and ω. (By the symbol ω we denote the set $\{0, 1, 2, \ldots\}$. So far we use the natural numbers and operations with them only intuitively.)

22. Using the Cantor–Bernstein theorem for "intuitive" notions of the sets $\mathbb{N}$ and $\mathbb{Q}$ prove that $\mathbb{N} \approx \mathbb{Q}$. (We can represent the rational numbers for example as fractions in their basic form).

23. Using the Cantor–Bernstein theorem prove that the segment $[0, 1]$ and the square $[0, 1] \times [0, 1]$ have the same cardinality. Use for example the decadic expansion of the real numbers (considered intuitively).

24. Try to define "x is a finite set" (by at least two different ways; for example using orderings or mappings).

7. lecture, 2. 4. 2025

25. Write a formula for $\mathrm{Fin}(x)$ using Tarski's definition of finiteness (every nonempty subset of $\mathcal{P}(x)$ has a maximal element with respect to inclusion).

26. Verify that in every linearly ordered set the lower sets form a chain with respect to inclusion.

27. Let R be a linear ordering on a set A, S a linear ordering on a set B, and let F, G be initial embeddings from A to B with respect to R and S. Is it necessarily true that $F \subseteq G$ or $G \subseteq F$?

28. Verify that if R is a class of mappings linearly ordered by inclusion (so it forms a chain with respect to $\subseteq$), then $\bigcup R$ is again a mapping.

29. Verify that in every ordered set the union of an arbitrary system of lower sets is again a lower set.

8. lecture, 9. 4. 2025

30. Verify the uniqueness in the theorem about comparing well-orderings: that is, if A and B are well-ordered sets, then the mapping F that is an isomorphism of A and a lower set in B or an isomorphism of a lower set in A and B is unique.

31. Prove that for every finite set x and every mapping f with domain x we have $\mathrm{Rng}(f) \preccurlyeq x$. You can use the induction principle for finite sets.

32. Prove that every finite set can be well-ordered. Again, you can proceed by induction for finite sets.

9. lecture, 16. 4. 2025

33. Prove that the set ω of all natural numbers (and in general every inductive set) is dedekind-infinite. (It is sufficient to show that the successor function is injective and its range is not the whole ω.)