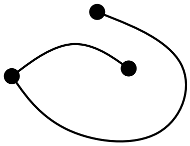
Counterexamples to the thrackle conjecture on higher genus surfaces

César Hernández-Vélez, Jan Kynčl,
Gelasio Salazar and Kebin Velasquez

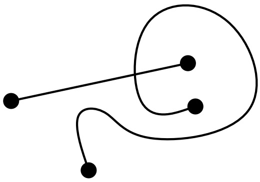
Charles University, Prague

Thrackles

Thrackle: a drawing of a graph where every pair of edges have **exactly** one point in common: an endpoint or a crossing



or



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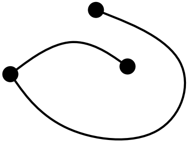

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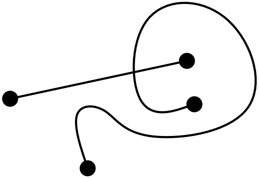
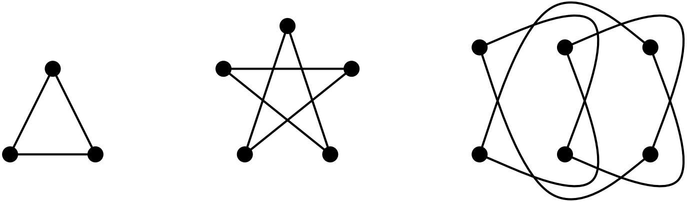


Examples:



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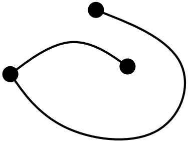

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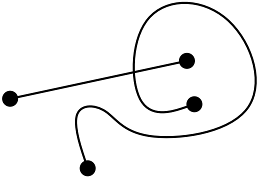

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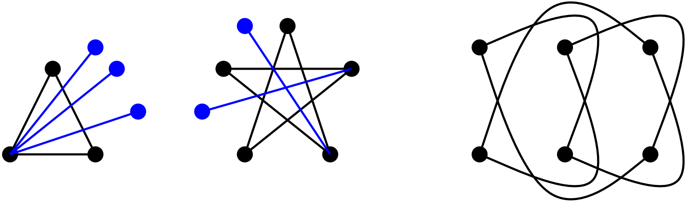
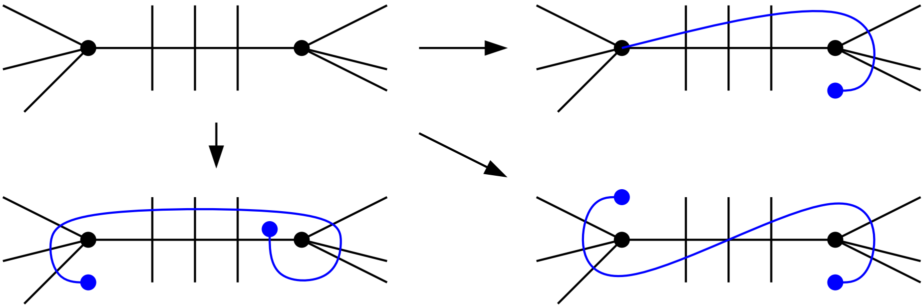


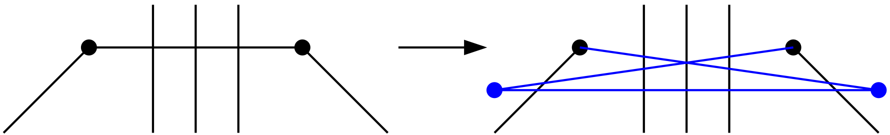
New thrackles from old

Adding an edge:



New thrackles from old

Subdividing a path:



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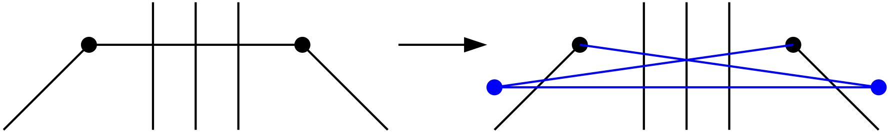
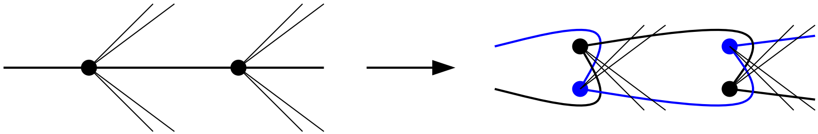


Conway's doubling:



Conway's conjecture

Let G be a graph with n vertices and m edges that has a thrackle drawing in the plane.

Conjecture: (Conway) $m \leq n$.

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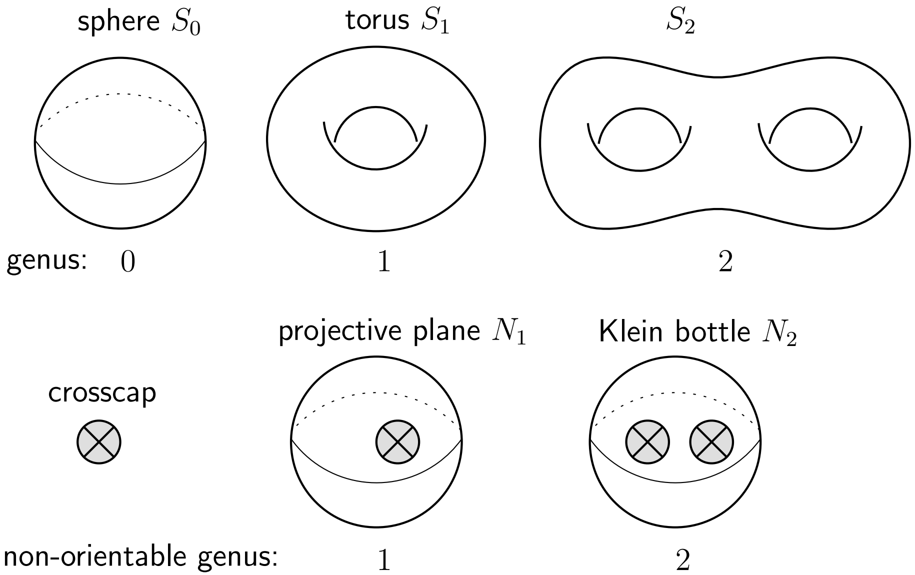
Conway's conjecture is known in special cases:

- geometric thrackles (exercise)
- monotone thrackles (Pach–Sterling, 2011)
- outerplanar thrackles (Cairns–Nikolayevsky, 2012)
- annular and pants thrackles (Misereh–Nikolayevsky, 2018)

Surfaces

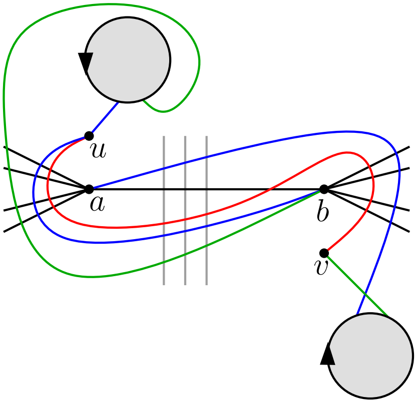


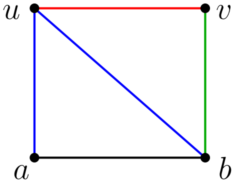
sphere S_0
torus S_1
S_2
genus: 0
1
2
projective plane N_1
Klein bottle N_2
crosscap
non-orientable genus: 1
2

Thrackles on surfaces

Cairns–Nikolayevsky, 2000

adding 1 handle, 2 vertices and 4 edges:





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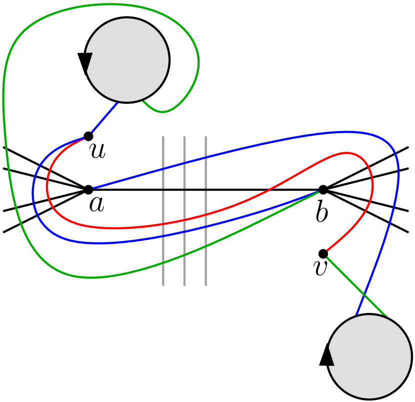

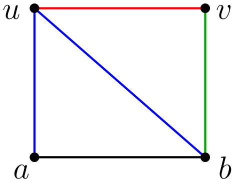

Let G be a graph with n vertices and m edges that has a thrackle drawing on S_g.

Conjecture: (Cairns–Nikolayevsky, 2000) $m \leq n + 2g$.

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Theorem: (Cairns–Nikolayevsky, 2000) If G is **bipartite** with k components with at least three vertices, then $m \leq 2n - 4k + 4g$.

Theorem: (Cairns–Nikolayevsky, 2009) If G is **connected**, then $m \leq 2n - 2 + 4g$.

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Theorem: If G has k components with at least three vertices, then $m \leq 2n - 4k + 4g + 2$. If G has instead a thrackle drawing on N_g, then $m \leq 2n - 4k + 2g + 2$.

(using an equivalence between generalized thrackles and X-parity embeddings (Pelsmajer–Schaefer–Štefankovič, 2009))

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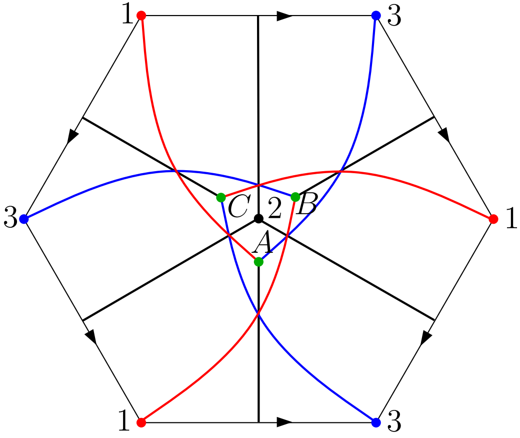
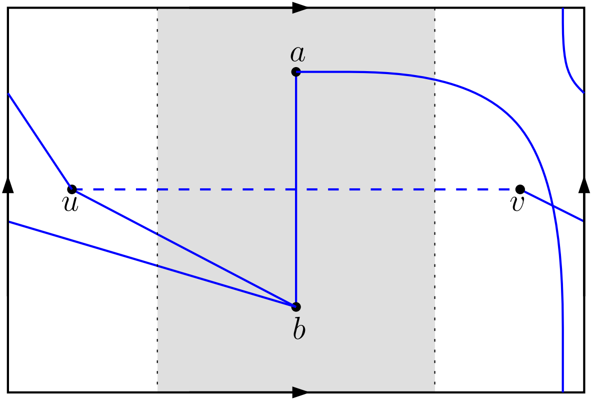
Theorem: (Cairns–McIntyre–Nikolayevsky, 2004)
The conjecture is true for $K_{3,3}$ and K_5.
In particular, $K_{3,3}$ has no thrackle drawing on the torus.

Counterexamples / lower bounds



Cairns–Nikolayevsky's toroidal iterator



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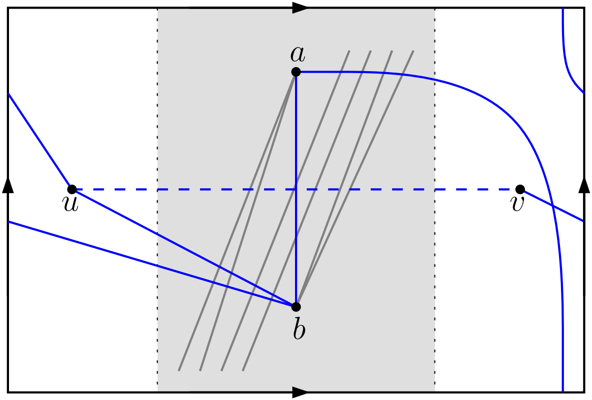
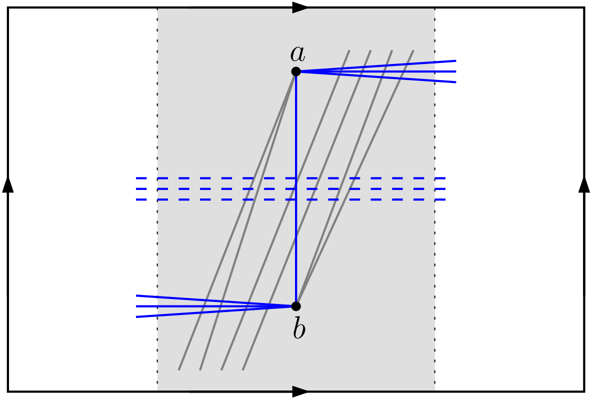
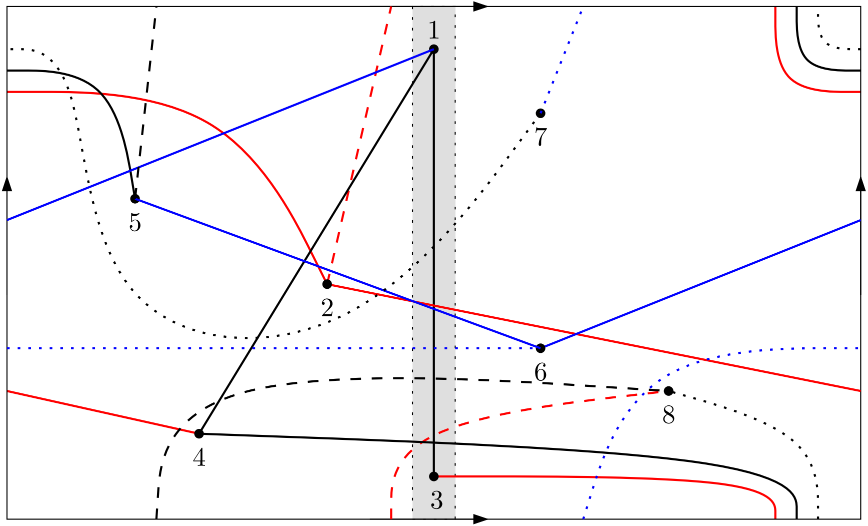


General toroidal iterator



Toroidal iterator with $n = 8$ and $m = 11$



adds 1 handle, 6 vertices and 10 edges

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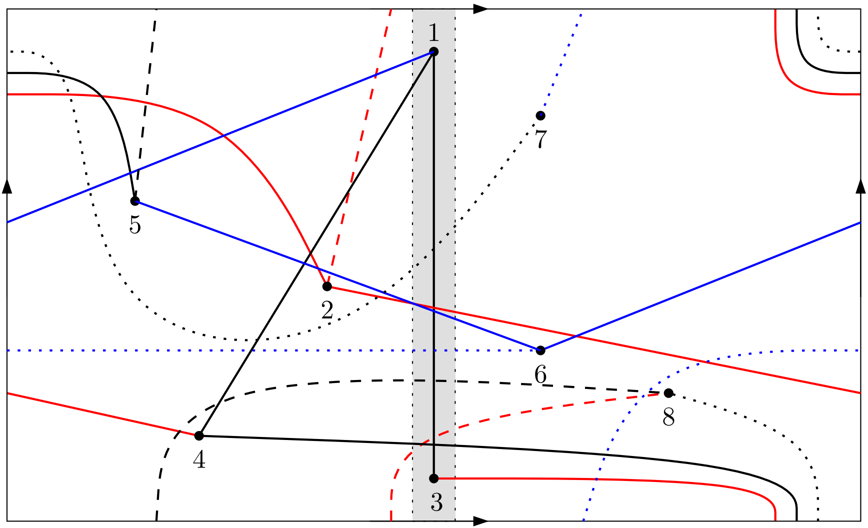
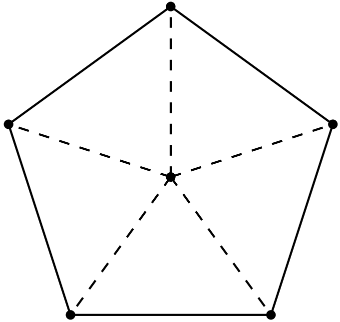
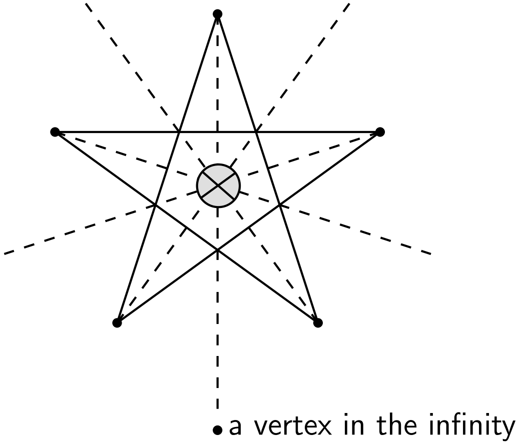


adds 1 handle, 6 vertices and 10 edges

creates thrackles on S_g with $m = n + 4g$.

Thrackled W_5 on the projective plane





In general, W_{2k+1} has $n = 2k + 1$ and $m = 2n - 2$.

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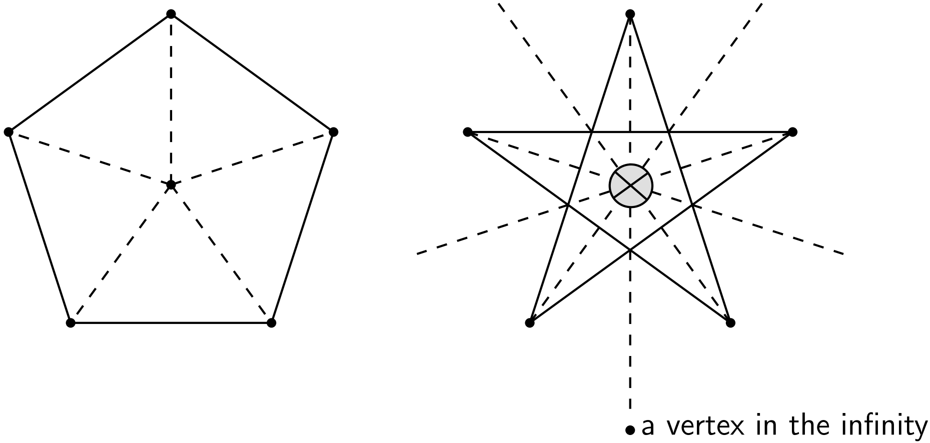
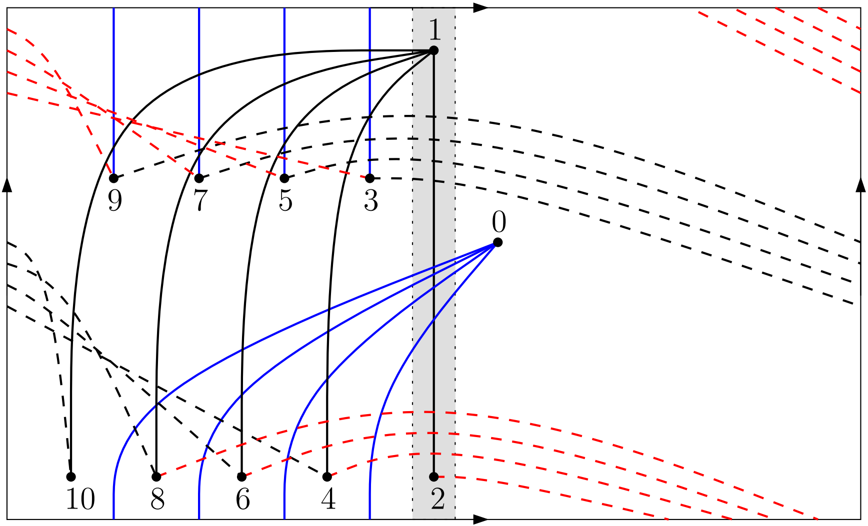


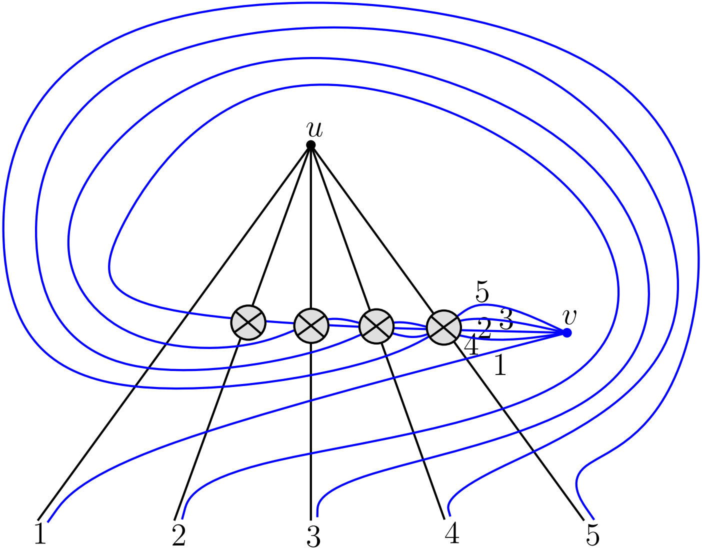
In general, W_{2k+1} has $n = 2k + 1$ and $m = 2n - 2$.

(without the crosscap: Figure 12 in Cairns–Nikolayevsky, 2000)

Thrackled sparse double-fan on the torus, with $m = 2n - 5$



Cloning a star, nonorientable surfaces



adding g crosscaps, 1 vertex and $g+1$ edges

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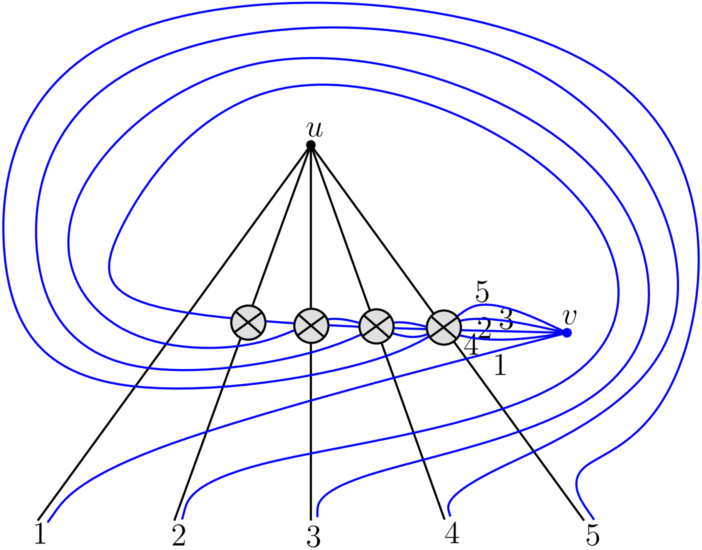

adding g crosscaps, 1 vertex and $g+1$ edges
creates thrackles on N_g with $m = 2n + g - 4$.

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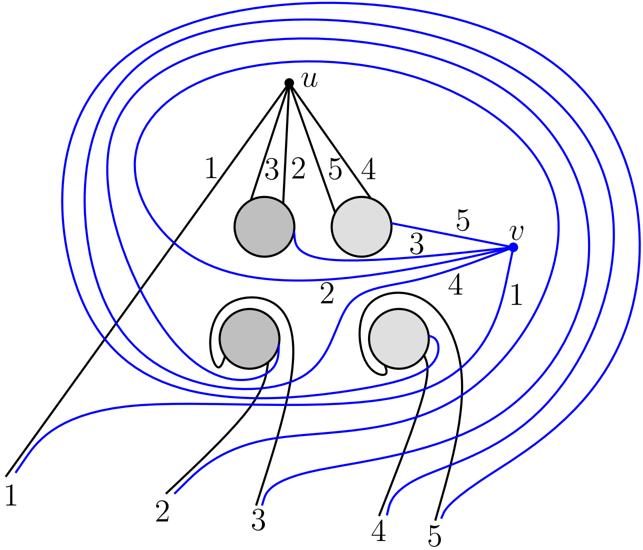

adding g handles, 1 vertex and $2g+1$ edges

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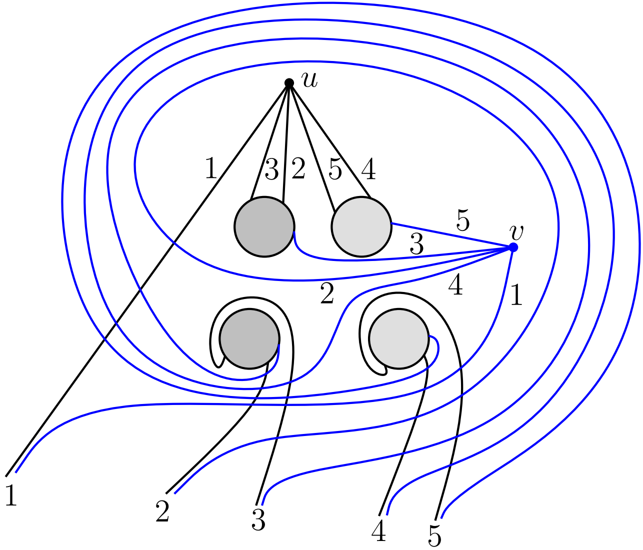

adding g handles, 1 vertex and $2g + 1$ edges
creates thrackles on S_g with $m = 2n + 2g - 8$.