Exercise problems for Infinite sets
WS 2019/2020

1. lecture, 3. 10. 2019

1. Prove that every lower set in $(\mathrm{On}, <)$ is an ordinal number.
2. Is every well-ordered proper class isomorphic to $(\mathrm{On}, <)$?

2. lecture, 10. 10. 2019

3. Verify that in every ordered set, the union of a system of arbitrarily many lower sets is a lower set.
4. Prove by transfinite recursion that whenever $(A, \prec)$ is a well-ordered proper class such that for every $x \in A$ the lower part $(\leftarrow, x)$ is a set, then $(A, \prec)$ is isomorphic to $(\mathrm{On}, <)$.
5. Derive the well-ordering principle from the axiom of choice.
6. Derive the well-ordering principle from the maximality principle. (Consider well-orderings on subsets and compare them by the relation of "extension".)

3. lecture, 17. 10. 2019

7. Let F be a normal ordinal function. Prove that the preimage of a closed class is closed in $\mathrm{Dom}(F)$; that is, it has the form $X \cap \mathrm{Dom}(F)$ where X is a closed class.
8. Let F be a normal ordinal function and $\lambda \in \mathrm{Dom}(F)$ a limit ordinal. Prove that $F(\lambda)$ is also a limit ordinal.
9. Verify that the class of all fixed points of an ordinal function defined on On is closed.
10. Verify that for every ordinal α the function $F(\xi) = \alpha + \xi$ is continuous.

4. lecture, 24. 10. 2019

11. * Prove the existence of an uncountable ordinal in ZF (without the axiom of choice).
12. Assign an ordinal to each hydra in such a way that after cutting a head by Hercules and regeneration in round n, the ordinal assigned to the regenerated hydra is strictly smaller than the ordinal asigned to the original hydra.

5. lecture, 31. 10. 2019

13. Compute the following functions of the Hardy hierarchy:

    (a) $H_{\omega+\omega}(n) =$

    (b) $H_{\omega\cdot\omega}(n) =$

    (c) $H_{\omega^3}(n) =$

6. lecture, 7. 11. 2019

14. Prove that for every countable limit ordinal α we have $\mathrm{cf}(\alpha) = \omega$.

15. Prove that for every limit ordinal α we have $\mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha)$.

7. lecture, 14. 11. 2019

16. Prove that in ZFC, for every ordinal α the sum of at most $\aleph_\alpha$ sets of cardinality at most $\aleph_\alpha$ has cardinality at most $\aleph_\alpha$. (Define an injective mapping of the union to $\omega_\alpha \times \omega_\alpha$, similarly as in the countable case.)

17. Prove that every infinite singular cardinal κ is the supremum of $\mathrm{cf}(\kappa)$ regular cardinals.

18. Verify by the definition of the cardinal power that $\kappa^{\mu+\nu} = \kappa^\mu \cdot \kappa^\nu$ and $(\kappa^\mu)^\nu = \kappa^{\mu\cdot\nu}$.

9. lecture, 5. 12. 2019

19. Verify that every chain in a tree is a well-ordered set.

20. Derive König's theorem about trees from the maximality principle (without using recursion).

10. lecture, 12. 12. 2019

21. Derive infinite Hall's theorem using the compactness principle.

22. Is a variant of the infinite Hall's theorem for systems of countable sets true?

23. Show that in every finite partition $\mathbb{Q} = X_1 \cup X_2 \cup \cdots \cup X_n$ of the rationals some of the sets X_i contains an isomorphic copy of $\mathbb{Q}$ with respect to the ordering $\leq$. (It is sufficient to show that X_i is dense in some nonempty open interval).

11. lecture, 19. 12. 2019

24. Using infinite Ramsey's theorem show that every infinite ordered set has an infinite chain or an infinite antichain.

25. Using infinite Ramsey's theorem show that every infinite linearly ordered set has an infinite increasing or decreasing sequence (a subset with order type ω nebo ω^*).

26. Show that every infinite cardinal κ satisfies $2^\kappa \nrightarrow (3)^2_\kappa$.

27. Using general Sierpinski's theorem show that every weakly compact cardinal is inaccessible.

28. By a suitable coloring of finite subsets of ω show that ω is not Ramsey cardinal.

12. lecture, 9. 1. 2020

29. Show that for every $n \in \mathbb{N}$ the unit disc can be partitioned into $n+2$ parts and the parts can be reassembled into a unit disc and n copies of the half-open unit segment $(0,1]$.

30. Using the statements and theorems from the lecture show that (in ZFC):

    (a) Every two balls A, B (of different sizes) in $\mathbb{R}^3$ are equidecomposable using a finite number of parts (that is, there is an $n \in \mathbb{N}$ such that $A \overset{n}{\cong} B$).

    (b) If A, B are bounded subsets of $\mathbb{R}^3$ with a nonempty interior, then there is an $n \in \mathbb{N}$ such that $A \overset{n}{\cong} B$.