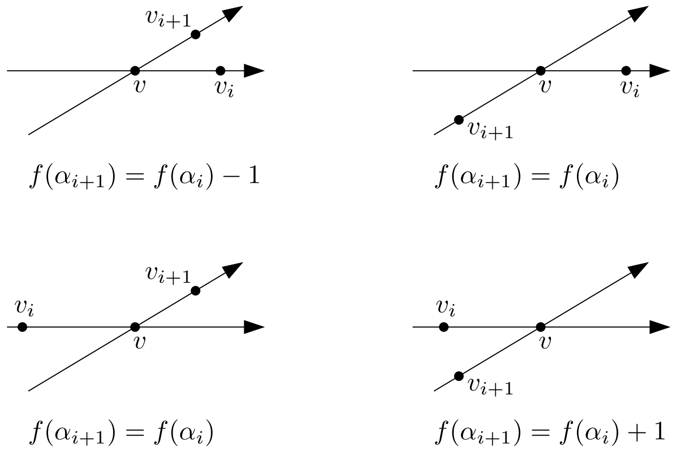

Figure 1: The four possibilities for the change of the value of the function f.

If P is a set of n points in the plane in general position, a segment s connecting two points of P is a *halving segment* if each open halfplane determined by s contains $\lfloor (n-2)/2 \rfloor$ or $\lceil (n-2)/2 \rceil$ points of P.

1. Let P be a set of an even number of points in the plane in general position, and let v be a point in P. Show that the halving segments and their reflections through v alternate around v. That is, between any pair of halving segments starting at v, there is a reflection of another halving segment starting at v.

   Solution: For $\alpha \in \mathbb{R}$, let $r(\alpha)$ be the ray starting at v with direction α (measured counter-clockwise). Let $h(\alpha)$ be the oriented line extending $r(\alpha)$. Note that $r(\alpha) = r(\alpha + 2\pi)$ and $h(\alpha) = h(\alpha + 2\pi)$ for every α. Let $f(\alpha)$ be the number of points of P in the left open half-plane determined by $h(\alpha)$. The value of f changes every time the line $h(\alpha)$ meets a point of $P \setminus \{v\}$. Assume that the number of points in P is $2k+2$. Let $\alpha_0, \alpha_1, \ldots, \alpha_{4k+1}$ be those angles such that every line $h(\alpha_i)$ contains a point of $P \setminus \{v\}$ and $0 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{4k+1} < 2\pi$. Let v_i be the point such that $h(\alpha_i) \cap P = \{v, v_i\}$. Observe that for every $i = 0, 1, \ldots, 2k$, we have $\alpha_{2k+1+i} = \alpha_i + \pi$ and $v_{2k+1+i} = v_i$.

   The cyclic sequence $f(\alpha_0), f(\alpha_1), \ldots, f(\alpha_{4k+1})$ satisfies $|f(\alpha_{i+1})| - |f(\alpha_i)| \in \{-1, 0, 1\}$ for every i (the indices are taken modulo $4k+2$). More precisely, there are four basic cases that depend on which "side" of v the point v_i is. See Figure 1.

   a) $v_i \in r(\alpha_i)$ and $v_{i+1} \in r(\alpha_{i+1})$. In this case, $f(\alpha_{i+1}) = f(\alpha_i) - 1$.
   b) $v_i \in r(\alpha_i)$ and $v_{i+1} \notin r(\alpha_{i+1})$. In this case, $f(\alpha_{i+1}) = f(\alpha_i)$.
   c) $v_i \notin r(\alpha_i)$ and $v_{i+1} \in r(\alpha_{i+1})$. In this case, $f(\alpha_{i+1}) = f(\alpha_i)$.
   d) $v_i \notin r(\alpha_i)$ and $v_{i+1} \notin r(\alpha_{i+1})$. In this case, $f(\alpha_{i+1}) = f(\alpha_i) + 1$.

Let vx and vy be two consecutive halving segments of P starting at v. That is, there exist $i, j \in \{0, 1, \ldots, 4k+1\}$ such that $x = v_i \in r(\alpha_i)$, $y = v_j \in r(\alpha_j)$, $j - i \leq 2k$, $f(\alpha_i) = f(\alpha_j) = k$, and no other $l \in \{i+1, i+2, \ldots, j-1\}$ satisfies $v_l \in r(\alpha_l)$ and $f(\alpha_l) = k$. We need to show that there exists $l \in \{i+i, i+2, \ldots, j-1\}$ such that $v_l \notin r(\alpha_l)$ and $f(\alpha_l) = k$.

Let $q \in \{1, 2, \ldots, j-i\}$ be the smallest number such that $f(\alpha_{i+q}) = k$; that is, $h(\alpha_{i+q})$ is a halving line. If $q = 1$, then $f(\alpha_{i+1}) = f(\alpha_i)$ and $v_i \in r(\alpha_i)$, so we are in case b), and thus $v_{i+1} \notin r(\alpha_{i+1})$. We can therefore take $l = i+1$.

Otherwise, for v_i and v_{i+1} we are in case a), and so $f(\alpha_{i+t}) < k$ for every $t \in \{1, 2, \ldots, q-1\}$. This implies that $f(\alpha_{i+q-1}) + 1 = f(\alpha_{i+q}) = k$, so v_{i+q-1} and v_{i+q} satisfy case d). In particular, $v_{i+q} \notin r(\alpha_{i+q})$ and thus $i + q < j$. Therefore, we can take $l = i + q$.

In both cases, vv_{i+q} is the halving segment we were looking for.

2. Suppose that n is even. Let G be the geometric graph determined by the halving segments of n points in the plane in general position. Let c be the number of crossing in G. Prove that

$$c + \sum_{v \in V(G)} \binom{(d(v)+1)/2}{2} = \binom{n/2}{2}.$$

Advice: start with n points in convex position and move them one by one to the vertices of G.

Solution: First observe that if the points are in convex position, then each point is incident to exactly one halving segment. So we have $n/2$ halving segments and every pair of them cross. Thus $k = \binom{n}{2}$ and each of the summands $\binom{(d(v)+1)/2}{2}$ is equal to $\binom{1}{2} = 0$.

Let P be the vertex set of G. By moving the points continuously one by one from the convex position, we want to achieve the configuration P. We show that during this process, the identity will still hold. The graph of the halving segments changes only when three points p_i, p_j, p_k become collinear at one moment and the orientation of their triangle changes. See Figure 2. During this event, the graph of the halving segments can change only within the triangle $p_i p_j p_k$; no halving segment incident with other vertex is affected. Assume that p_j lies in the segment $p_i p_k$ when the points p_i, p_j, p_k become collinear. Then there are three possibilities:

a) no segment in the triangle $p_i p_j p_k$ is halving before the collinearity,

b) $p_i p_k$ is a halving segment before the collinearity,

c) $p_i p_j$ and $p_j p_k$ are halving segments before the collinearity.

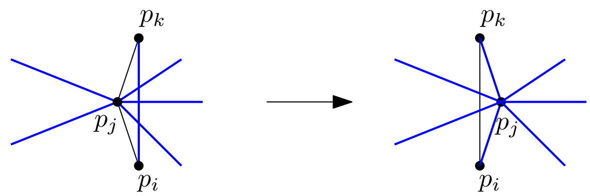

Figure 2: An elementary change in the graph of halving segments during a continuous motion of the points; case b).

In case a), the graph of halving segments does not change during the collinearity event.

In case b), p_ip_j and p_jp_k are not halving segments before the collinearity. After passing through the collinearity, both p_ip_j and p_jp_k become halving segments, and p_ip_j will cease to be halving. The degrees of all vertices but p_j in the halving graph do not change, and the degree of p_j increases by 2.

Now we investigate the change in the crossings of the halving graph. Every change in crossings must involve an edge of the triangle $p_ip_jp_k$. First, every crossing of halving edges xy and p_ip_k, where $x, y \neq p_j$, before the collinearity is replaced by a crossing of xy with p_ip_j or p_jp_k after the collinearity. So the only difference in the number of crossings is due to the edges incident to p_j.

Consider the configuration after the collinearity and draw a line h through p_j parallel to p_ip_k. Suppose there are a halving segments incident to p_j lying in the same open halfplane determined by h as p_i and p_k, and b halving segments in the opposite halfplane. The number of crossings between halving segments thus decreases by b during the collinearity event.

An easy consequence of problem 4 is that $a = b + 1$. Therefore, the degree of p_j is $a+b-2 = 2b-1$ before the collinearity and $a+b = 2b+1$ after the collinearity. The term $\binom{(d(p_j)+1)/2}{2}$ thus changes from $\binom{b}{2}$ to $\binom{b+1}{2}$ during the event, and the resulting difference in the sum is $\binom{b+1}{2} - \binom{b}{2} = b$. Therefore, the changes in the left-hand side of the identity cancel out.

Finally, we observe that case c) is just case b) played backwards.

Remark: The original version of this proof appeared in the paper "Results on k-sets and j-facets via continuous motion" by Andrzejak et al. You may find it, for example, at the following addresses: `http://dl.acm.org/citation.cfm?id=276906` or `http://sarielhp.org/research/papers/msc_phd/archive/kset.ps`.