Exercises for Combinatorial and Computational Geometry 2

Series 3 — Intersection patterns of convex sets

deadline 22. 4. 2025

1. (Carathéodory's theorem, cone version.)
   Let $A \subseteq \mathbb{R}^d$ and $x \in \operatorname{cone}(A)$; that is, x can be expressed as $x = \sum_{i\in[n]} \alpha_i a_i$ for some $n \in \mathbb{N}$, $a_i \in A$ and $\alpha_i \geq 0$. Prove that there is a subset $B \subseteq A$ of at most d points such that $x \in \operatorname{cone}(B)$. **[2]**

2. (Colorful Carathéodory's theorem, cone version.)
   Let $x \in \mathbb{R}^d \setminus \{0\}$ and $A_1, \dots, A_d \subseteq \mathbb{R}^d$ such that for every $i \in [d]$, we have $x \in \operatorname{cone}(A_i)$. Prove that there are points $a_1, \dots, a_d$ with $a_i \in A_i$ such that $x \in \operatorname{cone}(\{a_1, \dots, a_d\})$. **[2]**

3. (Colorful Carathéodory's theorem, stronger version in the plane.)
   Let X_1, X_2, X_3 be finite point sets in the plane such that for every $i, j \in \{1, 2, 3\}, i \neq j$, we have $0 \in \operatorname{conv}(X_i \cup X_j)$. Prove that there are points x_1, x_2, x_3 with $x_i \in X_i$ such that $0 \in \operatorname{conv}(\{x_1, x_2, x_3\})$. **[3]**

4. (Helly's theorem, colored version.)
   Let $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every choice $C_i \in \mathcal{C}_i$, $i \in [d+1]$, we have $\bigcap_{i\in[d+1]} C_i \neq \emptyset$. Prove that there exists $i \in [d+1]$ such that $\bigcap \mathcal{C}_i \neq \emptyset$ (that is, there is a point common to all sets of certain color). Hint: consider lexicographic minima of the intersections and show that one of them is the desired point. **[3]**

5. Prove that the quantity $(r-1)(d+1)+1$ in Tverberg's theorem cannot be improved: for every $d, r \geq 2$, there are $(r-1)(d+1)$ points in $\mathbb{R}^d$ with no Tverberg r-partition. **[2]**

6. (a) Let S be a set of n points in the plane in general position and let $x \in \operatorname{conv}(S)$. Prove that there are at least $n-2$ triangles with vertices in S that contain x in their convex hull. **[2]**

   (b) Let S be a set of n points in the plane in general position and let $x \in \mathbb{R}^2$ be an arbitrary point. Prove that the number of triangles with vertices in S that contain x in their convex hull is at most $n^3/24 + O(n^2)$. Hint: count the triangles that do not contain x. **[3]**
