

Exercises for Combinatorial and Computational Geometry

Series 3 — Crossing numbers and incidences

deadline 10:40 28.11.2025

1. Prove that a graph with n vertices that has a rectilinear drawing in the plane with no three pairwise crossing edges has $O(n^{3/2})$ edges. You may use the crossing lemma. (A *rectilinear drawing* is a drawing where every edge is drawn as a straight-line segment.) [2]
2. Let $I_{\text{circ}}(n, m)$ be the maximum number of incidences of n points and m unit circles in the plane. Show that $I_{\text{circ}}(n, n) = O(n^{4/3})$. [3]
3. Let $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ be a system of subsets of an n -element set N (that is, $\forall i \in [n] M_i \subseteq N$) such that every pair of sets M_i, M_j has at most one common element. The number of incidences of N and \mathcal{M} is defined as $I(N, \mathcal{M}) := \sum_{i=1}^n |M_i|$. Determine whether necessarily $I(N, \mathcal{M}) = O(n^{4/3})$. [2]
4. Find an n -point set in \mathbb{R}^4 with $\Omega(n^2)$ unit distances. [3]
5. Let P be an n -point set in the plane.
 - (a) Let $k > 1$. Show that there are at most $O(n^2/k^3 + n/k)$ lines such that each of them contains at least k points of P , and that the number of incidences of these lines with P is at most $O(n^2/k^2 + n)$. [3]
 - (b) Let $\alpha \in (0, \pi)$. Show that P determines at most $O(n^{7/3})$ triangles with at least one angle of size α . (Hint: split the triangles ABC with angle α at A into two groups according to whether the line AC contains more than $n^{1/3}$ points of P .) [3]