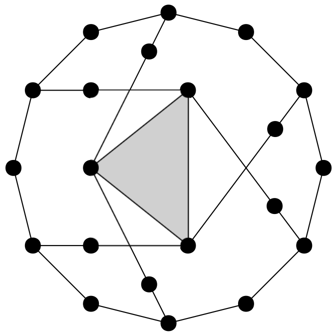
Exercises for Combinatorial and Computational Geometry 2

Series 4 — d-representability, d-collapsibility and triangulations

deadline 13. 5. 2025

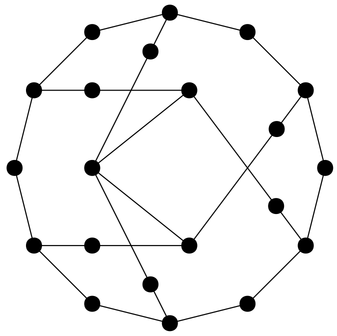
1. Determine whether the following simplicial complexes are 2-representable.

a)



[1]

b)



[2]

2. Let H be a d-collapsible simplicial complex with n vertices such that every $(d+1)$-tuple of vertices forms a face in H. Prove that H is a *combinatorial* $(n-1)$*-simplex*, that is, every nonempty subset of vertices forms a face of H. **[2]**

3. Without using Wegner's theorem prove that the boundary of the $(d+1)$-dimensional crosspolytope is not d-representable. (You can use the hyperplane separation theorem.) **[2]**

4. Finish the proof of the optimal version of the fractional Helly theorem from the lecture. That is, prove that for every $d \geq 1$, $n \geq d+1$ and $\alpha \in (0,1]$ the following statement is true: Let $\mathcal{F}$ be a system of n convex sets in $\mathbb{R}^d$ such that at least $\alpha \cdot \binom{n}{d+1}$ of (unordered) $(d+1)$-tuples of sets of $\mathcal{F}$ have a nonempty intersection. Then at least $\beta \cdot n$ sets from $\mathcal{F}$ have a nonempty common intersection where $\beta = 1-(1-\alpha)^{1/(d+1)}$.

   Use the following proposition proved during the lecture: Let $d \geq 1$, $n \geq d+1$ and $r \geq 0$. Let $\mathcal{F}$ be a system of n convex sets in $\mathbb{R}^d$ such that every $(d+r+1)$-tuple of sets of $\mathcal{F}$ has an empty intersection. Then the number of $(d+1)$-tuples of sets of $\mathcal{F}$ with empty intersection is at least $\binom{n-r}{d+1}$. **[2]**

5. Let K be a d-collapsible simplicial complex. Show that there exists a sequence of elementary d-collapses starting with K and ending with a simplicial complex of dimension smaller than d such that every elementary d-collapse in this sequence is determined by a free face of dimension exactly $d-1$ and removes at least two faces (in other words, the free face is not maximal). **[3]**

6. Let P be a finite set of points in the plane, with no three on the same line and no four on the same circle. We define a graph DT on P (called *Delaunay triangulation*): two points a, b are joined by an edge if and only if there exists a disc having a and b on its boundary and no other point of P in the interior.

   Let T be a minimum-weight spanning tree in the complete graph on P where the weights of the edges are their Euclidean lengths. Prove that $T \subseteq DT$. **[2]**