Exercises for Combinatorial and Computational Geometry 2

Series 2 — k-holes, halving lines and graph drawings

deadline 31.3.2023

1. Let X be a finite non-empty set of points in the plane in general position. Prove that the following identities hold for the numbers of k-holes in X:

   (a) $\sum_{k=1}^{|X|}(-1)^k \cdot \#k\text{-holes} = -1,$ **[2]**

   (b) If $|X| \geq 2$, then
   $$\sum_{k=1}^{|X|}(-1)^k \cdot k \cdot \#k\text{-holes} = -\#\text{points inside } \operatorname{conv}(X).$$
   [2]

   Hint: move points continuously along curves into a suitable configuration.

2. Let X be a set of n points in the plane in general position. Prove that the number of 4-holes is at least quadratic in n. **[2]**

3. Let P be a finite set of points in the plane that is not necessarily in general position and that contains no 5-hole. Prove that every convex 5-gon Q determined by points of P contains at least 1 point of P in the closed "inner" 5-gon that is determined by the diagonals of Q. **[2]**

4. For n even, let P be a set of n points in the plane in general position. Furthermore, let $k \leq n/2$ and let h be a line that does not intersect P and splits the plane into two halfplanes such that one of them contains exactly k points of P. Show that h intersects exactly k halving segments of P. **[2]**

5. Let P be a set of n points in the plane in general position. A pair of points in P is a *k-edge* if the line determined by these points separates exactly k points from P in one of the open hyperplanes. Let $E_k(P)$ be the number of k-edges in P and furthermore let $\overline{cr}(P)$ be the number of unordered 4-tuples of points in P in convex position.

   Prove the following identity:
   $$\overline{cr}(P) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} E_k(P) \cdot k \cdot (n-k-2).$$

   Hint: count in two ways the number of ordered triples $(a, \overline{bc}, d)$ such that a, b, c, d are pairwise different points of P, a lies to the left of the line $\overline{bc}$ and d lies to the right of the line of $\overline{bc}$. **[2]**

6. Two edges in a graph are *independent* if they do not share a vertex. In a *drawing* of a graph we assume that edges have only finitely many points in common, and that every common point of two edges is either their common endpoint or a crossing.

   Determine for which values of m and n, in every drawing of the graph $K_{m,n}$ the total number of crossings of independent pairs of edges is odd. Hint: what happens during a continuous deformation of the edges? **[3]**

web: `https://kam.mff.cuni.cz/~kvgweb/kvg/eng.html`