Exercises for Combinatorial and Computational Geometry

Series 6 — bonus problems

deadline: 2. 2. 2023

Please submit the solutions of problems 1,2,3 separately from problems 4,5; each subset will be handled by a different corrector.

1. Let $\mathcal{C}$ be the set of all cells (faces of dimension 2) in an arrangement of n lines in the plane. We denote the number of vertices of a cell C by $f_0(C)$. Prove that $\sum_{C\in\mathcal{C}} f_0(C)^2 = O(n^2)$. **[2]**

2. Let S be a set of n geometric objects in the plane. The *intersection graph of S* is a graph on n vertices that correspond to the objects in S. Two vertices are connected by an edge if and only if the corresponding objects intersect.

   (a) The total number of all graphs on n given vertices is $2^{\binom{n}{2}} = 2^{n^2/2+O(n)}$. Prove that the total number of all intersection graphs of n line segments in the plane is only $2^{O(n\log n)}$. (Be careful and consider also collinear line segments!) Use the theorem about the number of sign patterns. **[3]**

   (b) Show that the number of intersection graphs of n simple curves in the plane is at least $2^{\Omega(n^2)}$. If you wish, you can solve this exercise for n convex sets instead of simple curves. **[2]**

3. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of n points in the plane. We say that points x, y have the *same view* of P if the points of P are visible in the same cyclic order from x and y. That is, if we rotate light rays that emanate from x and y, respectively, the points of P are lit in the same order by these rays. We assume that neither x nor y is in P and that neither of them can see two points of P in occlusion.

   Show that there exists a point set P such that there are $\Omega(n^4)$ other points in the plane with mutually distinct views of P. **[3]**

4. (a) Show that for every positive irrational number α there are infinitely many pairs of numbers $m, n \in \mathbb{N}$ such that
$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}.$$
   Use Theorem 2.1.3 from the lecture notes. **[1]**

   (b) Prove that for $\alpha = \sqrt{2}$ there are ony finitely many pairs $m, n \in \mathbb{N}$ that satisfy
$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{4n^2}.$$
   [2]

   (c) Let α_1, α_2 be real numbers. Prove that for every $N \in \mathbb{N}$ there exist $m_1, m_2 \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \le N$ such that for every $i \in \{1, 2\}$, we have
$$\left|\alpha_i - \frac{m_i}{n}\right| < \frac{1}{n\sqrt{N}}.$$
   [2]

5. A point set P *pierces the triangles* of a point set M if every triangle determined by three points of M contains at least one point of P in its interior.

(a) Prove that for every $n \geq 3$ and every n-point set $M \subset \mathbb{R}^2$ in general position there is a set P of $2n-5$ points that pierces the triangles of M. **[2]**

(b) For every $n \geq 3$, construct an n-point set $M \subset \mathbb{R}^2$ in general position such that no set P of $2n-6$ points pierces the triangles of M. **[2]**

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web: `https://kam.mff.cuni.cz/~kvgweb/kvg/eng.html`