

# Exercises for Combinatorial and Computational Geometry

## Series 4 — Duality and polytopes

hint 14. 12. 2022, deadline 9:00 21. 12. 2022

1. (Missing pieces of the proof that every  $V$ -polytope is also an  $H$ -polytope.)
  - (a) Let  $C \subseteq \mathbb{R}^d$  be a convex set. Prove that  $C^*$  is bounded if and only if 0 lies in the interior of  $C$ . [2]
  - (b) Show that for every set  $X \subset \mathbb{R}^d$ , the second dual set  $(X^*)^*$  is the closure of  $\text{conv}(X \cup \{0\})$ . [2]
  - (c) Let  $P \subset \mathbb{R}^d$  be a  $V$ -polytope containing 0 in its interior. Show that  $P^*$  is the intersection of halfspaces dual to the vertices of  $P$ . [1]
2. A *convex body* is a bounded closed convex set in  $\mathbb{R}^d$  whose interior contains 0. A convex body is *smooth* if for each point on its boundary there is exactly one tangent hyperplane. A convex body is *strictly convex* if its boundary contains no straight-line segment of positive length. Prove that a convex body  $K$  is strictly convex if and only if  $K^*$  is smooth. [1]
3. Let  $v_1, \dots, v_n$  be linearly independent vectors in  $\mathbb{R}^n$ . Let  $C$  be the convex hull of the rays  $p_1, \dots, p_n$  that are determined by the vectors  $v_1, \dots, v_n$  and start in the origin (that is,  $p_i = \{x \in \mathbb{R}^n; x = \lambda v_i, \lambda \geq 0\}$ ).  
Prove that there is a ray in  $C$  that forms an acute angle with every ray  $p_i$ . [3]
4. Consider  $n$  line segments in the plane such that each of them is contained in a line passing through the origin, but none of these line segments contains the origin. Show that if every triple of the line segments can be intersected by a common line, then all  $n$  line segments can be intersected by a common line. (By intersecting we mean that the line segment and the line have at least one point in common. In particular, a line containing a line segment intersects this line segment.) [3]
5. Prove that every polytope  $P \subset \mathbb{R}^d$  is an orthogonal projection of some  $k$ -dimensional regular simplex in  $\mathbb{R}^n$  for suitable  $k, n$ . (An *orthogonal projection* is a mapping  $\pi$  from the space  $\mathbb{R}^n$  to a subspace  $M \cong \mathbb{R}^d$  embedded in  $\mathbb{R}^n$  such that for every  $x \in \mathbb{R}^n$  the vector  $\pi(x) - x$  is orthogonal to  $M$ . A simplex is *regular* if all its edges have the same length.) [4+hint]

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web: <https://kam.mff.cuni.cz/~kvgweb/kvg/eng.html>