Exercises for Combinatorial and Computational Geometry

Series 3 — Crossing numbers and incidences

deadline 9:00 7.12.2022

1. Let c be a positive real number and let G be a graph with n vertices and cn^2 edges. Prove that in every rectilinear drawing of G in the plane there is a matching M with $\Omega(n)$ edges and an edge intersecting all edges of M. (Edges of M can intersect, but no two can share a vertex since M is a matching.)

   You can use crossing lemma. A *rectilinear drawing* is a drawing where every edge is drawn as a straight-line segment. **[2]**

2. Let $I_{1\text{circ}}(n, m)$ be the maximum number of incidences of n points and m unit circles in the plane. Show that $I_{1\text{circ}}(n, n) = O(n^{4/3})$. **[3]**

3. Let $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$ be a system of subsets of an n-element set N (that is, $\forall i \in [n]\ M_i \subseteq N$) such that every pair of sets M_i, M_j has at most one common element. The number of incidences of N and $\mathcal{M}$ is defined as $I(N, \mathcal{M}) := \sum_{i=1}^{n} |M_i|$. Determine whether necessarily $I(N, \mathcal{M}) = O(n^{4/3})$. **[2]**

4. Find an n-point set in $\mathbb{R}^4$ with $\Omega(n^2)$ unit distances. **[3]**

5. Let P be an n-point set in the plane.

   (a) Let $k > 1$. Show that there are at most $O(n^2/k^3 + n/k)$ lines such that each of them contains at least k points of P, and that the number of incidences of these lines with P is at most $O(n^2/k^2 + n)$. **[3]**

   (b) Let $\alpha \in (0, \pi)$. Show that P determines at most $O(n^{7/3})$ triangles with at least one angle of size α. (Hint: split the triangles ABC with angle α at A into two groups according to whether the line AC contains more than $n^{1/3}$ points of P.) **[3]**