Exercises for Combinatorial and Computational Geometry

Series 1 — Convex sets

hints 26.10.2022, deadline 9:00 2.11.2022

Please choose a nickname that will be used in the list of scores on the webpage of the exercises. If you have already chosen a nickname, you can just sign your solutions with either your name or your nickname.

1. Find a set $M \subset \mathbb{R}^2$ that is a union of two convex sets such that $\mathbb{R}^2 \setminus M$ consists of five pairwise disjoint connected components. **[2]**

2. Prove Carathéodory's theorem (you may use Radon's theorem or a part of its proof). **[2]**

3. Let $M = \{x_1, y_1, x_2, y_2, \ldots, x_{d+1}, y_{d+1}\}$ be a set of $2d+2$ points in $\mathbb{R}^d$. Prove that M can be partitioned into two subsets A and B such that each of these subsets contains, for every $i = 1, 2, \ldots, d+1$, exactly one point from $\{x_i, y_i\}$, and the convex hulls of A and B have a nonempty intersection. (You may use the fact that the $(d+1)$-tuple of vectors $x_i - y_i$ is linearly dependent and then use an approach similar to the proof of Radon's theorem.) **[2]**

4. Let M be a finite set of at least four points in the plane such that each point is either red or blue. In addition, for every 4-tuple V of points of M there is a line strictly separating the red points of V from the blue points of V. Prove that there is a line strictly separating all the red points of M from all the blue points of M. **[3]**

5. Let $X_1, X_2, \ldots, X_{d+1}$ be finite point sets in $\mathbb{R}^d$ such that for every $i \in \{1, 2, \ldots, d+1\}$ the origin lies in $\operatorname{conv}(X_i)$. Prove that there exist $d+1$ points $x_1, x_2, \ldots, x_{d+1}$, with $x_i \in X_i$, such that $\operatorname{conv}(\{x_1, x_2, \ldots, x_{d+1}\})$ contains the origin. **[4, hint]**