

Exercises for Combinatorial and Computational Geometry

Series 4 — Duality and polytopes

hint 8. 12. 2021, deadline 15. 12. 2021

1. (Missing pieces of the proof that every V -polytope is also an H -polytope.)
 - (a) Let $C \subseteq \mathbb{R}^d$ be a convex set. Prove that C^* is bounded if and only if 0 lies in the interior of C . [2]
 - (b) Show that for every set $X \subset \mathbb{R}^d$, the second dual set $(X^*)^*$ is the closure of $\text{conv}(X \cup \{0\})$. [2]
 - (c) Let $P \subset \mathbb{R}^d$ be a V -polytope containing 0 in its interior. Show that P^* is the intersection of halfspaces dual to the vertices of P . [1]
2. A *convex body* is a bounded closed convex set in \mathbb{R}^d whose interior contains 0. A convex body is *smooth* if for each point on its boundary there is exactly one tangent hyperplane. A convex body is *strictly convex* if its boundary contains no straight-line segment of positive length. Prove that a convex body K is strictly convex if and only if K^* is smooth. [1]
3. Let v_1, \dots, v_n be linearly independent vectors in \mathbb{R}^n . Let C be the convex hull of the rays p_1, \dots, p_n that are determined by the vectors v_1, \dots, v_n and start in the origin (that is, $p_i = \{x \in \mathbb{R}^n; x = \lambda v_i, \lambda \geq 0\}$).
Prove that there is a ray in C that forms an acute angle with every ray p_i . [3]
4. Consider n line segments in the plane such that each of them is contained in a line passing through the origin, but none of these line segments contains the origin. Show that if every triple of the line segments can be intersected by a common line, then all n line segments can be intersected by a common line. (By intersecting we mean that the line segment and the line have at least one point in common. In particular, a line containing a line segment intersects this line segment.) [3]
5. Prove that every polytope $P \subset \mathbb{R}^d$ is an orthogonal projection of some k -dimensional regular simplex in \mathbb{R}^n for suitable k, n . (An *orthogonal projection* is a mapping π from the space \mathbb{R}^n to a subspace $M \cong \mathbb{R}^d$ embedded in \mathbb{R}^n such that for every $x \in \mathbb{R}^n$ the vector $\pi(x) - x$ is orthogonal to M . A simplex is *regular* if all its edges have the same length.) [4+hint]