

# Exercises for Combinatorial and Computational Geometry

## Series 3 — Crossing numbers and incidences

deadline 24.11.2021

1. Prove that a graph with  $n$  vertices that has a rectilinear drawing in the plane with no three pairwise crossing edges has  $O(n^{3/2})$  edges. You may use the crossing lemma. (A *rectilinear drawing* is a drawing where every edge is drawn as a straight-line segment.) [2]
2. Let  $I_{\text{circ}}(n, m)$  be the maximum number of incidences of  $n$  points and  $m$  unit circles in the plane. Show that  $I_{\text{circ}}(n, n) = O(n^{4/3})$ . [3]
3. Let  $\mathcal{M} = \{M_1, M_2, \dots, M_n\}$  be a system of subsets of an  $n$ -element set  $N$  (that is,  $\forall i \in [n] M_i \subseteq N$ ) such that every pair of sets  $M_i, M_j$  has at most one common element. The number of incidences of  $N$  and  $\mathcal{M}$  is defined as  $I(N, \mathcal{M}) := \sum_{i=1}^n |M_i|$ . Determine whether necessarily  $I(N, \mathcal{M}) = O(n^{4/3})$ . [2]
4. Find an  $n$ -point set in  $\mathbb{R}^4$  with  $\Omega(n^2)$  unit distances. [3]
5. Let  $P$  be an  $n$ -point set in the plane.
  - (a) Let  $k > 1$ . Show that there are at most  $O(n^2/k^3 + n/k)$  lines such that each of them contains at least  $k$  points of  $P$ , and that the number of incidences of these lines with  $P$  is at most  $O(n^2/k^2 + n)$ . [3]
  - (b) Let  $\alpha \in (0, \pi)$ . Show that  $P$  determines at most  $O(n^{7/3})$  triangles with at least one angle of size  $\alpha$ . (Hint: split the triangles  $ABC$  with angle  $\alpha$  at  $A$  into two groups according to whether the line  $AC$  contains more than  $n^{1/3}$  points of  $P$ .) [3]