

Exercises for Combinatorial and Computational Geometry

Series 2 — Helly-type theorems and the ham sandwich theorem

hints 3.11.2021, deadline 10.11.2021

1. Let C_1, \dots, C_n be a collection of at least three convex sets in the plane and let K be the line segment $[0, 1] \times \{0\}$. Show that if the intersection of every triple of sets from C_1, \dots, C_n contains a translated copy of K , then also the intersection of all the sets C_1, \dots, C_n contains a translated copy of K . [2]
2. Let $M \subset \mathbb{R}^2$ be a closed polygon with area $S(M)$, not necessarily convex. Prove that there is a point $x \in \mathbb{R}^2$ such that every line that contains x divides M into two parts, each with area at least $S(M)/3$. [2]
3. A collection $\mathcal{C} = \{C_1, \dots, C_n\}$ of convex sets in the plane has (p, q) -property if $n \geq p$ and every p -tuple of sets from \mathcal{C} contains a q -tuple with a nonempty intersection. The *piercing number* $s(\mathcal{C})$ of the collection \mathcal{C} is the minimum size of a set $X \subset \mathbb{R}^2$ such that every $C_i \in \mathcal{C}$ contains at least one point from X .
 - (a) Prove that every finite collection \mathcal{C} of closed axis-parallel rectangles with the $(2, 2)$ -property satisfies $s(\mathcal{C}) = 1$. [2]
 - (b) Prove that every finite collection \mathcal{C} of closed axis-parallel rectangles with the $(4, 3)$ -property satisfies $s(\mathcal{C}) \leq 2$. [3]
4. For $n \geq 4$, let C_1, \dots, C_n be a collection of convex sets in the plane. Show that if the intersection of every 4-tuple of sets from C_1, \dots, C_n contains a ray, then the intersection of all the sets C_1, \dots, C_n contains a ray. [4, hint]
5. The ham sandwich theorem says that for every collection of disjoint finite sets $A_1, \dots, A_d \subset \mathbb{R}^d$ there is a hyperplane h such that every open halfspace determined by h contains at most $\left\lfloor \frac{|A_i|}{2} \right\rfloor$ points of every A_i .

Let A_1, A_2, \dots, A_d be disjoint sets in \mathbb{R}^d such that every A_i contains n points and the points in $\bigcup_{i=1}^d A_i$ are in general position (that is, no hyperplane contains more than d points from this union). Show that the points from $\bigcup_{i=1}^d A_i$ can be partitioned into n *rainbow* d -tuples (that is, sets $\{x_1, x_2, \dots, x_d\}$ with $x_i \in A_i$) such that their convex hulls are disjoint.

If you wish to use a different version of the ham sandwich theorem than the one stated here, prove it first. [2]