Exercises for Combinatorial and Computational Geometry II

Series 5 — bonus problems

deadline: 16. 7. 2020

1. (One-dimensional second selection lemma.)
   Let $X \subset \mathbb{R}$ be a set of n real numbers, let $\alpha > 0$ and let F be a set of $\alpha\binom{n}{2}$ X-intervals. Prove that there is a point common to at least $\Omega(\alpha^2\binom{n}{2})$ intervals from F. **[2]**

2. Let X be a set of n points in the plane in general position and let $x \in \mathbb{R}^2$ be an arbitrary point. Prove that the number of X-triangles that contain x in their convex hull is at most $n^3/24 + O(n^2)$. Hint: count the triangles that do not contain x. **[3]**

3. a) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be families of convex sets in $\mathbb{R}^d$ such that $A \cap B \neq \emptyset$ for every $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$. Prove that $\mathcal{C}_1$ has non-empty intersection or there exist d hyperplanes that pierce $\mathcal{C}_2$; that is, every $B \in \mathcal{C}_2$ has a non-empty intersection with at least one of the d hyperplanes. **[2]**

   b) Find an example of families $\mathcal{C}_1$ and $\mathcal{C}_2$ in the plane satisfying the same assumptions as in part a), such that $\mathcal{C}_1$ has empty intersection and no line pierces $\mathcal{C}_2$. **[1]**

   c) (Colored Helly's theorem, stronger version in the plane.)
   Let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ be finite families of convex sets in the plane such that for every choice $C_i \in \mathcal{C}_i$, $i \in [3]$, we have $C_1 \cap C_2 \cap C_3 \neq \emptyset$. Prove that there are two different indices $i, j \in [3]$ such that $\bigcap \mathcal{C}_i \neq \emptyset$ and $\bigcap \mathcal{C}_j \neq \emptyset$ or there exist 4 lines that pierce $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$. **[2]**

4. Find an example of a set X of six points in the plane with a partition $X = X_1 \cup X_2 \cup X_3$ such that (X_1, X_2, X_3) has same-type transversals, but the intersection of the convex hulls of all the transversals (that is, "rainbow triples") is empty. **[1]**

5. Let $X_1, X_2, \ldots, X_{d+1}$ be disjoint finite sets in $\mathbb{R}^d$, assume that their union is in general position, and let $C_i = \mathrm{conv}(X_i)$. Prove that the following conditions are all equivalent:

   (a) There is no hyperplane intersecting all $C_1, C_2, \ldots, C_{d+1}$ simultaneously.

   (b) For every nonempty $I \subset [d+1]$, the sets $\bigcup_{i \in I} C_i$ and $\bigcup_{i \in [d+1] \setminus I} C_i$ can be strictly separated by a hyperplane.

   (c) $(X_1, X_2, \ldots, X_{d+1})$ has same-type transversals.

   [6]