

# Exercises for Combinatorial and Computational Geometry

## Series 5 — Polytopes, arrangements, and Voronoi diagrams

deadline 4. 1. 2018

1. Count the number of  $k$ -dimensional faces, for  $k = 1, 2, 3$ , of a 4-dimensional cyclic polytope on  $n$  vertices. [2]
2. Count the number of 1- and 2-dimensional faces in an arrangement of  $n$  planes in general position in  $\mathbb{R}^3$ . [2]
3. Let  $P = \{p_1, p_2, \dots, p_n\}$  be a set of  $n$  points in the plane. We say that points  $x, y$  have the *same view* of  $P$  if the points of  $P$  are visible in the same cyclic order from  $x$  and  $y$ . That is, if we rotate light rays that emanate from  $x$  and  $y$ , respectively, the points of  $P$  are lit in the same order by these rays. We assume that neither  $x$  nor  $y$  is in  $P$  and that neither of them can see two points of  $P$  in occlusion.

Show that the maximum number of points with mutually distinct views of  $P$  is  $O(n^4)$ . [2]

4. (a) How many cells are there in the arrangement of  $\binom{d}{2}$  hyperplanes in  $\mathbb{R}^d$  with equations  $x_i = x_j$ , where  $1 \leq i < j \leq d$ ? [2]  
(b) How many cells are there in the arrangement of hyperplanes in  $\mathbb{R}^d$  with equations  $x_i + x_j = 0$  and  $x_i = x_j$ , where  $1 \leq i < j \leq d$ ? [2]
5. Show that for  $n \geq 2$  the Voronoi diagram of a  $2n$ -point set  $A_{2n} := \{(i, 0, 0) : i = 1, 2, \dots, n\} \cup \{(0, n, j) : j = 1, 2, \dots, n\}$  in  $\mathbb{R}^3$  has at least  $cn^2$  vertices for some positive constant  $c$ . [2]

6. Let  $P$  be a finite point set in the plane with no three points on a line and no four points on a circle. Define a graph  $DT$  (called the *Delaunay triangulation*) on  $P$  as follows: two points  $a, b$  are connected by an edge if and only if there exists a circular disk with both  $a$  and  $b$  on the boundary and no point of  $P$  in its interior.

Prove that  $DT$  is a connected plane graph where every inner face is a triangle. [3]