Exercises for Combinatorial and Computational Geometry

Series 4 — Duality and polytopes

hint 12. 12. 2017, deadline 19. 12. 2017

1. (a) Let $C \subseteq \mathbb{R}^d$ be a convex set. Prove that C^* is bounded if and only if 0 lies in the interior of C. **[2]**

   (b) Show that for every set $X \subset \mathbb{R}^d$, the second dual set $(X^*)^*$ is the closure of $\mathrm{conv}(X \cup \{0\})$. **[2]**

   (c) Let $P \subset \mathbb{R}^d$ be a V-polytope containing 0 in its interior. Show that P^* is the intersection of halfspaces dual to the vertices of P. Using the previous parts together with that fact that every H-polytope is also a V-polytope, prove that every V-polytope is also an H-polytope. **[1]**

2. Let $v_1, \ldots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. Let C be the convex hull of the rays $p_1, \ldots, p_n$ that are determined by the vectors $v_1, \ldots, v_n$ and start in the origin (that is, $p_i = \{x \in \mathbb{R}^n; x = \lambda v_i, \lambda \geq 0\}$).

   Prove that there is a ray in C that forms an acute angle with every ray p_i. **[3]**

3. Consider n line segments in the plane such that each of them is contained in a line passing through the origin, but none of these line segments contains the origin. Show that if every triple of the line segments can be intersected by a common line, then all n line segments can be intersected by a common line. (By intersecting we mean that the line segment and the line have at least one point in common. In particular, a line containing a line segment intersects this line segment.) **[3]**

4. Prove that every polytope $P \subset \mathbb{R}^d$ is an orthogonal projection of some k-dimensional regular simplex in $\mathbb{R}^n$ for suitable k, n. (An *orthogonal projection* is a mapping π from the space $\mathbb{R}^n$ to a subspace $M \cong \mathbb{R}^d$ embedded in $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ the vector $\pi(x) - x$ is orthogonal to M. A simplex is *regular* if all its edges have the same length.) **[4+hint]**