Exercises for Combinatorial and Computational Geometry

Series 6 — bonus problems

deadline: 16. 2. 2016

1. Let $\mathcal{C}$ be the set of all cells (faces of maximum dimension) in an arrangement of n lines in the plane. Prove that $\sum_{C\in\mathcal{C}} f_0(C)^2 = O(n^2)$, where $f_0(C)$ denotes the number of vertices of a cell C. **[3]**

2. Let S be a set of n geometric objects in the plane. The *intersection graph of S* is a graph on n vertices that correspond to the objects in S. Two vertices are connected by an edge if and only if the corresponding objects intersect.

   (a) The total number of all graphs on n given vertices is $2^{\binom{n}{2}} = 2^{n^2/2+O(n)}$. Prove that the total number of all intersection graphs of n line segments in the plane is only $2^{O(n\log n)}$. (Be careful and consider also collinear line segments!) Use the theorem about the number of sign patterns. **[3]**

   (b) Show that the number of intersection graphs of n simple curves in the plane is at least $2^{\Omega(n^2)}$. If you wish, you can solve this exercise for n convex sets instead of simple curves. **[3]**

3. Prove that for fixed d the number of *unbounded* cells in an arrangement of n hyperplanes in $\mathbb{R}^d$ is at most $O(n^{d-1})$. **[2]**

4. (a) How many cells are there in the arrangement of $\binom{d}{2}$ hyperplanes in $\mathbb{R}^d$ with equations $x_i = x_j$, where $1 \le i < j \le d$? **[3]**

   (b) How many cells are there in the arrangement of hyperplanes in $\mathbb{R}^d$ with equations $x_i + x_j = 0$ and $x_i = x_j$, where $1 \le i < j \le d$? **[2]**

5. Show that for every positive irrational number α there are infinitely many pairs of numbers $m, n \in \mathbb{N}$ such that
   $$\left|\alpha - \frac{m}{n}\right| < \frac{1}{n^2}.$$
   Use Theorem 2.1.3 from the lecture notes. **[1]**

6. Prove that for $\alpha = \sqrt{2}$ there are ony finitely many pairs $m, n \in \mathbb{N}$ that satisfy
   $$\left|\alpha - \frac{m}{n}\right| < \frac{1}{4n^2}.$$
   [3]

7. Let α_1, α_2 be real numbers. Prove that for every $N \in \mathbb{N}$ there exist $m_1, m_2 \in \mathbb{Z}$, $n \in \mathbb{N}$, $n \le N$ such that
   $$\left|\alpha_i - \frac{m_i}{n}\right| < \frac{1}{n\sqrt{N}}, \quad i = 1, 2.$$
   [3]