Exercises for Combinatorial and Computational Geometry

Series 4 — Duality and polytopes

hints 15. 12. 2015, deadline 22. 12. 2015

1. For a set $C \subseteq \mathbb{R}^d$, show that $C = C^*$ if and only if C is the closed unit ball with the center in the origin. **[2]**

2. For every set $X \subset \mathbb{R}^d$, show that $(X^*)^*$ is equal to the closure of $\mathrm{conv}(X \cup \{0\})$. **[2]**

3. Let $v_1, \dots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. Consider the convex hull C of rays $p_1, \dots, p_n$ that start in the origin and that are determined by these vectors (that is, $p_i = \{x \in \mathbb{R}^n; x = \lambda v_i, \lambda \geq 0\}$).

   Prove that there is a ray in C that forms an acute angle with every ray p_i. **[3]**

4. Consider n line segments in the plane such that each of them is contained in a line passing through the origin, but none of these line segments contains the origin. Show that if every three of the line segments can be intersected by a line, then all the n line segments can be intersected by a line. (By intersecting we mean that the line segment and the line have at least one point in common. In particular, a line containing a line segment intersects this line segment.) **[3]**

5. Find a compact convex set $C \subseteq \mathbb{R}^3$ such that the set $\mathrm{ex}(C) = \{x \in C; \mathrm{conv}(C \setminus \{x\}) \neq C\}$ is not closed. **[3]**

6. Prove that every polytope $P \subset \mathbb{R}^d$ is an orthogonal projection of some k-dimensional regular simplex in $\mathbb{R}^n$ for suitable k, n. (An *orthogonal projection* is a mapping π from the space $\mathbb{R}^n$ to a subspace $M \cong \mathbb{R}^d$ that is embedded in $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ the vector $\pi(x) - x$ is orthogonal to M.) **[4+hint]**