Problems from Combinatorial and Discrete Geometry

Homework #4 - Cyclic polytopes and Voronoi diagrams

hints 15.12.2014, deadline 22.12.2014

1. Generalization of the Erdős-Szekeres Theorem for d-dimensional cyclic polytopes.

   (a) Let $x_1, \dots, x_n$ be points in $\mathbb{R}^d$. Let y_i denote the vector arising by appending 1 as the $(d+1)$st component of x_i. Show that if the determinant of all matrices with columns $y_{i_1} \dots, y_{i_{d+1}}$, for all choices of indices $i_1 < \dots < i_{d+1}$, have the same non-zero sign, then $x_1, \dots, x_n$ form the vertex set of a convex polytope combinatorially equivalent to the n-vertex cyclic polytope in $\mathbb{R}^d$. **[4+hint]**

   (b) Show that for any integers n and d there exists N such that among any N points in $\mathbb{R}^d$ in general position, one can choose n points forming the vertex set of a convex polytope combinatorially equivalent to the n-vertex cyclic polytope in $\mathbb{R}^d$. **[3]**

2. Let V be a set of n points on the moment curve $\{(t, t^2, \dots, t^d) \in \mathbb{R}^d : t \in \mathbb{R}\}$. Let W be a subset of V containing at most $\lfloor \frac{d}{2} \rfloor$ elements. Show that conv(W) is a face of conv(V). Determine the number of $\leq k$-dimensional faces of d-dimensional cyclic polytope for $k = 0, 1, \dots, \lfloor \frac{d}{2} \rfloor$. **[3]**

3. (a) Show that for $n \geq 2$ the Voronoi diagram of a $2n$-point set $A_{2n} := \{(i, 0, 0) : i = 1, 2 \dots n\} \cup \{(0, n, j) : j = 1, 2 \dots n\}$ in $\mathbb{R}^3$ has at least cn^2 vertices, where c is some positive constant. **[3]**

   (b) Show that for $n \geq k$ the Voronoi diagram of a $2n$-point set in $\mathbb{R}^{2k-1}$ can have up to $c_k n^k$ vertices, where c_k is some positive constant. **[2]**

Information about practicals can be found at `http://kam.mff.cuni.cz/kvg`