Problems from Combinatorial and Discrete Geometry

Homework #3 - Polytopes and duality

hints 1.12.2014, deadline do 8.12.2014

1. For a set $X \subseteq \mathbb{R}^d$, we define the set dual to X, denoted by X^*, as $X^* = \{y \in \mathbb{R}^d \colon \langle x, y\rangle \leq 1 \text{ for all } x \in X\}$.

   (a) Let $C \subseteq \mathbb{R}^d$ be a convex set. Prove that C^* is bounded if and only if the origin 0 lies in the interior of C. **[2]**

   (b) Let $C = \operatorname{conv}(X) \subseteq \mathbb{R}^d$. Prove that $C^* = \bigcap_{x \in X} \mathcal{D}_0^-(x)$ where $\mathcal{D}_0^-(x) = \{y \in \mathbb{R}^d \colon \langle x, y\rangle \leq 1\}$. **[2]**

   (c) Show that for any set $X \subset \mathbb{R}^d$, $(X^*)^*$ equals the closure of $\operatorname{conv}(X \cup \{0\})$. **[2]**

   (d) Using the previous parts together with that fact that every H-polytope is also a V-polytope, prove that every V-polytope can be expressed as an intersection of finitely many half-spaces. In other words, show that every V-polytope is a H-polytope. **[1]**

2. Find a compact convex $C \subseteq \mathbb{R}^3$ for which $\operatorname{ex}(C) = \{x \in C \colon \operatorname{conv}(C \setminus \{x\}) \neq C\}$ is not closed. **[3]**

3. Show that every polytope $P \subset \mathbb{R}^d$ can be expressed as an orthogonal projection of some k-dimensional regular simplex in $\mathbb{R}^n$ for suitable k, n. (An orthogonal projection is a mapping π from $\mathbb{R}^n$ to a subspace $M \cong \mathbb{R}^d$ that is embedded in $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ the vector $\pi(x) - x$ is orthogonal to M.) **[4+hint]**

Information about practicals can be found at `http://kam.mff.cuni.cz/kvg`