Problems from Combinatorial and Discrete Geometry

Homework #2 – Helly type theorems and counting incidences

hints 10.11.2014, deadline 24.11.2014

1. A family $\mathcal{C} = \{C_1, \ldots, C_n\}$ of convex sets in the plane has a (p, q)-property if $n \geq p$ and from every p-tuple of sets of $\mathcal{C}$ we can always choose q with a nonempty intersection. The piercing number $s(\mathcal{C})$ of the family $\mathcal{C}$ is the cardinality of a minimal set of points from X such that every $C_i \in \mathcal{C}$ contains at least one point from X.

   (a) Prove that if $\mathcal{C}$ is a finite family of axis parallel closed rectangles with a $(4, 3)$-property, then $s(\mathcal{C}) \leq 2$. **[3]**

   (b) Find a family $\mathcal{C}$ of some axis parallel closed rectangles with a $(3, 2)$-property for which $s(\mathcal{C}) = 3$. **[2]**

2. (a) Let $r < \frac{\pi}{3}$ and let A be a set of at least three points on a sphere such that every three points from A can be covered by a spherical disk with a radius r. Prove that all points from A can be covered by a spherical disk with a radius r. **[4+hint]**
   Sphere is a boundary of a ball in $\mathbb{R}^3$. *Spherical disk* with a center in x and a radius r is a set of points of the sphere, which are—when looking from the center of the ball, in a degree distance at most r from x.

   (b) Prove that in the case (a) the condition $r < \frac{\pi}{3}$ cannot be replaced by $r < \frac{\pi}{2}$. **[2]**

3. Find an n-point set in $\mathbb{R}^4$ with $\Omega(n^2)$ unit distances. **[3]**

4. In a *drawing* of a graph G, vertices of G correspond to distinct points in the plane and edges of G are represented by continuous curves connecting corresponding vertices. A *crossing* of two edges is their common point which does not represent a vertex. We assume that no three edges have a common crossing, every pair of edges has at most finite number of points in common and no edge contains other vertices than its own ending vertices. For every finite graph G show that in any drawing of G, which has the smallest number of crossings, no two edges contains more than one point in common. **[2]**

Information about practicals can be found at `http://kam.mff.cuni.cz/kvg`