Contents

Chapter 1

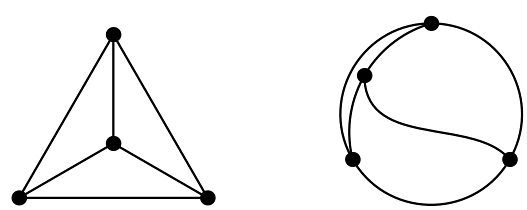
Geometric and Topological graphs

Here, I would like to show beautiful proofs, which I really like.
Some of them are real pearls in the theory.

1.1 Introduction

An *(abstract) graph* G is a pair $(V(G), E(G))$ where $V(G)$ is the set of *vertices* and $E(G)$ is the set of *edges* $\{u, v\}$ each joining two vertices $u, v \in V(G)$.

Drawing $D(G)$ of a graph G into the plane is a mapping f, which assigns to each vertex a distinct point in the plane and to each edge uv assigns a continues arc connecting points $f(u)$ and $f(v)$ (continues arc is an image of a closed interval under continues mapping). The point, which is image of a vertex, is called *vertex of drawing* $D(G)$ and the arc connecting $f(u)$ and $f(v)$ is called the *edge* uv *of* $D(G)$. The mapping f must also satisfy the following conditions: (1) No edge of $D(G)$ passes through a vertex (except for the endpoints of the edge). (2) Two edges of $D(G)$ have a finite number of intersection points. (3) Any intersection of two edges of $D(G)$ is a proper crossing (they cannot just touch). (4) No three edges of $D(G)$ have a common intersection point.



Because we use the drawing of G very often, we simply say "topological graph" instead of "drawing of a graph". Or we say "geometric graph" instead of "drawing of a graph by straight-line segments". More precise definitions are below.

A *geometric graph* is a graph G drawn in the plane by straight line segments. It is defined as a pair $(V(G), E(G))$, where $V(G)$ is a finite set of points in general position in the plane, i.e. no three points are collinear, and $E(G)$ is a set of line segments with endpoints in $V(G)$. $V(G)$ and $E(G)$ are the *vertex set* and the *edge set* of G, respectively. Let H and G be two geometric graphs, we say that H is a *(geometric) subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A *topological graph* is defined similarly. It is a graph drawn in the plane in such a way that edges are Jordan curves. Two of these curves share a finite number of points and no curve passes through a vertex. Obviously, geometric graphs are a subclass of topological graphs. Topological graph G is *simple* if every two edges of G share at most one point.

We say that two edges *cross* each other if they have an interior point in common. Two edges are *disjoint* if they have no point in common.

A *geometric thrackle* is a geometric graph, such that every two edges intersect. A *crossing family* is a set of edges of the geometric graph, such that every two edges cross each other.

1.2 Plane graphs

Planar graph is an abstract graph, which can be drawn in the plane with no crossing edges. *Plane graph* is a topological graph, the drawing of a graph in the plane.

Theorem 1.2.1 (Euler formula[1]). *Let $G = (V, E)$ be a connected planar graph and $D(G)$ be its plane drawing. Put $v := |V|$, $e = |E|$ and $f := |F|$, where F is the set of faces of $D(G)$ (number of connected regions, which we get after cutting the plane along the edges E). Then*

$$v + f = e + 2.$$

Corollary 1.2.2. *Let $G = (V, E)$ be a planar graph. Then $|E| \leq 3|V| - 6$. That further implies that G has a vertex v with $deg(v) \leq 5$ (in fact there are at least 4 such vertices).*

Proof. The proof of Euler formula and inequality $|E| \leq 3|V| - 6$ is for example in Matoušek, Nešetřil [22]. Here we show just the extension, that there are at least 4 vertices with degree at most 5 in G.

We can assume that G is a plane triangulation. Otherwise we add an edge to each face which is not triangle. Adding edges can only increase the degrees of vertices. Thus the minimum degree of a vertex in G is at least 3.

Suppose for a contradiction, that $\deg(v) \geq 6$ for every vertex except for at most 3 vertices, which have $3 \leq \deg(v) \leq 5$. Thus $\sum_{v \in V} \deg(v) \geq 6n - 9$. On

[1] This formula holds also for convex polytopes in $\mathbb{R}^3$. Vertices are vertices of the polytope, similarly edges and faces. In fact, that is where we took to notation for graphs. Euler also discovered this formula for polytopes. It is interesting, that the formula can be generalized for polytopes in higher dimensions ($\mathbb{R}^d$, $d \geq 4$). Then there are also higher dimensional faces and we get more than one equation.

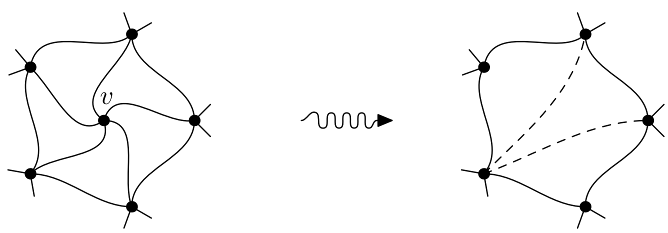
the other hand by the corollary of Euler formula $\sum_{v \in V} \deg(v) = 2|E| \leq 6n - 12$. That is a contradiction. □

Theorem 1.2.3 (Fáry's theorem [12]). *Any planar graph G admits a drawing into the plane such that the edges are straight line segments, which do not cross.*

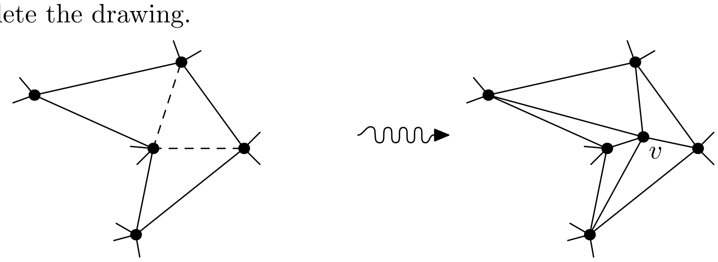
The original Fáry's proof uses a decomposition of G into 3 connected trees and then barycentric coordinates. Here we present much simpler proof.

Proof. [2] Let $G = (V, E)$ be a planar graph with n vertices. We may assume that G is maximal planar graph, thus its drawing $D(G)$ is a plane triangulation. Otherwise we add a new edge into each face which is not triangle.

Choose some three vertices $a, b, c \in V$ forming a triangular face. We prove by induction on n that there exists a straight-line drawing of G in which the triangle abc is the outer face. The result is trivial for $n = 3$, because a,b, and c are the only vertices in G. In other cases, all vertices in G have at least three neighbors.



Suppose that $n > 3$. By the corollary of Euler formula, G has a vertex v of degree at most 5. Moreover there are at least 4 such vertices, thus at least one is different from a, b, c. Let G' be planar graph, which we get by removing v from G and retriangulating the face formed by removing v. By induction hypothesis, G' has a straight line drawing with outer face abc. Remove the added edges in G', forming a polygon P with at most five sides into which v should be placed to complete the drawing.



Now, it is just a simple case analysis to show, that every polygon P with at most 5 sides contains an interior point p, which sees all vertices (imagine the polygon as walls). Then place v into point p. The edges incident with v do not cross any other edges because of visibility from p to all vertices of P. That completes the proof. □

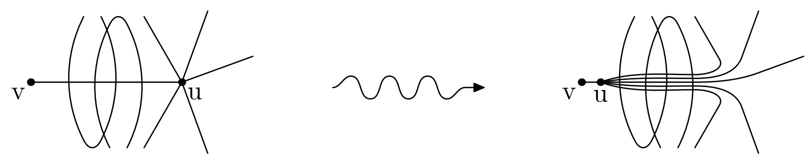
[2] The author of this proof is unknown. You can find this proof for example on Wikipedia.

Theorem 1.2.4 (Hanani-Tutte). *If graph G has drawing in the plane such that every pair of edges cross even number times then G is planar.*

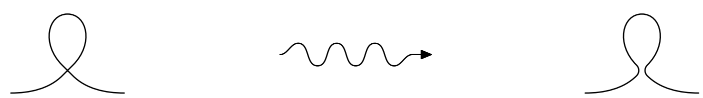
Proof (Pelsmajer, Schaefer, Štefankovič [34]). [3] [4]

Let $D(G)$ be the drawing of G where each pair of edges cross even number times. Assume that G has only one connected component otherwise we can show the theorem for each component separately.

Pull operation: To pull a vertex u towards a vertex v along the edge uv means to redraw the vertex u on the edge uv closer to the vertex v, shorten the edge uv to the part between new u and v and redraw all other edges incident with u in such a way that they follow their original drawing up to close neighborhood of u where they form an infinitesimally narrow beam together with the other edges incident with u and then they follow the original drawing of edge uv until they join vertex u.



We always pull u so close to v, that there is no edge crossing uv. If we create some self-crossing edges, we replace their self-crossings as shown on the next figure.



The property of pulling operation is that after the pulling, each pair of edges cross even number times. Indeed, the edge uv does not cross anything, which is an even number. To the other edges incident with u we have added all crossings of the original edge uv, which is again an even number.

Take arbitrary spanning tree T of G and choose one vertex r as a root. Pull the neighbors of r in T towards the root r. Then pull the neighbors of these neighbors towards the root and continue this way until the whole spanning tree T is in the neighborhood of r and is not crossed by any edge of G.

Now, we want to redraw the other edges to get plane drawing of G. For a tree T, we denote the unique path in T between x and y by xTy. Put

[3] Actually, the Hanani-Tutte theorem is much stronger: If graph G has a drawing in the plane such that every pair of *independent* edges cross even number times then G is planar. Two edges are independent, if they have no vertex in common. Pelsmajer, Schaefer, Štefankovič [34] prooved the stronger version. Here we show just the weak version.

[4] Once I was thinking about this proof. I discovered this simple proof. I got enthusiastic. I told several people and they were surprised, how easy it is to proof Hanani-Tutte theorem. But I got disappointed a little later, when I searched the web, because the authors of [34] were faster by a few months. The simple proof was not known for 35 years and I missed that by a few months. That's life. Here in this proof I present my version of the proof.

$F := E(G) \setminus E(T)$. All edges of F starts and ends on the spanning tree T. Consider an edge $e = xy \in F$. The edge ye together with the unique path xTy forms a cycle C_e. There is no edge $f \in F$ starting inside of C_e and ending outside of C_e, because e and f would have odd number of crossings (f does not cross xTy).

In other words, if you imagine an infinitisimaly thin cycle C_T around T, then the endpoints of edges F form a valid system of parenthesis.

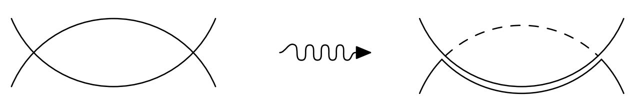
Let us start with $i = 0$ and $F_0 = F$. Take an edge $e_i = xy \in F_i$ such that there is no edge of F_i starting and ending inside C_{e_i} (all edges start and end on T). Such an edge exists because we can take e_i with shortest $C_{e_i} \cap T$. Redraw $e_i = xy$ along xTy so close to the the path xTy or to the already redrawn edges, that it does not intersect any other edge of F_i. Set $F_{i+1} = F_i \setminus \{e_i\}$ and repeat this step until $F_i \neq \emptyset$. The invariant is that at every step i all edges $E \setminus F_i$ are drawn without crossings. We finish with a plane drawing of the graph G. □

Alternative proof. [5]

First, we follow the first proof. Pull the vertices towards the root r until we get all vertices to a small neighborhood of r. Then the whole spanning tree T is not crossed by any other edge.

Now we proceed by induction on the number of crossings. Until there are two crossing edges e, f in $D(G)$, we repeat the following.

The edges e, f cross. Thus they have at least two crossings. Take two crossings $p, q \in e \cap f$ that are consecutive on e. Denote the part of e between p and q by $e_{p,q}$. Similarly, denote the part of f between p and q by $f_{p,q}$. Curves $e_{p,q}$ and $f_{p,q}$ form a Jordan cycle C, because p, q are consecutive on e. Since the whole spanning tree T (and all vertices V) is either inside C or outside C, every edge $h \in E$, which cross the cycle C, has to cross it even number times. Thus the parity of $|h \cap e_{p,q}|$ and $|h \cap f_{p,q}|$ is the same. Let us say that $f_{p,q}$ contains less or equal number of crossings with other edges than $e_{p,q}$. (Otherwise swap the names for e and f). Then redraw the part $e_{p,q}$ of edge e along $f_{p,q}$ and get rid of the crossings p and q.



If we create self-crossing edges, we replace their self-crossings as we did in the pulling phase (see second figure in the first proof of Hanani-Tutte Theorem).

All pairs of edges cross even number times and we decreased the number of crossings. □

[5] Joke: A professor at a lecture hall explains some theorem. Suddenly some student raises his arm and says: "Mr. professor, I have a counter example!". Professor answers: "It does not matter. I have two proofs."

1.3 Crossing numbers

If we get a drawing of a graph G, we can easily count the number of crossings. The number of crossings in a drawing $D(G)$ is the crossing number $\textsc{cr}(D(G))$ of the drawing $D(G)$. *Crossing number* of an abstract graph G, denoted $\textsc{cr}(G)$, is the smallest number of crossings in any drawing of G.

The first motivation for crossing number comes from 50s [18, 16] and the task was to determine $\textsc{cr}(K_n)$ and $\textsc{cr}(K_{n,n})$. There are some partial results and famous conjectures for $\textsc{cr}(K_n)$ and $\textsc{cr}(K_{n,n})$, but the exact values were not determined yet.

The breakthrough in applications came with the following theorem, which was discovered by Ajtai, Chvátal, Newborn, Szemeredi [4] and independently by Leighton [20]. This theorem is a basic ingredient in quite simple proofs for many geometric problems (for example Szemerédi-Trotter theorem for line-point incidences, upper bounds for unit distances in the plane). Author of the crossing number method is Székely [37]. For a survey of applications of crossing number method, see Matoušek [21]).

Theorem 1.3.1 (Crossing lemma). *Let G be a graph with n vertices and m edges. If $m \geq 4n$ then*

$$\textsc{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Proof. Let G be a topological graph (drawing of an abstract graph G) with the minimum $\textsc{cr}(G)$.

Simple estimate on $\textsc{cr}(G)$: Graph G has $\textsc{cr}(G)$ crossings. We remove each crossing by deleting one edge involved in the crossing. After removing all crossings one by one, we end up with a graph with no crossings. That is a plane graph. This plane graph has at most $m - \textsc{cr}(G) \leq 3n - 6$ edges. Thus we get

$$\textsc{cr}(G) \geq m - 3n. \tag{1.1}$$

Probabilistic strengthening: Let $0 < p < 1$ be a fixed parameter. We will choose the suitable value of p later.

Choose a random subgraph $H \subseteq G$ by choosing each vertex of G at random, independently and uniformly, with probability p. Set of chosen vertices determines an induced subgraph H. Put $n_H := |V(H)|$, $m_H := |E(H)|$. The expected number of vertices in H is $\mathbb{E}[n_H] = pn$. The expected number of edges in H is $\mathbb{E}[m_H] = p^2 m$, because each edge of G stays in H iff both of its end-points stay in H. Similarly, the expected crossing number of topological subgraph H is $\mathbb{E}[\textsc{cr}(H)] \leq p^4\textsc{cr}(G)$.[6] Each crossing of G remains in H is both crossing edges remains in H, that means end-vertices of these edges must stay in H.

Because every possible topological subgraph H satisfies the simple estimate (1.1), the expected values have to satisfy the same inequality. Thus $\mathbb{E}[\textsc{cr}(H)] \geq$

[6] Note that the crossing number of abstract subgraph H is smaller than or equal to the crossing number of the particular drawing of H, inherited from the topological graph G.

$\mathbb{E}[m_H] - 3\mathbb{E}[n_H]$, what is

$$p^4 \text{CR}(G) \geq p^2 m - 3pn.$$

Now we set $p := 4n/m$ and after some algebraic operations we get

$$\text{CR}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Note that the choice $p = 4n/m < 1$, because $4n \leq m$. Thus it corresponds to a probability. □

One improvement of the theorem is due to Pach, Tóth [30]: If $m \geq 7.5n$ then $\text{CR}(G) \geq \frac{1}{33.75}\frac{m^3}{n^2}$. Currently the best constant in the theorem is due to Pach, Radoičić, Tardos, Tóth [26]: If $m \geq \frac{103}{6}n$ then $\text{CR}(G) \geq \frac{1024}{31827}\frac{m^3}{n^2}$.

All improvements of the constant in Crossing lemma use the same probabilistic proof. They just improve the simple estimate on $\text{CR}(G)$,[7] which immediately leads to a better constant.

The crossing lemma is asymptotically tight in general case. But for special classes of graphs, we can get better lower bounds.

Pach, Spencer, Tóth [28] showed a better bound for graphs with monotone graph properties. Graph property $\mathcal{P}$ is *monotone* if whenever a graph G satisfies $\mathcal{P}$, then every subgraph of G also satisfies $\mathcal{P}$, and whenever G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union satisfies $\mathcal{P}$, too.

Theorem 1.3.2 (Pach, Spencer, Tóth [28]). *Let $\mathcal{P}$ be monotone graph property such that every graph G with n vertices, satisfying property $\mathcal{P}$, has at most $O(n^\alpha)$ edges, for some fixed $\alpha > 0$.*

Then there exist constants $c, c' > 0$ such that every graph G with n vertices and $m \geq cn \log^2 n$ edges, satisfying property $\mathcal{P}$ has

$$\text{CR}(G) \geq c' \frac{m^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

Particularly, for graphs with $girth(G) > 2r$, we get the following theorem. Székely found a quite simple proof for this theorem.

Theorem 1.3.3 (Pach, Spencer, Tóth [28], Székely [36]). *Let G be a simple topological graph with n vertices and m edges. If $m \geq 4n$ and $girth(G) > 2r$ then*

$$\text{CR}(G) \geq c_r \cdot \frac{m^{r+2}}{n^{r+1}},$$

where $r > 0$ is a fixed integer and constant c_r depends on r.

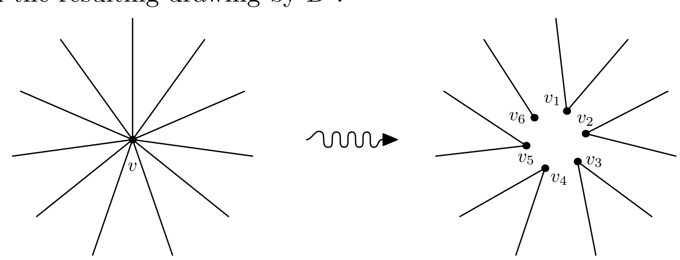
The proof of this theorem is a nice illustration of *embedding method.* The basic idea of embedding method is the following. Let G be a graph, for which we need an upper bound on $\text{CR}(G)$. The method starts with an optimal drawing

[7] Which stopped being simple and becomes more and more complicated.

of a connected subgraph $H \subseteq G$, which is has the same vertex set and known $\text{CR}(H)$. Then we embed each edge $e = uv \in E(G) \setminus E(H)$ "infinitesimally close" to a path in H connecting u and v. Now we can easily count the added crossings (they lie either close to an original crossing in the drawing of H or close to a vertex) and find an upper bound $\text{CR}(G) \leq F(\text{CR}(H), n, m)$ for some function F. For application of embedding method see for example Leighton [20], Richter and Thomassen [35], Székely [36], Fox and Tóth [14], Černý, Kynčl and Tóth [9].

Proof of the theorem 1.3.3. Let G be a graph with n vertices, m edges, crossing number $\text{CR}(G)$ and $girth(G) > 2r$. Let D be the drawing of G realizing $\text{CR}(G)$. Let ρ denote the average degree of G, that is $\rho := 2m/n$.

Auxiliary graph G' with maximum degree ρ: We split every vertex of G whose degree exceeds ρ into vertices of degree at most ρ, as follows. Let v be a vertex with degree $\deg(v) > \rho$, and let vw_1, vw_2, $\ldots$, $vw_{deg(v)}$ be edges incident with v, listed in the clockwise order. Replace v by $\lceil deg(v)/\rho \rceil$ new vertices v_1, v_2, $\ldots$, $v_{\lceil deg(v)/\rho \rceil}$, placed in clockwise order on a very small circle around v. Without introducing any new crossings, connect w_j with v_i, if and only if $\rho(i-1) < j \leq \rho i$, for $1 \leq j \leq \rho$ and $1 \leq i \leq \lceil deg(v)/\rho \rceil$. Repeat this procedure with every vertex whose degree exceeds ρ. Denote the resulting graph by G' and the resulting drawing by D'.



Observe, that we did not add any crossings. Thus $\text{CR}(G') \leq \text{CR}(G)$. The girth of G' still exceeds $2r$. Maximum degree of G' is at most $\Delta := \lfloor \rho \rfloor$. Further we have $|V(G')| \leq \sum_{v \in V} \lceil \deg(v)/\rho \rceil \leq n + \sum_{v \in V} \deg(v)/\rho = 2n$. Thus the average degree of G' is at least $\rho/2 = m/n$.

r-distance graph G'': Now, we define a graph G'' with $V(G'') := V(G')$, and $E(G'') =$ pairs of vertices from $V(G')$, whose distance in G' is exactly r.

Since $girth(G) > 2r$, the r-neighborhood of every vertex $v \in V(G)$ is a tree. (We will use this local property very often in this proof.) Thus maximum degree of a vertex in G'' is at most $\Delta(\Delta - 1)^{r-1}$.

Embedding method: We construct the drawing D'' of G'' by embedding method. We use the drawing D' of G' as a plan for embedding. Every edge $e \in E(G'')$ correspond to a unique r-path P_e in G'. Draw e as a curve "infinitesimally close" to this unique path P_e. Repeat this for every edge in $E(G'')$ and finally obtain the drawing D''.

We can split crossing in D'' into two categories. A crossing of the *first category* lies very close to the original crossing of two edges e_1, e_2 of D', which are parts of two different r-paths. The number of r-paths containing a fixed edge

e is at most $r(\Delta-1)^{r-1} \le r\Delta^{r-1}$ (there are r choices for the edge on r-path and for each vertex on a "free" end of the path we have at most $\Delta - 1$ choices). Therefore, every crossing of D' corresponds to at most $r^2\Delta^{2r-2}$ crossings of D'' of the first category.

A crossing of the *second category* lies very close to a vertex of D', where two different r-paths, passing through vertex, cross. The number of different r-paths passing through a fixed vertex v is at most $\Delta \cdot r\Delta^{r-1}$, because there are at most Δ edges incident with v and there are at most $r\Delta^{r-1}$ r-paths containing a fixed edge. Thus around every vertex of G', there are at most $\binom{r\Delta}{2}$ crossings of the second category.

Altogether we have

$$\textsc{cr}(G'') \le r^2\Delta^{2r-2}\textsc{cr}(G) + 2n\binom{r\Delta^r}{2}.$$

That is

$$\textsc{cr}(G) \ge \frac{\textsc{cr}(G'')}{r^2\Delta^{2n-2}} - n\Delta^2. \tag{1.2}$$

Lower bound on $E(G'')$: Now we show that $E(G'') \ge c''_r \cdot \frac{m^r}{n^{r-1}}$, for some constant c''_r dependent on r.[8]

For any real $s > 0$, we construct a large subgraph $H' \subseteq G'$ with minimum degree at least $\rho/(4+s)$. Consequently we get $H'' \subseteq G''$ with many edges.

Delete all vertices of G' with degree smaller than $\rho/(4+s)$, together with edges incident with them. We iterate this for the resulting graph. More formally we construct a sequence $G' = G'_0 \supseteq G'_1 \supseteq \dots$ of induced subgraphs as follows. If G'_i has a vertex v_i of degree $\deg(v_i) < \rho/(4+s)$, we let $G'_{i+1} := G'_i \setminus v_i$. If not, we terminate the sequence and set $H' := G'_i$. The average degree of H' is still at least $\rho/2$, because with each vertex we had deleted at most $\rho/(4+s) < \rho/4$ edges. Since G' has at most $2n$ vertices, altogether we had deleted at most $4m/(4+s)$ edges. Therefore at least $sm/(4+s)$ edges are left in H'. Because the maximum degree in H' is at most $\Delta = \lfloor \rho \rfloor$, there are at least $sn/(4+s)$ vertices in H'.

Let H'' be the r-distance graph of H'. Clearly, $H'' \subseteq G''$. For every $v \in V(H')$, there are at least $(\frac{\rho}{(4+s)})^r$ paths of length r starting in v. Thus $E(G'') \ge E(H'') \ge \frac{sn}{4+s} \cdot \frac{1}{2}(\frac{\rho}{4+s})^r \ge s(\frac{1}{2(4+s)})^{r+1} \cdot \frac{m^r}{n^{r-1}}$.

Last step: By Applying the crossing lemma (Theorem 1.3.1) on G'', we obtain that $|E(G'')| \le 8n$, or

$$\textsc{cr}(G'') \ge \frac{c}{(2n)^2}\left(c''_r \cdot \frac{m^r}{n^{r-1}}\right)^3. \tag{1.3}$$

[8]Erdős and Simonovits ([11], combination of formula (13) and Theorem 5) showed even that $|E(G'')| \ge (\frac{1}{2} - o(1)) \cdot \frac{m^r}{n^{r-1}}$. By using their lower bound, we can improve the constant c_r in the statement of this theorem.

Finally, by combination of inequalities (1.2) and (1.3) we get

$$\text{CR}(G) \geq \frac{cc_r''^3}{2^{2r}r^2} \cdot \frac{m^{r+2}}{n^{r+1}} - \frac{4m^2}{n} \geq C(r) \cdot \frac{m^{r+2}}{n^{r+1}} - \frac{4m^2}{n},$$

where $C(r)$ is a constant depending on r. The term $\frac{4m^2}{n}$ is little-oh of the main term. Hence we finished the proof. □

1.3.1 Different definitions of a crossing number

It is clear, what is a crossing of two edges. But there are four definitions of crossing number. Let G be a simple graph.

- The *crossing number* of G, $\text{CR}(G)$, is the minimum number of crossings over all drawings of G.
- The *rectilinear crossing number* of G, $\text{LIN-CR}(G)$, is the minimum number of crossings in any drawing of G, where edges are represented by straight-line segments.
- The *pairwise crossing number* of G, $\text{PAIR-CR}(G)$, is the minimum number of pairs of crossing edges over all drawings of G.
- The *odd crossing number* of G, $\text{ODD-CR}(G)$, is the minimum number of pairs of edges with odd number of crossing over all drawings of G.

The following lemma is in fact a simple observation.

Lemma 1.3.4. *For any simple graph, we have*

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G).$$

There is a difference between CR and LIN-CR, because Guy [17] has shown that $\text{LIN-CR}(K_8) = 19 > 18 = \text{CR}(K_8)$. Moreover, Bienstock and Dean [6] constructed graphs G_n with $\text{CR}(G_n) = 4$, but arbitrary large $\text{LIN-CR}(G_n)$.

There is also difference between PAIR-CR and ODD-CR. Pelsmajer, Schaefer, Štefankovič [33] have shown that there is a graph G with

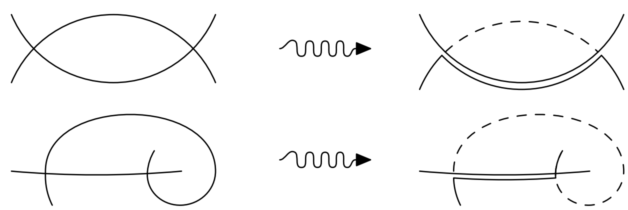
$$\text{ODD-CR}(G) \leq \left(\frac{\sqrt{3}}{2} + o(1)\right) \text{PAIR-CR}(G).$$

Tóth [39] improved this bound a little to $\text{ODD-CR}(G) \leq 0.855 \cdot \text{PAIR-CR}(G)$. Pach and Tóth [32] have shown for every graph G that

$$\text{ODD-CR}(G) \leq \text{CR}(G) \leq 2(\text{ODD-CR}(G))^2.$$

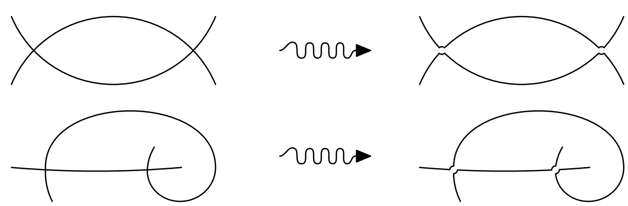
Lemma 1.3.5. *Let G be a graph and $D(G)$ be its drawing into a plane with minimum number of crossings (witnessing $\text{CR}(G)$). Then each pair of edges e, f cross at most once.*

Proof. Suppose on the contrary, that e, f are two edges that cross at least twice. Take two crossing $p, q \in e \cap f$ which are consecutive on e. Denote the part of e between p and q by $e_{p,q}$. Similarly the part of f between p and q by $f_{p,q}$. Curves $e|_{p,q}$ and $f_{p,q}$ form a cycle C, because p, q are consecutive on e. Let us say that $f_{p,q}$ contains less or equal number of crossings with the other edges than $e_{p,q}$. (Otherwise swap the names for e and f). Then redraw the part $e_{p,q}$ of edge e along $f_{p,q}$ as shown on the next figures. There are two possibilities – either e crosses f in the points p, q from the same side of f or from different sides.



If we create self-crossings on edge e, we shorten the edge e by deleting loops. We reduced the total number of crossings at least by one, what is a contradiction. □

Note: Another option in the previous proof was to swap parts $e_{p,q}$, $f_{p,q}$ between edges e and f and make a small distance between e, f at points p, q, where e, f touch. As shown on the next figure.



That removes only crossings p, q, but it is sufficient.

One might think, that previous lemma shows, that CR = PAIR-CR, but it is not true. Why? We start with the optimal drawing for $\text{CR}(G)$. Then we might redraw some edge e so, that the new edge e contains more than one crossing, but on the other hand the redrawing reduces the number of crossing pairs. In other words the optimal drawing of a graph G for $\text{CR}(G)$ need not to be optimal drawing for $\text{PAIR-CR}(G)$.

Open problem 1 (Pach, Tóth [31]). *Is it true, that for every graph G, $\text{CR}(G) = \text{PAIR-CR}(G)$?*

Tóth [39] improved the result of Valtr [42] and showed currently the best upper bound, that every graph G with $k := \text{PAIR-CR}(G)$ has

$$\text{CR}(G) \leq \frac{9k^2}{\log^2 k}.$$

1.3.2 Decay of crossing number

In this subsection we look what happens, if we delete some edges from a graph. How much does the crossing number decrease?

Richter and Thomassen [35] conjectured that there is a constant c such that every graph G has an edge e with $\text{CR}(G \setminus e) \geq \text{CR}(G) - c\sqrt{\text{CR}(G)}$. They only proved that G has an edge e with $\text{CR}(G \setminus e) \geq \frac{2}{5}\text{CR}(G) - O(1)$. The following theorem together with Crossing lemma shows, that the conjecture holds for graphs with $\Omega(n^2)$ edges.

Theorem 1.3.6 (Pach, Tóth [31]). *Let G be a connected graph on n vertices and $m \geq 1$ edges. For every edge $e \in E(G)$ we have*

$$\text{CR}(G \setminus e) \geq \text{CR}(G) - m + 1.$$

Proof. Take a drawing of G with the minimum number of crossings. The chosen edge e cross any other edge at most once by Lemma 1.3.5. Thus by removing e we loose at most $m - 1$ crossings. □

What if we want to delete more edges? Let us say some positive fraction. Some edges are more responsible for the crossing number, because some edge might contain more crossings than other edge. Thus Fox and Cs. Tóth [14] asked what is the minimum decrease of crossing number in the case, when the theorem can choose which edges will be deleted.

Theorem 1.3.7 (Fox, Cs. Tóth [14]). *For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\text{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \text{CR}(G).$$

We improved their result to the following.

Theorem 1.3.8 (Černý, Kynčl, Tóth [9]). *For every $\varepsilon, \gamma > 0$, there is an $n_{\varepsilon,\gamma}$ such that every graph G with $n(G) \geq n_{\varepsilon,\gamma}$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon\gamma}{2394}\right) m(G)$$

and

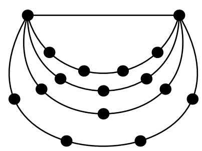
$$\text{CR}(G') \geq (1 - \gamma)\text{CR}(G).$$

1.3.3 Proof of the theorem on decay of $\mathrm{CR}(G)$

Because this is my thesis and I am co-author of the proof of Theorem 1.3.8, I show the proof here. The proof is taken from [9] almost without changes. It is quite simple proof and it also illustrate the use of embedding method.

The proof is based on the argument of Fox and Tóth [14], the only new ingredient is Lemma 1.3.10.

Definition 1.3.9. *Let $r \geq 2, p \geq 1$ be integers. A $2r$-earring of size p is a graph which is a union of an edge uv and p edge-disjoint paths between u and v, each of length at most $2r-1$. Edge uv is called the main edge of the $2r$-earring.*



5-earring of size 4

Lemma 1.3.10. *Let $r \geq 2, p \geq 1$ be integers. There exists n_0 such that every graph G with $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges contains at least $m/3pr$ edge-disjoint $2r$-earrings, each of size p.*

Proof. By the result of Alon, Hoory, and Linial [5], for some n_0, every graph with $n \geq n_0$ vertices and at least $n^{1+1/r}$ edges contains a cycle of length at most $2r$.

Suppose that G has $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges. Take a *maximal* edge-disjoint set $\{E_1, E_2, \ldots, E_x\}$ of $2r$-earrings, each of size p. Let $E = E_1 \cup E_2 \cup \cdots \cup E_x$, the set of all edges of the earrings and let $G' = G \setminus E$. Now let E'_1 be a $2r$-earring of G' of maximum size. Note that this size is less than p. Let $G'_1 = G' \setminus E'_1$. Similarly, let E'_2 be a $2r$-earring of G'_1 of maximum size and let $G'_2 = G'_1 \setminus E'_2$. Continue analogously, as long as there is a $2r$-earring in the remaining graph. We obtain the $2r$-earrings $E'_1, E'_2, \ldots, E'_y$, and the remaining graph $G'' = G'_y$ does not contain any $2r$-earring. Let $E' = E'_1 \cup E'_2 \cup \cdots \cup E'_y$.

We claim that $y < n^{1+1/r}$. Suppose on the contrary that $y \geq n^{1+1/r}$. Take the main edges of $E'_1, E'_2, \ldots, E'_y$. We have at least $n^{1+1/r}$ edges so by the result of Alon, Hoory, and Linial [5] some of them form a cycle C of length at most $2r$. Let i be the smallest index with the property that C contains the main edge of E'_i. Then C, together with E'_i would be a $2r$-earring of G'_{i-1} of greater size than E'_i, contradicting the maximality of E'_i.

Each of the earrings $E'_1, E'_2, \ldots, E'_y$ has at most $(p-1)(2r-1)+1$ edges so we have $|E'| \leq y(p-1)(2r-1) + y < (2pr-1)n^{1+1/r}$. The remaining graph, G'' does not contain any $2r$-earring, in particular, it does not contain any cycle of length at most $2r$, since it is a $2r$-earring of size one. Therefore, by [5], for the number of its edges we have $e(G'') < n^{1+1/r}$.

It follows that the set $E = \{E_1, E_2, \ldots, E_x\}$ contains at least $m - 2prn^{1+1/r} \geq \frac{2}{3}m$ edges. Each of $E_1, E_2, \ldots, E_x$ has at most $p(2r-1)+1 \leq 2pr$ edges, therefore, $x \geq m/3pr$. □

Lemma 1.3.11. Fox and Cs. Tóth [14] *Let G be a graph with n vertices, m edges, and degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Let ℓ be the integer such that $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. If n is large enough and $m = \Omega(n \log^2 n)$ then*

$$\text{CR}(G) \geq \frac{1}{65} \sum_{i=1}^{\ell} d_i^2.$$

Proof of the Theorem. Let $\varepsilon, \gamma \in (0,1)$ be fixed. Choose integers r, p such that $\frac{1}{r} < \varepsilon \leq \frac{2}{r}$ and $\frac{132}{p} < \gamma \leq \frac{133}{p}$. Then there is an $n_{\varepsilon,\gamma}$ with the following properties: (a) $n_{\varepsilon,\gamma} \geq n_0$ from Lemma 1.3.10, (b) $(n_{\varepsilon,\gamma})^{1+\varepsilon} > 6pr \cdot (n_{\varepsilon,\gamma})^{1+1/r}$,

Let G be a graph with $n \geq n_{\varepsilon,\gamma}$ vertices and $m \geq n^{1+\varepsilon}$ edges.

Let $v_1, \ldots, v_n$ be the vertices of G, of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ and define ℓ as in Lemma 1.3.11, that is, $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. Let G_0 be the subgraph of G induced by $v_1, \ldots, v_\ell$. Observe that G_0 has at least $m/3$ edges. Therefore, by Lemma 1.3.10 G_0 contains at least $m/9pr$ edge-disjoint $2r$-earrings, each of size p.

Let M be the set of the main edges of these $2r$-earrings. We have $|M| \geq m/9pr \geq \frac{\varepsilon\gamma}{2394} m$. Let $G' = G \setminus M$ and $G'_0 = G_0 \setminus M$.

Take an optimal drawing $D(G')$ of the subgraph $G' \subset G$. We have to draw the missing edges to obtain a drawing of G. Our method is a randomized variation of the embedding method. For every missing edge $e_i = u_i v_i \in M \subset G_0$, e_i is the deleted main edge of a $2r$-earring $E_i \subset G_0$. So there are p vertex-disjoint paths in G_0 from u_i to v_i. For each of these paths, draw a curve from u_i to v_i infinitesimally close to that path. Call these p curves *potential $u_i v_i$–edges* and call the resulting drawing D.

To get a drawing of G, for each $e_i = u_i v_i \in M$, choose one of the p potential $u_i v_i$–edges at random, independently and uniformly, with probability $1/p$, and draw the edge $u_i v_i$ as that curve.

There are two types of new crossings in the obtained drawing of G. First category crossings are infinitesimally close to a crossing in $D(G')$, second category crossings are infinitesimally close to a vertex of G_0 in $D(G')$.

The expected number of first category crossings is at most

$$\left(1 + \frac{2}{p} + \frac{1}{p^2}\right) \text{CR}(G') = \left(1 + \frac{1}{p}\right)^2 \text{CR}(G').$$

Indeed, for each edge of G', there can be at most one new edge drawn next to it, and that is drawn with probability at most $1/p$. Therefore, in the close neighborhood of a crossing in $D(G')$, the expected number of crossings is at most $(1 + \frac{2}{p} + \frac{1}{p^2})$. See figure 1.1(a).

In order to estimate the expected number of second category crossings, consider the drawing D near a vertex v_i of G_0. In the neighborhood of vertex v_i we have at most d_i original edges. Since we draw at most one potential edge along each original edge, there can be at most d_i potential edges in the neighborhood. Each potential edge can cross each original edge at most once, and any two potential edges can cross at most twice. See figure 1.1(b). Therefore, the total

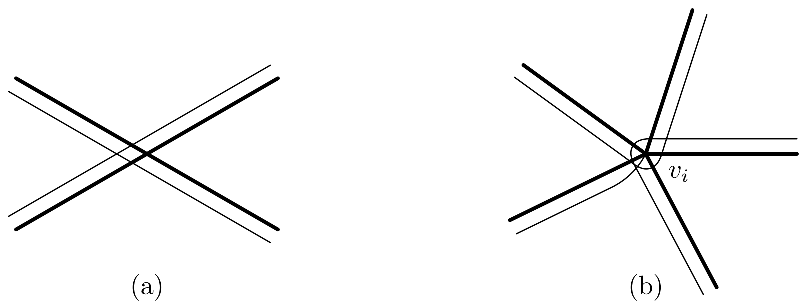


Figure 1.1: The thick edges are edges of G', the thin edges are the potential edges. Figure shows (a) a neighborhood of a crossing in $D(G')$ and (b) a neighborhood of a vertex v_i in G'.

number of first category crossings in D in the neighborhood of v_i is at most $2d_i^2$. (This bound can be substantially improved with a more careful argument, see e. g. [14], but we do not need anything better here.) To obtain the drawing of G, we keep each of the potential edges with probability $1/p$, so the expected number of crossings in the neighborhood of v_i is at most $\frac{1}{p}2d_i^2$.

Therefore, the total expected number of crossings in the random drawing of G is at most $(1+\frac{2}{p}+\frac{1}{p^2})\text{CR}(G')+\frac{2}{p}\sum_{i=1}^{\ell} d_i^2$.

There exists an embedding with at most this many crossings, therefore, by Lemma 1.3.11 we have

$$\text{CR}(G) \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G') + \frac{2}{p}\sum_{i=1}^{\ell} d_i^2 \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G') + \frac{130}{p}\text{CR}(G).$$

It follows that

$$\left(1-\frac{130}{p}\right)\text{CR}(G) \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G')$$

so

$$\left(1-\frac{130}{p}\right)\left(1-\frac{1}{p}\right)^2 \text{CR}(G) \leq \text{CR}(G')$$

consequently

$$\text{CR}(G') \geq \left(1-\frac{132}{p}\right)\text{CR}(G) \geq (1-\gamma)\text{CR}(G).$$

$\square$

1.4 Geometric graphs with no forbidden subgraphs

We investigate properties of subclasses of geometric or topological graphs with some geometrical constrains. One of the simplest questions is how to characterize graphs with no crossing edges. These graph are known as *plane* graphs and have been studied for more than hundred years.

Kupitz, Erdős and Perles initiated and many others continued in the investigation of the following general problem. Given a class $\mathcal{H}$ of so-called forbidden geometric subgraphs, determine or estimate the maximum number $t(\mathcal{H}, n)$ of edges that a geometric graph with n vertices can have without containing a subgraph belonging to $\mathcal{H}$.

There are many nice results for various forbidden classes—k pairwise disjoint edges, k pairwise "parallel" edges, k pairwise crossing edges, self-crossing paths, even cycles and many others. For a survey of results on geometric graphs see Pach [24], Felsner [13].

In the following subsections, we look at some forbidden classes more in detail.

1.4.1 $k+1$ pairwise disjoint edges

We focus on geometric graphs with no $k+1$ pairwise disjoint edges. For $k \geq 1$, let $\mathcal{D}_k$ denote the class of all geometric graphs consisting of k pairwise disjoint edges. Denote $d_k(n) = t(\mathcal{D}_{k+1}, n)$ the maximum number of edges of a geometric graph on n vertices with no $k+1$ pairwise disjoint edges.

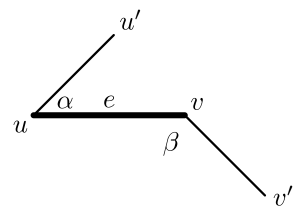
Let's look at the history of this problem. One of the first investigations on geometric graphs, besides planar graphs, was motivated by repeated distances in the plane. Erdős asked how many times can the maximum distance among n points in the plane be repeated. Connect each pair of points with the maximum distance by an edge. It's clear that the resulting graph cannot have two disjoint edges. Suppose on the contrary that there are two disjoint edges e, f. The convex hull of endpoints of e and f forms either a triangle or a quadrilateral. In both cases there is a distance longer than the length of the edge. That's a contradiction. The former question turns to the following: How many edges can have a geometric graph with no two disjoint edges? Erdős proved that at most n.

Theorem 1.4.1 (Erdős [10]). *$d_1(n) = n$.*

The proof is a beautiful illustration of a discharging method and it is very simple.

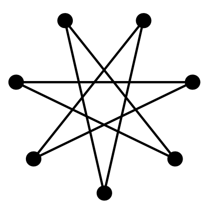
Proof (Perles). For each vertex, mark one edge incident to it. For vertices of degree one, there is no choice. At the other vertices, mark the left edge at the largest angle (For each such a vertex v, take a hen and place it close to v. The hen cannot sit in v nor on any edge incident to v, thus we place the hen to the wedge with largest angle. Then the hen looks from its place to v and marks the edge on the left).

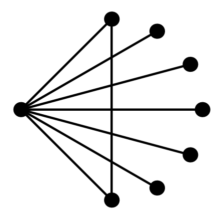
If there remain an unmarked edge $e = uv$, we have the situation as in the following figure:



There must be edges uu' and vv' because we marked the left edge at the largest angle at every vertex. Angles α and β are less than the largest angles at the vertices u, v, so they are less than π. But this implies that edges uu', vv' are disjoint and so there cannot be an unmarked edge in the graph. Thus, the number of edges is not more than the number of vertices.

On the other hand there exist graphs achieving this bound:





□

Definition 1.4.2.

- *A vertex v is pointed if all edges incident with it lie in a half-plane whose boundary contains the vertex v (see figure 1.2a).*
- *A vertex v which is not pointed is cyclic. This means that in every open half-plane determined by a line passing through the vertex v, there is an edge incident with v (see figure 1.2b).*
- *We say that an edge xy is to the left of an edge xz if the ray $\overrightarrow{xz}$ can be obtained from the ray $\overrightarrow{xy}$ by a clockwise turn of less than π. Similarly, we define that an edge is to the right of another edge (see figure 1.2c).*

The marking (deleting) idea from the proof of Theorem 1.4.1 can be generalized to marking in several rounds as in the proof of the following theorem.

Theorem 1.4.3. *$d_2(n) \leq 3n$.*

Proof (Goddard, Katchalski and Kleitman [15]). Let G be a geometric graph with no three pairwise disjoint edges. Set $G_0 = G$. We construct two subgraphs $G_i = (V, E_i)$ of graph G for $i \in \{1, 2\}$ as follows.

For each pointed vertex x in G_0 delete the leftmost edge at x. Denote the resulting graph by G_1. For each pointed vertex x in G_1 delete two rightmost edges at x (if any). Denote the resulting graph by G_2.

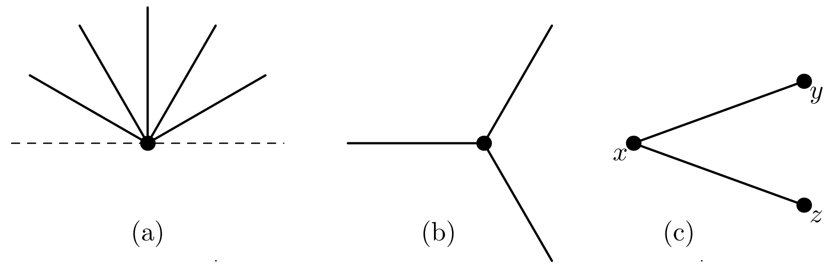
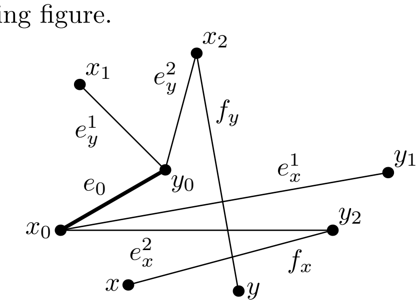


Figure 1.2: (a) an example of a pointed vertex, (b) an example of a cyclic vertex, (c) there are edges xy, xz where the edge xy is to the left of the edge xz.

Since we have deleted at most 3 edges at each vertex, we had deleted at most $3n$ edges. We want to show, that G_2 contains no edge. Assume for the contradiction that the edge $e_0 = x_0y_0$ was not deleted.

Then there exist two edges $e_x^1 = x_0y_1$, $e_x^2 = x_0y_2$ to the right of x_0y_0 and two edges $e_y^1 = y_0x_1$, $e_y^2 = y_0x_2$ to the right of y_0x_0. There is also an edge $f_x = x_2y$ to the left of x_2y_0 and there is also an edge $f_x = y_2x$ to the left of y_2x_0. See the following figure.



There are two cases. Either f_x, f_y are two disjoint edges, but then f_x, f_y, e_0 would be three disjoint edges, or f_x, f_y cross. In the second case, we may without loss of generality assume, that the supporting lines of f_x and f_y cross on the same side of e_0 as y_2. Then y_0x_2, x_0y_1, y_2x are three pairwise disjoint edges. Edges y_0x_2, y_2x are disjoint because they lie on opposite sides of x_0y_1. In both cases we got a contradiction. □

Theorem 1.4.4. *$d_2(n) \le 2.5n$.*

This upper bound is tight up to an additive constant. It is a nice exercise to find a lower bound showing $d_2(n) \ge \lceil 2.5n \rceil - 3$. Take an inspiration from the next figure.

Sketch of the proof (Černý [8]). Let $G = (V, E)$ be a geometric graph with no three disjoint edges. Denote the number of cyclic vertices in G by γ. Using case analysis it is not difficult to show that $\gamma \le 2$.

We construct two subgraphs $G_i = (V_i, E_i)$, $i = 1, 2$ as follows. For each pointed vertex in G delete the rightmost edge. Denote the resulting graph by

G_1. For each pointed vertex in G_1 delete the leftmost edge (if any). If there are two vertices c, d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of the segment cd. The vertex cyclic in G_1 must be one of the vertices c, d because $\gamma \leq 2$. Denote the resulting graph by G_2. Deleting in the second round is for each pointed vertex in G_1 (not in G)!

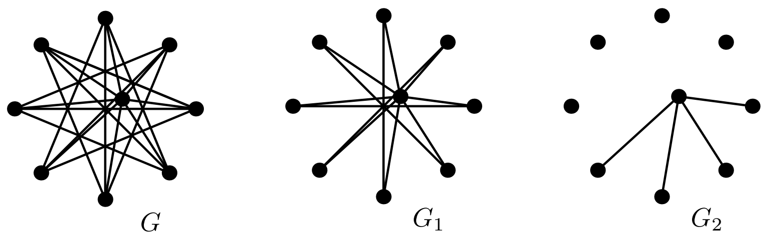


Figure 1.3: (Example) Graphs G, G_1 and G_2. In the first round delete the rightmost edge at each pointed vertex of G and obtain graph G_1. In the second round delete the leftmost edge at each pointed vertex of G_1 and if there are two vertices c, d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of the segment cd. We obtain graph G_2.

We have deleted at most $2n - \gamma$ edges to get the graph G_2. The deleting scheme reminds the one from the proof of Theorem 1.4.3, but one round is missing. Thus one can show that the graph G_2 contains no two disjoint edges (again by case analysis).

Finally, use a discharging method to show, that graph G_2 has at most $(n + 2\gamma)/2$ edges (as a surprise, again by case analysis). We then conclude that G has at most $(2n-\gamma)+(n+2\gamma)/2 \leq 2.5n$ edges. The use of discharging method again (as a surprise) need several lemmas, which can be proved by case analysis.

We do not prove the technical steps (case analysis), the reader can find it in the paper [8]. □

Theorem 1.4.5 (Pach-Törőcsik). *$d_k(n) \leq k^4 n$.*

Proof. This proof is again a nice application of binary relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$ and Dilworth theorem (see chapter **??**). Each relation compare some disjoint segments (edges of geometric graph). Each pair of disjoint edges is comparable by at least on of these binary relations.

Let G be a geometric graph. Edges $E(G)$ are segments in the plane. For fix $i \in \{1, 2, 3, 4\}$, there is no $k+1$ chain in partial ordering $\prec_i$, because it would correspond to $k+1$ disjoint edges. Further by Dilworth theorem, $E(G)$ can be partitioned into at most k classes so that two edges from the same class are not comparable by $\prec_i$.

By overlaying four partitions of $E(G)$ (for $i \in \{1, 2, 3, 4\}$) we obtain a decomposition of $E(G)$ into at most k^4 classes E_j $(1 \leq j \leq k^4)$. Two edges from the same class are comparable by none of $\prec_i$, thus they must cross. Hence by theorem 1.4.1 $|E_j| \leq n$. Therefore $|E(G)| = \sum_{j=1}^{k^4} |E_j| \leq k^4 n$. □

Historical notes and survey of result

Alon and Erdős (1989) proved $d_2(n) \le 6n$. One year later O'Donnell and Perles (1990) improved it to $d_2(n) \le 3.6n + c$. Later Goddard et al. [15] (1993) showed $d_2(n) \le 3n$. At the end, Mészáros [23] improved that to $d_2 \le 3n-1$. Combining some of the ideas of the proof of Goddard et al. [15] with a discharging method Černý [8] (2003) showed the upper bound $d_2(n) \le \lfloor 2.5n \rfloor$. The best known lower bound $d_2(n) \ge \lceil 2.5n \rceil - 3$ is an unpublished result of Perles.

For $d_3(n)$ Goddard et al. [15] showed $3.5n \le d_3(n) \le 10n$. Later Tóth and Valtr [40] improved that to $4n - 9 \le d_3(n) \le 8.5n$. The proof uses similar ideas as the presented proof of Theorem 1.4.3.

The first general upper bound $d_k(n) = O(n(\log n)^{k-3})$ was given again by Goddard et al. [15]. In 1993 Pach and Törőcsik [29] introduced the order relations on disjoint edges and as an application of Dilworth's Theorem they showed that $d_k(n) \le k^4 n$. That was the first upper bound linear in n. Tóth and Valtr [40] added a concept of zig-zag and improved the bound to $d_k(n) \le k^3(n+1)$. Later Tóth [38] further improved the bound to $d_k(n) \le 256k^2 n$. Original constant in Tóth's proof was a bit bigger. This one is due to Felsner [13].

It is believed that the truth is about $d_k(n) \sim ckn$. It is also an interesting problem, if this is true for geometric graphs whose edges can be intersected by a line. That would give general upper bound $d_k(n) \le c(k \log k)n$. Just bisect the vertex set of the graph and count edges in both parts recursively.

Kupitz [19] proved that for every k and $n \ge 2k + 1$, geometric graph on n vertices in convex positions with no $k + 1$ disjoint edges has at most kn edges. This bound is tight, because there are graphs with so many edges.

Open problem 2. *All previous results were for geometric graphs. What can you say for simple topological graphs with no $k + 1$ pairwise disjoint edges?*

1.4.2 k pairwise crossing edges

A topological graph G is *k-quasi-planar* if G contains no k pairwise crossing edges. For $k = 2$ we get planar graphs and for $k = 3$ quasi-planar graphs. For $k \ge 1$, let $\mathcal{C}_k$ denote the class of all topological graphs consisting of k pairwise crossing edges.[9] Let $c_k = t(\mathcal{C}_k, n)$ be the maximum number of edges of a topological graph on n vertices with no k pairwise crossing edges.

From the subsection on planar graphs follows, that every topological graph on n vertices with no 2 crossing edges has at most $3n - 6$ edges.

Theorem 1.4.6. *Every topological graph G on n vertices with no 3 pairwise crossing edges (quasi-planar graph) has at most $10n - 20$ edges.*

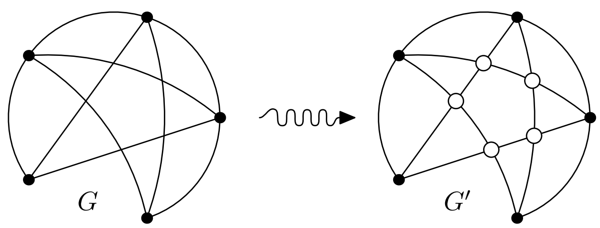
Proof (Ackerman and Tardos [2]). Let $G = (V, E)$ be a topological graph with n vertices and with no 3 pairwise crossing edges. Without loss of generality we might assume that G is a drawing with minimum $\textsc{cr}(G)$. Further we might assume that G is connected, otherwise we use the induction on components. We

[9] So $\mathcal{C}_k$ is a crossing family.

can assume that the minimum degree of a vertex is at least 5, otherwise delete such a vertex together with all edges incident with it, and use induction.

Making a plane graph G' from G: We start with the topological graph G. We add all crossing points X of edges E as a new vertices. Vertices X divides original edges of G into new edges of G'. Graph G' has two types of vertices – original vertices V and new vertices X. More precisely $G' = (V', E')$, where $V' = V \cup X$ and E' are parts of original edges E connecting 2 vertices of V' and not containing other vertex of V'.

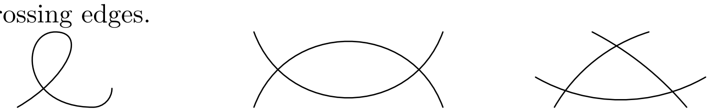
We use the notation that dashed variables correspond to the graph G' and undashed variables to the graph G. For example edge e' is an edge of G' and edge f is an edge of G.



There are no 3 edges crossing at one point, because they would be 3 pairwise crossing edges. Thus for every $x \in X$, $\deg_{G'}(x) = 4$. G' is plane graph, so we can denote its faces by F'. For a face $f \in F'$ define the size of f, denoted by $|f|$, as the number of edges on the boundary of f. (Let us note that some edge can appear on the boundary twice.) Further define $v(f)$ to be the number of original vertices on the boundary of f. (Similarly, some original vertex can be counted several times.)

We will need some more notation. Faces of size 3, 4, 5 are called triangle, quadrilateral, pentagon. As a shortcut for a face f with $v(f)$ original vertices on the boundary, we write simply $v(f)$-face f i.e. 1-triangle, 2-pentagon.

Graph G' contains no faces of size 1 or 2. Otherwise they correspond to a self-crossing edge or empty lens in G and thus we might reduce the number of crossings in G. G' also contains no 0-triangle, because it corresponds to 3 pairwise crossing edges.



self-crossing edge *empty lens* 0-*triangle*

Placing the charge: Each face $f \in F'$ receives a charge

$$ch(f) := |f| - v(f) - 4.$$

Now, we claim that the total charge on faces is $4n - 8$.

When we are adding the new vertices X to G one by one, then each new vertex (crossing point) adds exactly two "new" edges. Thus we have $|E'| = |E| + 2|X|$.

$$\sum_{f\in F'} ch(f) = \sum_{f\in F'} |f| - v(f) - 4 = 2|E'| - \sum_{v\in V} \deg(v) - 4|F'|.$$

By plugging $\sum_{v\in V}\deg(v) = 2|E| = 2|E'| - 4|X|$ into the previous formula we get $\sum_{f\in F'} ch(f) = 4(|E'|-|F'|-|X|) = 4(|V'|-|X|-2) = 4n-8$. The last but one equality holds by Euler formula, which is $|E'| - |F'| = |V'| - 2$. That finishes the proof of the claim.

Redistribution of the charge: We want to redistribute the charge so to get

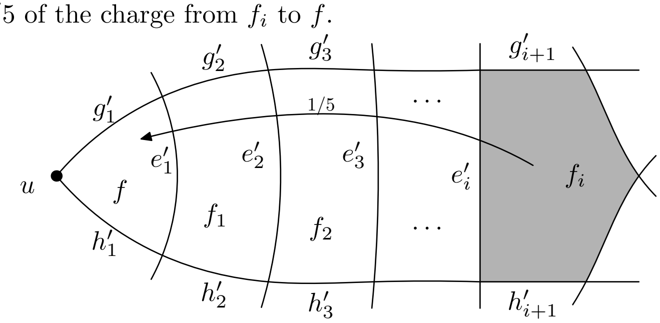
$$ch'(f) \geq v(f)/5 \tag{1.4}$$

for every $f \in F'$.

By definition of $ch(f)$, it already holds for every face f with $|f| \geq 4$. Moreover it holds for faces f which are k-triangles with $k \geq 2$, because $ch(f) = k-1$. There are no 0-triangles in G', thus the only problem is with 1-triangles.

Now we show how to recharge arbitrary 1-triangle f. 1-triangle f has one original vertex u and edges g'_1, h'_1, e'_1, such that g'_1, h'_1 are incident with u (see the next figure). Edges g'_1, h'_1 are parts of original edges $g, h \in E$. Put $f_0 := f$. Look at the face f_1 on the other side of e'_1. If f_1 has positive charge, we stop. Otherwise f_1 is 0-quadrilateral with edges e'_1, g'_2, e'_2, h'_2 such that $g'_2 \subset g$, $h'_2 \subset h$. Note that edge e'_1 has no original vertex as its end-point. Hence the face f_2 is not a triangle, because then f_2 would be necessary 1-triangle and therefore there would be multiple edges between two vertices of G.

But back to the case of 0-quadrilateral. Consider edge e'_2 instead of e'_1 and repeat the same idea. We repeat the idea until we find a face f_i with positive charge. We say that we recharge the 1-triangle f through edges e'_1, e'_2,... e'_i by sending 1/5 of the charge from f_i to f.



After recharging all 1-triangles, we get new charges $ch'(f_j)$ on faces. We have to check, that $ch'(f) \geq v(f)/5$ for all faces $f \in F'$. If f is 1-triangle, then $ch'(f) = 1/5$, as we wanted. If f is quadrilateral, then $v(f) \geq 1$, otherwise we did not change its charge and inequality (1.4) still holds. Such a quadrilateral can be charged at most twice, because we are recharging 1-triangles through edges with no original vertex. Thus $ch'(f) \geq 3/5 \cdot v(f)$. If $|f| \geq 5$ then f has lost at most $|f|/5$ of its charge. Thus $ch'(f) \geq 4/5 \cdot |f| + v(f) - 4 \geq v(f)$. Therefore inequality (1.4) holds.

Charging original vertices from faces: For every original vertex v we do the final recharging. Every face incident with original vertex v gives $1/5$ of its charge to v. Thus $ch(v) = 1/5 \cdot \deg(v)$ for every $v \in V$ and $ch''(f) \geq 0$ for every $f \in F'$ by inequality (1.4). Hence we get

$$4n - 8 \geq \sum_{f \in F'} ch''(f) + \sum_{v \in V} \deg(v)/5 \geq 2|E|/5.$$

That is $|E| \leq 10n - 20$. □

By further redistribution of charges we might get a better bound (as in [2]). Ackerman [1] used the same method to show $c_4 = O(n)$. But his proof contains more technical parts based on case analysis.

Open problem 3 (Pach [25]). *Is it true, that for any fixed k, $c_k = O(n)$?*

Historical notes and survey of result

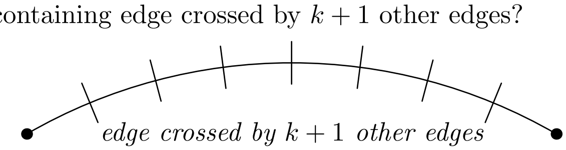
Pach [25] conjectured, that for every fixed k, $c_k = O(n)$. For $k = 2$ the conjecture is trivial, because in planar graphs $c_2 = |E| = 3n - 6$. Agarwal et. al [3] were first who proved that $c_3 = O(n)$. Later, Pach et. al. [27] simplified the proof and got better bound of $c_2 \leq 65n$. Recently Ackerman and Tardos [2] used discharging technique and showed $7n - O(1) \leq c_3 \leq 8n - 20$ and for simple topological graphs[10] they showed $c_3 = 6.5n - O(1)$. The last improvement was motivated by Ackerman [1], who showed $c_4 \leq 36(n-2) = O(n)$ using the same technique.

For general $k > 4$, the best upper bound is $O(n \log^{4k-16} n)$ as shown in [1]. For any fixed k, Valtr [41] obtained the upper bound $O(n \log n)$ for topological graphs with x-monotone edges.

Capoyleas and Pach [7] showed that every k-quasi-planar geometric graph on n vertices in convex position has at most $2(k-1)n - \binom{2k-1}{2}$ edges, which is tight.

1.4.3 $k+1$ crossings on an edge

What is the maximum number of edges that a simple topological graph can have with no containing edge crossed by $k+1$ other edges?



Gärtner, Thiele, and Ziegler showed a bound for $k = 1$. Pach and Tóth [30] generalized their result and showed the following theorem.

[10] Remark that the result distinguish between topological graphs and simple topological graphs. In other words, releasing the condition, that a pair of edges can cross at most once, allows us to get a graph with the same forbidden pattern, but with more edges.

Theorem 1.4.7 (Pach, Tóth [30]). *Let $G = (V, E)$ be a simple topological graph on n vertices with every edge crossed by at most k other edges. For $0 \le k \le 4$ we have $|E| \le (k+3)(n-2)$.*

Moreover they showed

Theorem 1.4.8 (Pach, Tóth [30]). *Let $G = (V, E)$ be a simple topological graph on n vertices with every edge crossed by at most k other edges. For $k \ge 1$ we have $|E| \le 4.108\sqrt{k}n$.*

These upper bounds leads to the improvement of the crossing lemma. In the folklore probabilistic proof of crossing lemma, we just make another simple estimates on $\text{CR}(G)$ using Theorem 1.4.8.

It can happen that many of the edges crossing one edge are emerging from the same vertex. Two edges are *independent* if they do not have a common end-vertex. Tóth and Pach asked what changes if the constrain is that G contains no edge crossed by $k+1$ independent edges.

Theorem 1.4.9 (Pinchasi). *Topological graph $G = (V, E)$ on n vertices with every edge crossed by at most k independent edges have at most $2.208 \cdot k^2 n$ edges.*

Proof. Take a maximum bipartite subgraph $H \subseteq G$. It is easy to show, that H has at least $|E(G)|/2$ edges.[11]

For every $e \in E(H)$, denote by $E_e \subseteq E(H)$ the set of edges crossing an edge e. Every edge $e \in E(H)$ is crossed by at most k independent edges. Thus E_e can be covered by at most k vertices (some of their endpoints) by classical König theorem.

We choose a random induced subgraph $H' \subseteq H$ by taking every vertex with probability p. We say that an edge e is *good* if it appears in H', but all of the edges E_e do not. An edge e is good with probability at least $p^2(1-p)^k$, because it suffice that both end-points of e are chosen and the k vertices covering E_e are not. That gives the first estimate

$$|E(H)| \cdot p^2(1-p)^k \le \mathbb{E}[\,\#\text{good edges}\,] \le 3n - 6$$

The second estimate holds because the subgraph of good edges is a planar graph. By taking $p = 1/k$ in the inequality we get $|E(H)| \le 3/e \cdot k^2 n$. □

[11] We start with sets A, B, which are empty at the beginning. Then in succesive steps, we place each vertex $v \in V$ either in A or B. For each vertex v, we choose such a set, that the number of edges leading from v to already placed veritices in the other set is maximal. Thus at least half of the edges from v to already placed vertices appears in the constructed bipartite graph.

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