Contents

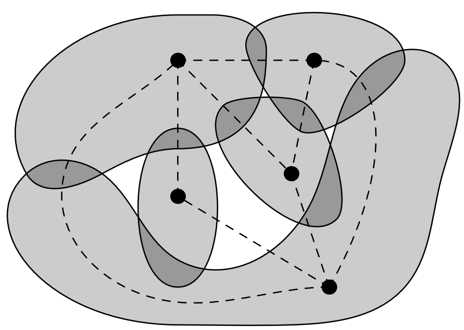
Chapter 1

Circle graphs

1.1 Intersection graphs

Intersection graph represents the pattern of intersection of a family of sets. The vertices of intersection graph are sets and two vertices are joint by an edge if and only if the sets intersects.

For example look at the following figure.



On the other hand every graph G can be represented as an intersection graph of some family of set. For every edge $e = \{u, v\}$ take witness $w_{\{u,v\}}$. As a set representing vertex v take $S_v := \{w_{\{x,v\}} \mid \{x, v\} \in E(G)\}$ for every $v \in V(G)$.

It is more interesting to investigate intersection graphs of some geometric objects. For various geometric objects we get various classes of intersection graphs.

Interval graphs (denoted by INT) are intersection graphs of intervals on a real line. *Circle-arc graphs* (CA) are intersection graphs of arcs on a circle. *Circle graphs* (CIR) are intersection graphs of chords of a circle. *Polygon-circle graphs* (POLYGON-CIR) are intersection graphs of convex polygons inscribed in a circle. *Chordal graphs* (CHOR) are intersection graphs of subtrees in a tree. *Function graphs* (FUN) are intersection graphs of functions on closed interval $[0, 1]$. *Permutation graphs* (PER) are intersection graphs of segments, which are linear functions on closed interval $[0, 1]$. *Segment graphs* (SEG) are intersection graphs of segments in the plane. *String graphs* (STRING) are intersection graphs of strings (connected curves) in the plane.[1] Graph has *boxicity* k if it is an intersection graph of k-dimensional boxes. *Line graph* $L(G)$ is an intersection graph of edges of a graph G, where by edge we mean set of two vertices. To close the list of graph classes we also mention *comparability graphs* (denoted by CO). Vertices of comparability graph are elements of poset and two elements are joint by an edge if and only if they are comparable.

If $\mathcal{A}$ is some class of graphs, then $\overline{\mathcal{A}} := \{\overline{G} \mid G \in \mathcal{A}\}$ is class of complements of graphs from $\mathcal{A}$.

A graph G is *perfect* if and only if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$. Perfect graphs are important because many problems like coloring, finding maximum clique, finding maximum independent set can be solved in polynomial time in perfect graphs. Many classes of intersection graphs are perfect graphs (INT, CHOR, CA, PER, line graphs for example).

There are two famous theorems about perfect graphs:

- **Perfect graph theorem** (Lovász 1972): graph is perfect if and only if its complement is perfect.
- **Strong perfect graph theorem** (Chudnovsky, Robertson, Seymour, Thomas 2002): An induced cycle of odd length at least 5 is called an odd hole. An induced subgraph that is complement of an odd hole is called an odd anti-hole. Graph is perfect if and only if it contains neither odd hole nor odd anti-hole.

The second theorem was called Berge conjecture, before it was proved.

Many classes of intersection graphs are well characterized. Many classes of intersection graphs (INT, CIR, CHOR) are recognizable in polynomial time. That means we can find the representation of the intersection graph in polynomial time. It is NP-complete for some other classes like STRING.

We list some basic results. Some of them are clear, other need some work. $\text{PER} = \overline{\text{PER}}$, $\text{INT} = \text{CHOR} \cap \overline{\text{CO}}$, $\text{INT} \subset \text{CA}$, $\text{CIR} \subset \text{POLYGON-CIR}$, $\text{PER} \subset \text{FUN}$, $\text{PER} \subset \text{CIR} \subset \text{SEG} \subset \text{STRING}$.

Interesting recent result is $\text{PLANAR} \subseteq \text{SEG}$. In other words, every planar graph can be represented as an intersection graph of segments in the plane. For a proof see [2].

For an overview of results on intersection graphs see for example [8].

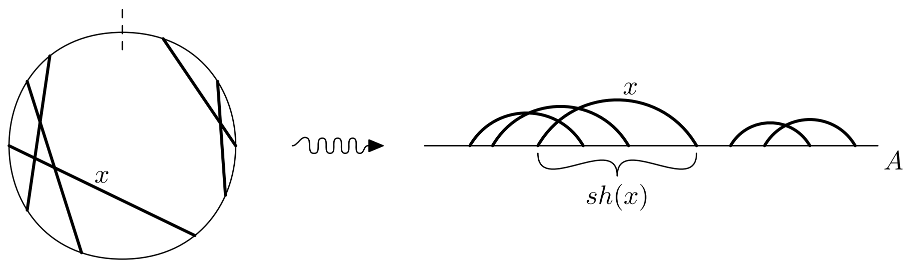
1.2 Circle graphs

Circle graphs (denoted by CIR) are intersection graphs of chords of a circle. We assume that no two chords have a common endpoint.

[1] String graphs are sometimes called "spaghetti" graphs, because they look like spaghetties on the plane (or on the plate?).

Circle graphs can be equivalently defined as *overlap graphs* of intervals on a line. Two intervals overlap if they intersect, but none of them is subinterval of the other. Overlap graph is a graph whose vertices are intervals and two vertices are joined by an edge if the corresponding intervals overlap.

Another representation is somewhere in between. We start with the chords of a circle. Cut the supporting circle at one point to get the arc A. The arc A is called the *supporting arc.* Then imagine, that the circle and chords are from a rubber. Unwrap the cut circle to a line as shown on the following figure.



We will often use this *unwrapped representation*, because it is suitable for drawing figures and also it enable us to speak about things like on the left, on the right, under a chord. We number the endpoints of chords along the arc A. The direction from smaller numbers to larger numbers is to the right, the opposite direction is to the left.

Let $G \in$ CIR and $V = V(G)$ be its set of chords. *Shadow* $sh(s)$ of chord $s \in V$ is the part of the arc A lying between the endpoints of s. We say that $B \subseteq A$ *lies under* the chords $W \subset V$ if $B \subseteq \bigcap_{s \in W} sh(s)$. Particularly a point p lies under the chord s if $p \in sh(s)$. Sometimes we say B *pierces* W instead of B lies under W. It has the same meaning, but it might be more expressible.

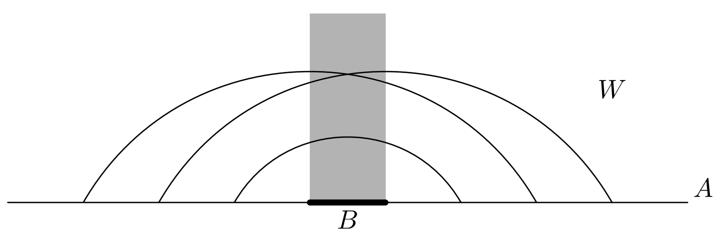


Figure 1.1: Arc $B \subseteq A$ pierces chords W (all chords in the figure). Conversely we say, that W covers B.

Conversely we say that $W \subset V$ *covers* an arc $B \subseteq A$ if and only if B lies under W. Denote by $V|B$ the chords induced by arc B. That are chords with both endpoints in B. Chord v lies under the chord w if $sh(v) \subset sh(w)$. Remind that the chords cannot have a common endpoint.

If A is an arc and $P \subset A$ then $conv\{P\}$ is the smallest arc $B \subseteq A$ containing P (in unwrapped representation).

1.3 Coloring circle graphs

We denote the clique and the chromatic number of the graph G by $\omega(G)$, $\chi(G)$, respectively. Obviously, $\chi(G) \geq \omega(G)$. Gyarfás [4],[5] defined the notion of binding function: f is a binding function for χ and for the class of graphs $\mathcal{A}$ iff $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{A}$. The main question of this paper is to determine or estimate the following binding function

$$f(k) = \max\{\, \chi(G) \mid G \in \textsc{cir} \ \&\ \omega(G) \leq k\}.$$

Clearly $f(1) = 1$. Ageev [1] showed that $f(2) = 5$. Kostochka and Kratochvíl [7] showed $f(k) \leq 50 \cdot 2^k - 32k - 64$ for general k. The best lower bound is $f(\omega) = \Omega(\omega \log \omega)$ due to Kostochka [6]. (See Section 1.4 of this thesis for the proof of lower bound). We were told by several people that many scientists tried to improve the upper bound, but they were not able to decrease it even by an additive constant. In this paper we modify the proof of Kostochka and Kratochvíl and get a slightly better upper bound

$$f(k) \leq 16 \cdot 2^k - 16k - 16.$$

In the discussion with Jan Kára, we observed $\chi(G) \leq \omega\lceil \log n \rceil$, where n is the number of vertices of the graph. This upper bound shows, that the circle graphs with small maximum clique and large chromatic number must be very large.

Esperet and Ochem [3] have proven an interesting observation: every circle graph G with girth $g \geq 5$ contains either a vertex of degree at most one, or a chain of $g - 4$ vertices of degree 2. That implies the result of Ageev for $g = 5$.

1.3.1 The proof of the upper bound

This section presents the proof of the following theorem.

Theorem 1.3.1. *$f(k) \leq 16 \cdot 2^k - 16k - 16$ for $k \geq 1$.*

In all proofs in this section, we use the unwrapped representation of circle graphs. First we prove the following lemma, which is one of the basic tools for coloring chords.

Lemma 1.3.2. *Let $G \in \textsc{cir}$, $p \in A$ point on the supporting arc. If $W \subseteq V(G)$ is a set of chords pierced by p then W can be colored by $\omega(G)$ colors.*

Proof. For any two disjoint chords, one must be under the other. That is a partial ordering on disjoint chords. Any two disjoint chords are comparable, thus incomparable chords form a clique. By Dilworth theorem all chords can be covered by at most $\omega(G)$ chains, which are the color classes. □

The trick of the main proof is to add something what looks useless at first sight. During the coloring of a circle graph, we work with sub-arcs of the supporting arc A.

Let $G = (V, E)$ be the circle graph and A supporting arc of its representation. Let $\mathcal{C}$ be a family $\mathcal{C}$ of sub-arcs of A. The family $\mathcal{C}$ *pierces* the set of chords $W \subseteq V$ if and only if

i) $C_1 \cap C_2 = \emptyset$ for every $C_1, C_2 \in \mathcal{C}$ and

ii) for every chord $w \in W$ there is an arc $C_w \in \mathcal{C}$ piercing w.

In other words, the arcs must be disjoint and every chord must be pierced by some arc. We denote by $\overline{\omega}(V, \mathcal{C})$ the maximum size of a clique $Q \subseteq V$ pierced by some arc $C \in \mathcal{C}$.

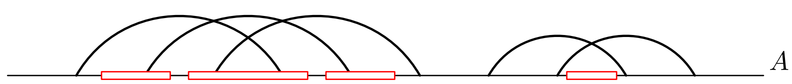


Figure 1.2: Here is an example. The set V contains 5 chords and the family $\mathcal{C}$ contains four disjoint arcs. Family $\mathcal{C}$ pierces V, $\overline{\omega}(V, \mathcal{C}) = 2$ but $\omega(G) = 3$.

Note that $\omega(G)$ can be three times larger than $\overline{\omega}(V, \mathcal{C})$. For example look at the left component in the figure 1.2.

It is easy to find a family of arcs piercing V. We can for example choose $\mathcal{C}$ as the endpoints of all chords.[2] Another option is the set of points of A lying between two consecutive endpoints of chords.

We define an auxiliary binding function

$$g(k) = \max\{\, \chi(G) \mid \quad G \in \textsc{cir} \text{ and there exists } \mathcal{C} \text{ piercing } V(G) \text{ such that } \overline{\omega}(V(G), \mathcal{C}) \le k\},$$

where $\mathcal{C}$ is set of sub-arcs of the supporting arc.

Clearly $f(k) \le g(k)$. If there is no $k+1$ clique in the graph, then there is no $k+1$ clique pierced by an arc. Moreover we show $f(k) \le 2g(k-1)$ in Lemma 1.3.8. Hence it is sufficient to show the upper bound on $g(k)$.

Theorem 1.3.3. *$g(k) \le 16 \cdot 2^k - 8k - 16$ for $k \ge 1$.*

For the proof of Theorem 1.3.3 we first show the recurrence $g(k) \le 2 \cdot (g(k-1) + 4k)$ (Lemma 1.3.4). The theorem follows from the recurrence by induction.

Let us look at Theorem 1.3.3 from algorithmic point of view. We want to color a circle graph $G = (V, E)$ such that there are arcs $\mathcal{C}$ piercing G with $\overline{\omega}(V, \mathcal{C})$. We find the coloring by recursive procedure `color(`$V, \mathcal{C}$`)`. Parameter V is a subset of chords and $\mathcal{C}$ is a set of sub-arcs of the supporting arc A, which is piercing V. Following Lemma 1.3.4 corresponds to one call of the procedure. Recursive calls corresponds to the proof by induction. At the beginning of the algorithm $\overline{\omega}(V, \mathcal{C})$ is at most $\omega(G)$. For each recursive call `Color(`$W, \mathcal{D}$`)`, where $W \subseteq V$, we decrease $\overline{\omega}$. Thus the depth of the recursion is $\overline{\omega}(V, \mathcal{C})$.

[2] Remind that according to the definition the endpoints of a chord *ch* lie under *ch*. It is just technical detail which comes useful in implementation of the coloring.

Lemma 1.3.4. *$g(k) \leq 2 \cdot (g(k-1) + 4k)$ for $k \geq 1$.*

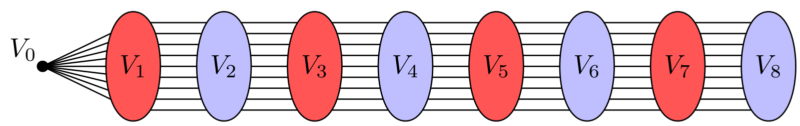
Proof. Let $G = (V, E)$ be the circle graph and $\mathcal{C}$ be the set of sub-arc of A witnessing that $\overline{\omega}(V, \mathcal{C}) \leq k$. We want to color the vertices V.

We find the coloring by recursive procedure `color(`V`, `$\mathcal{C}$`)`. Parameter V is a subset of chords and $\mathcal{C}$ is a set of sub-arcs of the supporting arc A, which is piercing V. Now we are ready to sketch the idea of the coloring.

`Color(`$V,\mathcal{C}$`):`

- We can color all connected components by the same colors so assume that V are the vertices of one connected component.

- **BFS layers:** Take the chord $s \in V$ incident with the leftmost endpoint on A and run breadth first search (BFS) starting at s to get BFS layers $V_0, V_1, \ldots, V_t$. That is $V_0 = \{s\}$ and chords in layer V_i are the chords in distance i from s in G. Chords V_i cross only chords from V_{i-1} and V_{i+1}. So we can alternate two color sets, one for even layers and the other for odd layers.



- **Coloring layer V_i:** Color all connected components of V_i independently one by one. So we can assume that V_i is one connected component.

  First, we construct a new set $\mathcal{D}$ of disjoint sub-arcs of A. We split V_i into U_1 and U_2. The chords U_1 are chords pierced by some arc of $\mathcal{D}$ and chords U_2 are not.

  The chords U_1 can be colored recursively by calling `color(`U_1`,`$\mathcal{D}$`)`, because $\mathcal{D}$ pierces U_1. The recursion is possible because $\overline{\omega}(U_1, \mathcal{D}) < \overline{\omega}(V, \mathcal{C})$ by the construction of $\mathcal{D}$ (details are in Claim C1).

  The chords U_2 can be colored directly by $4 \cdot \overline{\omega}(V, \mathcal{C})$ colors (details are in Claim C2).

Now, we can advance to the **detailed proof.** First two steps of the procedure `Color` are clear. Let us look at the third point. Coloring of layers V_0 and V_1 is easy because V_0 contains only one chord and V_1 can be colored by Lemma 1.3.2. We want to color the layer V_i for $i \geq 2$ (assume that it has one component). Take the smallest arc $B \subseteq A$ containing all endpoints of chords V_i and let $P_B = B \cap \{\text{endpoints of } V_{i-1}\}$.

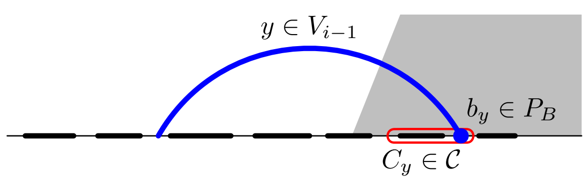
Claim 1.3.5. *Every chord $x \in V_i$ crosses a chord $y \in V_{i-1}$ for $i \geq 2$. Hence x covers a point $b \in P_B$, the endpoint of y. Moreover the second endpoint of y lies outside B.*

Every chord $x \in V_i$ has a neighbor $y \in V_{i-1}$ in the graph G. Being neighbors means that x crosses y and thus one endpoint of y must lie under x. No chord $x \in V_i$ can cross a chord $z \in V_0 \cup V_1 \cup \cdots \cup V_{i-2}$, because then x would be in distance $j < i$ from $s \in V_0$ and x would be in layer V_j.

Chords $V_0 \cup V_1 \cup \cdots \cup V_{i-2}$ are cutting the supporting circle into convex regions $\mathcal{R}$. Chords V_i lie fully in the regions $\mathcal{R}$. A chord $y \in V_{i-1}$ cannot lie completely in a convex region $R \in \mathcal{R}$, because it would not cross any chord of V_{i-2}. Thus the endpoints of chords V_{i-1} lies in different regions.

Figure 1.3: *On the left:* The black segments $V_0 \cup V_1 \cup \cdots \cup V_{i-2}$ cut the circle into regions $\mathcal{R}$. Dashed segments V_{i-1} must cross some black segment. *On the right:* Dashed segments of a component of V_i lie fully in region $R \in \mathcal{R}$. Grey segments V_{i-1} have at most one point in R.

Definition of new family of disjoint arcs: For every chord $y \in V_{i-1}$ with an endpoint $b_y \in P_B$, there is an arc $C_y \in \mathcal{C}$ lying under y. It holds because $\mathcal{C}$ pierces V. From all possible arcs lying under y we choose C_y to be the one, which is closest to b_y.



For every chord $y \in V_{i-1}$ with endpoint $b_y \in P_B$ and arc C_y define a new arc D as $conv\{C_y \cup \{b_y\}\}$ i.e. the shortest arc containing both C and b_y. Denote the set of all new arcs by $\mathcal{D}^*$. Choose $\mathcal{D} \subseteq \mathcal{D}^*$ to be the maximum subset of disjoint arcs such that their right ends are leftmost possible.

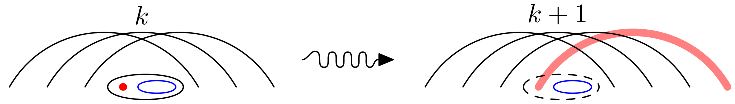
Further in this proof, $\mathcal{C}$ denotes the original arcs and $\mathcal{D}$ denotes the newly defined arcs.

Now we partition the set V_i into two sets U_1 and U_2. U_1 are the chords which are pierced by an arc from $\mathcal{D}$ and U_2 are the chords which are not.

We claim that $\chi(U_1) \leq g(k-1)$ (Claim C1) and $\chi(U_2) \leq 4k$ (Claim C2). It remains to prove the claims and that will finish the proof. □

Claim 1.3.6 (C1). *Arcs $\mathcal{D}$ pierce U_1 and $\overline{\omega}(U_1, \mathcal{D}) = k-1$. Hence $\chi(U_1) \leq g(k-1)$.*

Proof. Set $\mathcal{D}$ pierces U_1 by the choice of U_1. Suppose on the contrary that $Q \subset U_1$ is a clique of size k covering $D \in \mathcal{D}$. By the definition of new arcs, D consists of a point b_x and an original arc $C \in \mathcal{C}$. Point b_x is an endpoint of chord $x \in V_{i-1}$. Chord x crosses all chords in the clique Q because its other endpoint lies outside B. Arc C lies under all chords of Q and also under x. So $Q \cup \{x\}$ is a clique in V covering C and its size is $k+1$. That is a contradiction.



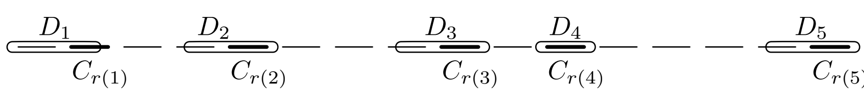
□

Claim 1.3.7 (C2). $\chi(U_2) \leq 4k$.

Proof. This proof is easy but technical. We want to color chords U_2, which are pierced by arcs $\mathcal{C}$. There is no chord of U_2 covering a new arc from $\mathcal{D}$. That brings us to the idea that there is no "long" chord in U_2 and thus all chords lie in the "gaps" between arcs $\mathcal{D}$. In fact their endpoints can lie on arcs $\mathcal{D}$ too. That makes the coloring difficult because such "gaps" overlap and chords from different gaps can cross above the arc of $\mathcal{D}$.

For $C \in \mathcal{C}$ define $S(C) = \{x \in U_2 \mid C \subseteq sh(x)\}$, what are the chords of U_2 pierced by the arc C. Similarly, we can define $S(p)$ for a point $p \in A$. Remind that by Lemma 1.3.2 the chords $S(C)$ can be colored by $\overline{\omega}(V, \mathcal{C})$ colors, which is equal to k. That will be our coloring tool in the proof of the claim.

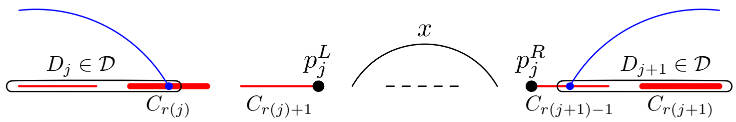
Every new arc $D \in \mathcal{D}$ is a convex hull of one original arc $C' \in \mathcal{C}$ and a point $b \in P_B$. The point b can lie on another arc $C'' \in \mathcal{C}$. There is no arc of $\mathcal{C}$ lying between C' and C'' because of the definition of new arcs $\mathcal{D}$. We conclude that every new arc D intersects at most two original arcs from $\mathcal{C}$. Denote the arcs of $\mathcal{C}$ by C_1, C_2,..., C_q as they appear from the left to the right. Similarly denote the new arcs of $\mathcal{D}$ by D_1, D_2,..., D_m. For $j \in \{1, \dots, m\}$, $r(j)$ is the index of the rightmost arc of $\mathcal{C}$ intersecting D_j. So $C_{r(j)}$ is that arc. See an example on the next figure. From technical reasons, we define $r(0) = 0$ and $r(m+1) = q+1$. You can imagine that as adding virtual arcs C_0 and C_{q+1} to the left and to the right of arcs $\mathcal{C}$.



Put $W := U_2 \setminus (\bigcup_{i=1}^{m} S(C_{r(i)}))$. That are all chords of U_2, which are not pierced by any arc $C_{r(j)}$ for $j \in \{1, \dots, m\}$. Now, we split W into sets W_j for $j \in \{0, \dots, m\}$. The set $W_j \subseteq U_2$ is induced by the arc $conv\{C_{r(j)}, C_{r(j+1)}\}$.

For every j, define p_j^L as the right endpoint of $C_{r(j)+1}$ and p_j^R as the left endpoint of $C_{r(j+1)-1}$ (see the next figure). If $C_{r(j)+1} = C_{r(j+1)-1}$ (there is exactly one arc of $\mathcal{C}$ between $C_{r(j)}$, $C_{r(j+1)}$), then define $p_j^R := p_j^L$. Moreover in case $C_{r(j)+1} = C_{r(j+1)}$ (there is no arc of $\mathcal{C}$ between $C_{r(j)}$, $C_{r(j+1)}$), define the points $p_j^L = p_j^R$ as arbitrary point on A lying between $C_{r(j)}$, $C_{r(j+1)}$.

Now we claim that each chord of W_j is pierced by at least one of the points p_j^L, p_j^R.



Suppose on the contrary that there is a chord $x \in W_j$ which is pierced neither by p_j^L nor by p_j^R. Chord x must be pierced by some arc of $\mathcal{C}$. It cannot lie to the left of p_j^L, because it is not pierced neither by $C_{r(j)}$ (otherwise $x \notin W$) nor by $C_{r(j)+1}$ (otherwise it is pierced by p_j^L). Symmetricaly, it cannot lie to the right of p_j^R. Thus it must lie between p_j^L, p_j^R. There is a point $b_x \in P_B$ under x. Point b_x together with either $C_{r(j)+1}$ or $C_{r(j+1)-1}$ forms a new arc D', which had to be considered during the choice of $\mathcal{D}$. The arc D' can be either added to $\mathcal{D}$ to enlarge it or it can replace arc D_{j+1}, because its right endpoint is more to the left. That is the contradiction with the definition of $\mathcal{D}$.

By now, we have shown, that each chord of U_2 is piereced by $C_{r(j)}$ or p_j^L or p_j^R for some $j \in \{0, 1, \ldots, m, m+1\}$. Now we proceed to the coloring step.

We color $\bigcup_{j=0}^{m+1} S(p_j^L)$ by $2k$ colors. We alternate two color sets. One for $S(p_0^L)$, second for $S(p_1^L)$, again the first one for $S(p_2^L)$, and so on.

Chord $x \in S(p_i^L)$ has the endpoints on $conv\{D_i, D_{i+1}\}$, otherwise it would be pierced by D_i or D_{i+1}. Hence two chords of the same color cannot cross.

Points p_i^L for $i \in \{0, 1, \ldots, m, m+1\}$ split the remaining uncolored chords into strips. Strips are well separated by the points p_i^L. Each strip contains only chords which are pierced by p_i^R or $C_{r(i+1)}$. Hence chords in each strip can be colored by $2k$ colors. Therefore we have colored all chords U_2 by $4k$ colors. That finishes the proof. □

Lemma 1.3.8. $f(k) \leq 2g(k-1)$.

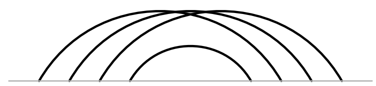
Proof. Let $G = (V, E)$ be a circle graph with $\omega(G) \leq k$. Put $\mathcal{C}$ to be a set of endpoints of all chords V. Now we follow the proof of Lemma 1.3.4, what you can imagine as a call of `Color(V,C)`. For every layer V_i we find points P_B such that every chord $x \in V_i$ has a point $b \in P_B$ under it. Each new arc will consist of only one point $b \in P_B$. Thus $U_2 = \emptyset$ in this case. □

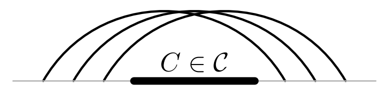
Proof of Theorem 1.3.3. For $k = 0$ is $g(0) = 0$ and the Lemma holds. For $k \geq 1$ we apply the recurrence from Lemma 1.3.4 and get $g(k) \leq 2 \cdot (16 \cdot 2^{k-1} - 8(k-1) - 16) + 8k = 16 \cdot 2^k - 8k - 16$. □

Proof of Theorem 1.3.1. By combination of Lemma 1.3.8 and Theorem 1.3.3 we get $f(k) \leq 2g(k-1) = 16 \cdot 2^k - 16k - 16$. □

If we find a better estimate on $g(k)$ for some small k, we can use it in the proof of Theorem 1.3.3 and further improve the upper bound.[3]

Note: What is the main difference between this proof and the proof from [7]? In fact the basic idea is the same. Kratochvíl and Kostochka were coloring circle graphs $\mathcal{H}(k)$ with no chord lying under a k-clique. We are coloring circle graph $G \in \mathcal{G}(k)$ for which there exists a set of arcs $\mathcal{C}$ piercing $V(G)$ with no arc $C \in \mathcal{C}$ lying under a k-clique. The forbidden configurations are on the following figures.



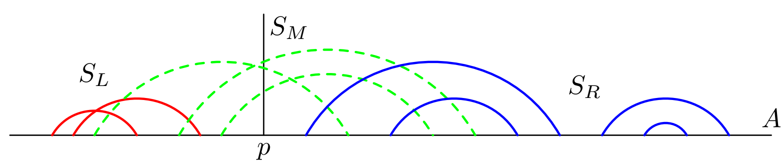


1.3.2 Second upper bound for $\chi(G)$

During the discussion with Jan Kára we found the following bound.

Lemma 1.3.9. *Any graph $G \in$ CIR can be colored by $\chi(G) \leq \omega\lceil \log n \rceil$ colors, where $\omega = \omega(G)$.*

Proof. We use the method divide et empera. First we split the problem into subproblems and then solve them recursively. The input of our problem is an arc A and n chords S with endpoints on A. For any point $p \in A$, we can split S into S_L (the chords to the left of p), S_M (chords whose shadow contains p) and S_R (the chords to the right of p).



By starting on the left end of A and going to the right we find a point $p \in A$ such that $S_L \leq n/2$ and $S_R \leq n/2$. That's easy because during the movement we can only add chords to S_L or remove chords from S_R. By Lemma 1.3.2 we color chords S_M by ω colors and we color S_L, S_R recursively. Note that S_L can be colored by the same colors as S_R. So in the second iteration we will need only another ω colors. The sets S_L, S_R become empty after at most $\lceil \log n \rceil$ iterations and the algorithm stops. Thus we had colored all chords with at most $\omega\lceil \log n \rceil$ colors. □

1.3.3 Generalization to polygon-circle graphs

All previous results can be generalized for polygon-circle graphs. The same proofs works. It is not a surprise because our proof is just a modification

[3] Theorem 1.3.3 gives us $g(1) \leq 8$. The lower bound is 3, because of 5-cycle C_5. We cannot show anything better.

of former proof by Kratochvíl and Kostochka [7], which is for polygon-circle graphs. We decided to present the proofs just for circle graphs to make it more simple.

1.4 The lower bounds

The aim of this section is to prove the following two theorems (Theorem 1.4.1 and Theorem 1.4.2). They were originally proven by Kostochka in [6]. His paper is in Russian, it was not reproduced in English and thus almost no one knows the proof. That is why we decided to reproduce the proofs in English.[4]

Theorem 1.4.1. *There is a circle graph G with $\alpha(G) \leq k$ and $\omega(G) \leq k$ having $\Omega(k^2 \log k)$ vertices.*

We use the simple bound $\chi(G) \geq |V(G)|/\alpha(G)$ to get the following theorem.

Theorem 1.4.2. *There is a circle graph G with $\chi(G) = \Omega(\omega \log \omega)$.*

Let C be a circle in the plane. Take $kt+2$ points P on the circle and denote them by numbers $0,\dots, kt+1$ in the order as they appear on C. Later, we will be counting with the points P modulo $kt+2$.

By $[i, j]$ we denote the segment connecting points $i, j \in P$. Segment $[i, j]$ splits the circle C into two arcs γ_1, γ_2. Let l_1, resp. l_2, be the number of points of P on γ_1, resp. γ_2. Then define the length of the segment $[i, j]$ as $\min\{l_1, l_2\}$.

Consider the following set of segments (chords of C):

$$H_i(k,t) := \{\, [j, j+it+1] \mid j \in \{0,\dots,kt+1\}\,\}$$
$$H(k,t) := \bigcup_{i=1}^{\lfloor k/2 \rfloor} H_i(k,t).$$

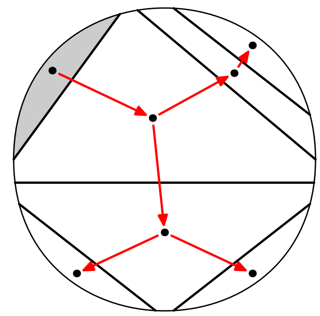
We say that $H_i(k,t)$ is the i-th layer of $H(k,t)$. All segments in one layer have the same length. For $i < k/2$, $|H_i(k,t)| = kt+2$, but for even k, $|H_{k/2}(k,t)| = (kt+2)/2$. The layers are disjoint. Hence $H(k,t)$ has at least $\lfloor (k-1)/2 \rfloor (kt+2)$ segments.

Let $G(k,t)$ be the intersection graph of $H(k,t)$. For a set R in the plane, ∂R denotes the boundary of R.

[4] The proofs might be different from original proofs, because I do not speak Russian enough to understand them.

Lemma 1.4.3. *Every subgraph $G' \subseteq G(k,t))$ has $\alpha(G') \leq k-1$*

Proof. Let $I \subseteq H(k,t)$ be an independent set in G'. All segments I cuts the circle into regions $\mathcal{R}$. Consider a graph G, whose vertices are the regions $\mathcal{R}$ and two regions are joint by an edge if they are neighbors. Graph G is a tree. Choose one leaf of the tree as a root and orient all edges away from root.



Every segment $s \in I$ cuts the circle C into two parts. Denote the part containing root region by $R(s)^+$ and the other part by $R(s)^-$.

Claim 1.4.4. *Boundary of every region $R \in \mathcal{R}$ contains at least $t+2$ points from P.*

First, let $R_r \in \mathcal{R}$ be the root region and r be the segment separating R_r. From the definition of $H(t,k)$ segment r has length at least t. Thus ∂R_r contains at least t points plus two endpoints of r.

Now consider $R \in \mathcal{R} \setminus \{R_r\}$. Let $S \subseteq I$ are the segments contained in ∂R. Let $r \in S$ be the segment separating R from the root region. Part $R(r)^-$ contains pt points for some natural number p. Regions $R(s)^-$ for $s \in S \setminus \{r\}$ are uncutting qt points from $R(r)^-$, for some natural number q. Since there are also the endpoints of $S \setminus \{r\}$ in $R(r)^-$ or because the minimum length of segment is t, we have $p > q$. Hence there remain at least t points of P in ∂R plus the endpoints of r.

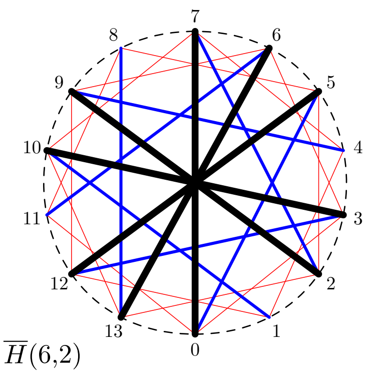
Let n be the number of regions in $\mathcal{R}$. Now we show that the union of ∂R for $R \in \mathcal{R}$ contains at least $nt+2$ points of P. We start with the root region, which has $t+2$ points of P, and we are adding neighboring regions one by one. Always when we add a neighboring region of already added regions, we add at least t new points of P. (Two points, endpoints of separating segment, were already counted). P has $kt+2$ points, thus $n \leq k$. Therefore there are at most $k-1$ cutting segments. □

We like the property $\alpha(G') \leq k$ for every subgraph $G' \subseteq G(k,t)$. On the other hand $G(k,t)$ has a large maximum clique. Thus we want to find a subgraph of $G(k,t)$, which has the size of maximum clique $O(k+t)$.

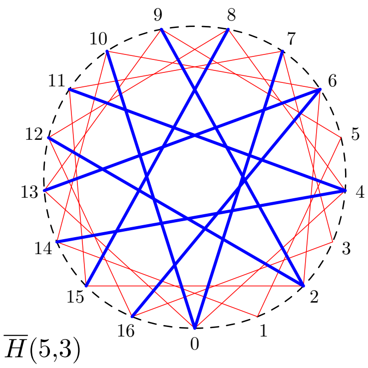
$$\overline{H}_i(k,t) := \{\, [j, j+it+1] \mid j \in \{0, i, 2i, \ldots, \lfloor (kt+1)/i \rfloor i\} \,\}$$

$$\overline{H}(k,t) := \bigcup_{i=1}^{\lfloor k/2 \rfloor} \overline{H}_i(k,t)$$

Look at the examples on the next figures.



$\overline{H}$(6,2)



$\overline{H}$(5,3)

In each layer $\overline{H}_i(k,t)$, we took only every i-th segment of $H_i(k,t)$. Every layer $\overline{H}_i(k,t)$ with $i < k/2$ contains $\lfloor (kt+1)/i \rfloor + 1\}$ segments. The last layer $\overline{H}_{k/2}(k,t)$ for k even contains a little less than $\lfloor (kt+1)/(k/2) \rfloor + 1$ segments. For simplicity, we just estimate the number of segments in $\overline{H}_i(k,t)$.

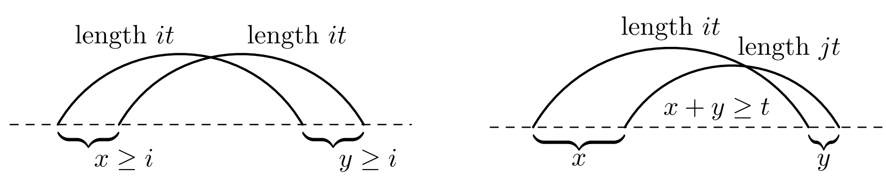
$\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \lfloor (kt+1)/i \rfloor + 1 \leq |\overline{H}(k,t)| \leq \sum_{i=1}^{\lfloor k/2 \rfloor} \lfloor (kt+1)/i \rfloor + 1$. Therefore, there are constants c_1 and c_2 such that $c_1 kt \log k \leq |\overline{H}(k,t)| \leq c_2 kt \log k$.

Denote the intersection graph of $\overline{H}(k,t)$ by $\overline{G}(k,t)$.

Lemma 1.4.5. $\omega(\overline{G}(k,t)) \leq 3k + 2t - 1$

Proof. In this proof, we will draw the segments of the circle graph in unwrapped representation.[5]

First, we make an easy observation. Let r and s be two crossing segments. Let $a < b < c < d$ be their endpoints on the circle C. Define $x := b - a$ and $y := d - c$. If r and s has the same length it, for some i, then both $x \geq i$ and $y \geq i$. It is because $\overline{H}_i(k,t)$ contains only every i-th segment of $H_i(k,t)$. If r and s have different lengths, then $x + y \geq t$, because the difference between lengths is at least t. Note that r, s can share an endpoint.

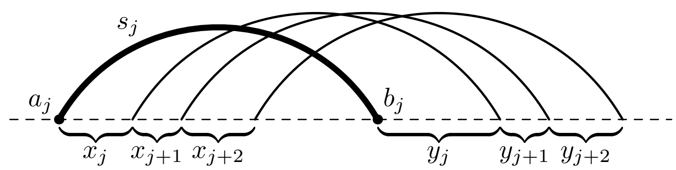


Now we are ready to start the proof. Let $Q \subseteq \overline{H}(k,t)$ be a clique. Let s_1, s_2, …, s_d be the segments of Q in the order as they appear from left to right. For $i \in \{1, 2, \ldots, d\}$ let a_i and b_i be the left and right endpoint of s_i,

[5] Imagine that the circle with segments is from a rubber. Cut it between points 0 and $kt+1$ and unwrap the circle to a line. All points 0,…, $kt+1$ will lie on a line and segments will become ”arcs” above the line.

respectively. Necessarily $a_1 < a_2 < \cdots < a_d < b_1 < b_2 < \cdots < b_d$. Further define $x_i := a_{i+1} - a_i$ and $y_i := b_{i+1} - b_i$ for $i \in \{1, \ldots, d-1\}$.

Consider s_j, the shortest segment in Q. Remind that s_j splits the supporting circle into two arcs. The length of s_j is the number of points from P on the arc, which contains less points of P. We can without loss of generality assume that the point 0 is not on the shorter arc. Otherwise we can renumber the points P so that the point 0 is in other half-plane determined by s_j. Thus the length of s_j is $mt = b_j - a_j$ for some $m \leq \lfloor k/2 \rfloor$.



Now we are going to prove that there are not too many segments of Q which lie to the right of a_j. We can get the bound on the segments of Q lying to the left of b_j from the symmetry.

Define *u-length*[6] of s_i as $b_i - a_i$. There are two types of pairs (s_i, s_{i+1}), for $i \in \{j, \ldots, d-1\}$. Either s_i and s_{i+1} has the same u-length, or not.

There are at most t pairs of the same u-length: From the observation and because s_j is the shortest segment in Q (with length mt), each such a pair (s_i, s_{i+1}) has $x_i \geq m$. Since there are mt points under s_j, and every s_i for $i \in \{j+1, \ldots, d\}$ has a left endpoint a_i under s_j, there are at most $mt/m = t$ pairs of the same u-length.

Every s_i for $i \in \{j+1, \ldots, d\}$ has a left endpoint a_i under s_j. Thus u-length of s_d is at least $b_{d-1} - b_j$, but at most $(k-1)t$, what is the maximum u-length of a segment in $H(k,t)$. Therefore we get $X = \sum_{i=j}^{d} x_i \leq mt$ and $Y = \sum_{i=j}^{d-1} y_i \leq (k-1)t$. Each pair of different u-lengths contribute to $X + Y$ by at least t. Hence there are at most $(m + k - 1)t/t \leq m + k - 1$ pairs of different u-lengths.

Therefore there are at most $t+m+k-1$ pairs (s_i, s_{i+1}) for $i \in \{j, \ldots, d-1\}$. From the symmetry, the same bound holds for pairs (s_i, s_{i+1}), for $i \in \{1, \ldots, j-1\}$. Thus there are at most $2t + 2m + 2(k-1) + 1 \leq 2t + 3k - 1$ segments in clique Q.

Moreover we have $m < \lfloor k/2 \rfloor$ otherwise all segments has the same maximum length and there are no pairs different lengths. □

Proof of theorem 1.4.1. Set $\overline{G} := \overline{H}(k,k)$. Because $\overline{G}$ is a subgraph of $G(k,k)$, we have by Lemma 1.4.3 that $\alpha(\overline{G}) \leq k$. By Lemma 1.4.5 is $\omega(\overline{G}) \leq 5k$. Graph $\overline{G}$ has at least $\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \lfloor (k^2+1)/i \rfloor + 1 \geq c \cdot k^2 \log k$ vertices for suitable constant c. □

[6] U-length of segment s is the number of points of P below s in unwrapped representation. It depends on the numbering of points P.

1.5 Size of circle graphs with $\omega \leq k$ and $\alpha \leq l$

In this section we will use the representation of circle graphs by chords of a circle (not the unwrapped representation as in the previous section). Chords are segments in the plane and thus we will often speak just about segments.

The goal is to determine or estimate the function

$$c_{k,l} = \max\{\, |V(G)| \,:\, G \in \textsc{cir} \;\&\; \omega(G) \leq k \;\&\; \alpha(G) \leq l \,\}.$$

Value $c_{k,l}$ is the maximum number of vertices of a circle graph G with $\omega(G) \leq k$ and $\alpha(G) \leq l$.

We show the upper bounds $c_{k,l} \leq \frac{3}{2}kl\log(l)$ and $c_{k,l} \leq 16(2^k - k - 1)l$. First bound is linear in k (for fixed l) and the second is linear in l (for fixed k). But it is interesting that there is no upper bound linear in both k and l. Moreover there is a lower bound by Kostochka [6], that shows $c_{m,m} = \Omega(m^2 \log m)$. For details see Theorem 1.4.1.

Later in this section we show almost tight constructions of lower bounds for small values of k, l.

1.5.1 General upper bounds

Permutation graph is an intersection graph of segments between two parallel lines. Hence it is a special case of circle graph.

Lemma 1.5.1. *Let G be a permutation graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. Then $|V(G)| \leq kl$.*

Proof. Assume that the supporting parallel lines are vertical. Two segments are independent if and only if one lies above the other. The binary relation "segment s lies above segment r" induce a partial order on the segments. Chains are sets of pairwise independent segments and antichains are sets of pairwise crossing segments. Hence the lemma follows from Dilworth's theorem. □

Let G be a circle graph. For a segment $s \in V(G)$ let $V_{cr}(s) = \{r \in V(G) \mid s \cap r \neq \emptyset\}$ be the set of segments crossing s.

Corollary 1.5.2. *Let G be a circle graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. Then $|V_{cr}(s)| \leq (k-1)l$ for every segment $s \in V(G)$.*

Proof. Segments $V_{cr}(s)$ determine a permutation graph. Moreover they contain at most $k-1$ pairwise crossing segments, because every clique in $V_{cr}(s)$ can be extended by segment s. Hence by Lemma 1.5.1 we get $|V_{cr}(s)| \leq (k-1)l$. □

Let G be the circle graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. We simply denote $\alpha = \alpha(G)$ and $\omega = \omega(G)$. The segment $s \in V(G)$ has *order* i if and only if there are exactly i independent segments of G in one half-plane determined by s and at least i independent segments of G in the other half-plane. In other words, s has order at least i, if it separates two independent sets of G, each of size at least i. Let $V_i \subseteq V$ be the set of segments of order i. Denote the graph $G|V_i$ induced by these segments by G_i and the size of independent set in G_i by α_i.

Let us define the *shell* $S(s)$ of a segment s. Segment s divides the supporting circle into two arcs. Shell of s is that of the two arcs, which induces smaller set of pairwise independent segments in G.

Observation 1.5.3. *There are no segments of order* $\lceil \alpha/2 \rceil$ *in a circle graph.*

Proof. Assume for a contradiction that there is a segment s of higher order. There will be at least $\lceil \alpha/2 \rceil$ independent segments in each half-plane determined by s so we would have $2\lceil \alpha/2 \rceil + 1 \geq \alpha + 1$ independent segments in G. □

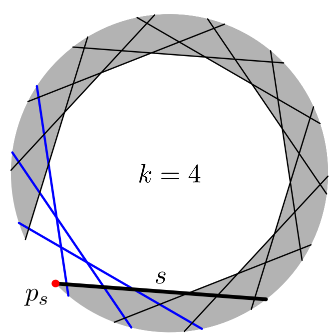
Observation 1.5.4. $\alpha(G_i) \leq \lfloor \alpha(G)/(i+1) \rfloor$.

Proof. Take independent segments $V_{\alpha_i} \subseteq V_i$. Shell of every $s \in V_{\alpha_i}$ contains no segment of V_i, but it contains i independent segments of G. All together we have $(i+1) \cdot |V_{\alpha_i}|$ independent segments of G. That gives $\alpha(G_i) \leq \alpha(G)/(i+1)$. Because $\alpha(G_i)$ is an integer, we take the lower integer part. □

Observation 1.5.5. *If* $\alpha_i = 1$ *then* $|V_i| \leq k$ *otherwise* $|V_i| \leq k(\alpha_i + 1) - 1$.

Proof. If $\alpha_i = 1$ then all segments V_i must cross each other. Thus $|V_i| \leq k$.

Assume that $\alpha_i > 1$. Let p_s be an endpoint of some segment $s \in V_i$. Cut the supporting circle at point p_s and unwrap it to a segment. (As we did with the unwrapped representation in section 1.2). Split V_i into two sets X, Y. Set $X \subseteq V_i$ contains the segments, whose shell strictly contains p_s. Set $Y \subseteq V_i$ contains the other segments. Segment s is in Y.



There are at most $k - 1$ segments in X, because all of them have to cross each other and segment s cross them too.

Two segments of Y can either lie next to each other (their shells are disjoint) or they cross. They cannot lie one under the other. We introduce a partial ordering on disjoint segments in Y. One segment is smaller than the other if and only if the segment lies to the left of the other.

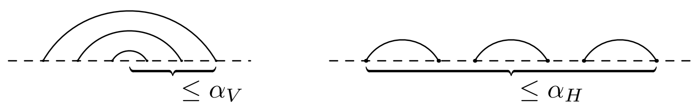
By Dilworth theorem there are at most $k\alpha_i$ segments in Y. Altogether there are at most $k\alpha_i + k - 1 = k(\alpha_i + 1) - 1$ segments in V_i. □

Remark: Observation 1.5.5 can be improved by one, if the independent set of G contains at least one segment of other order.

Remark: Assume that we have unwrapped representation. Observation 1.5.5 and corollary 1.5.2 bring two partial orderings on disjoint segments:

$u \prec_V v \iff u$ lies below v.

$u \prec_H v \iff u$ lies to the left of v.



Let α_V and α_V be the maximum length of a chain in orderings $\prec_V$, $\prec_H$ (for given circle graph G). We suggest to split the constrain $\alpha \leq l$ into two constrains $\alpha_V \leq l$ and $\alpha_H \leq l$. It might be interesting to investigate

$$c_{k,l_V,l_H} = \{\, |V(G)| \,:\, G \in \textsc{cir} \,\&\, \omega(G) \leq k \,\&\, \alpha_V(G) \leq l_V \,\&\, \alpha_H(G) \leq l_H \,\}.$$

Clearly $c_{k,l} \leq c_{k,l,l}$.

An easy upper bound from Dilworth theorem is $c_{k,l_V,l_H} \leq k \cdot l_V \cdot l_H$. The idea of proof is similar to the proof of Theorem **??**. There are at most l_V classes of segments incomparable by $\prec_V$ and similarly at most l_H classes of segments incomparable by $\prec_H$. Thus there are at most $l_V \cdot l_H$ classes of segments incomparable by $\prec_V$ or $\prec_H$. Each such a class contains pairwise crossing segments and thus its size is at most k.

That immediately shows $c_{k,l} \leq kl^2$.

Theorem 1.5.6. *$c_{k,l} \leq \frac{3}{2} kl \log(l)$ for $l \geq 2$.*

Proof. Proof of this theorem is adapted from Kratochvíl and Kostochka [7].

Remind $\alpha \leq l$. By the definition of the order of a segment and by Observation 1.5.3 we have $V = V_0 \cup V_1 \cup \cdots \cup V_{\lceil \alpha/2 \rceil -1}$. By Observations 1.5.4 and 1.5.5, the size of V_i is at most $k(\lfloor \alpha/(i+1) \rfloor + 1) - 1$ for $i \in \{0, \ldots, \lceil \alpha/2 \rceil - 1\}$. Moreover if α is odd then $\alpha_{\lceil \alpha/2 \rceil -1} = 1$ and $|V_{\lceil \alpha/2 \rceil -1}| \leq k$. Hence altogether we have

$$|V| \leq \sum_{i=0}^{\alpha/2-1} k\left(\left\lfloor \frac{\alpha}{i+1} \right\rfloor + 1\right) - 1$$

for α even. For α odd we have something slightly better:

$$|V| \leq k + \sum_{i=0}^{\lceil \alpha/2 \rceil -2} k\left(\left\lfloor \frac{\alpha}{i+1} \right\rfloor + 1\right) - 1$$

After a little more calculations we can get $|V| \leq k \cdot (\alpha \log(\alpha) + \alpha/2) - \alpha/2 \leq \frac{3}{2} k\alpha \log(\alpha)$ for $\alpha \geq 2$. □

For small values of k the recurrence from the proof of theorem 1.5.6 gives

$c_{k,1} \leq k$	$c_{k,5} \leq 10k - 2$
$c_{k,2} \leq 3k - 1$	$c_{k,6} \leq 14k - 3$
$c_{k,3} \leq 5k - 1$	$c_{k,7} \leq 16k - 3$
$c_{k,4} \leq 8k - 2$	$c_{k,8} \leq 20k - 4$

Theorem 1.5.7. $c_{k,l} \leq 16(2^k - k - 1)l$.

Proof. We proved in Theorem 1.3.1 that $\chi(G) \leq 16 \cdot 2^{\omega(G)} - 16\omega(G) - 16$ in circle graphs. Let G be the circle graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. Color the segments, which correspond to the vertices of the graph G. Each color class has at most l segments and we have at most $16(2^k - k - 1)$ colors. □

1.5.2 Better upper bounds for small l

Lemma 1.5.8. $c_{k,1} \leq k$ *and* $c_{k,2} \leq 3k - 1$.

Proof. If $\alpha(G) = 1$ then G is a clique and has at most k vertices.

Now consider $\alpha(G) = 2$. Take some segment $s \in V(G)$ and split $V(G)$ into two sets: segments S_1, which cross s and segments S_2, which do not. Set S_2 contains no two disjoint segments. Otherwise there are three disjoint segments (the two disjoint and s). Hence $|S_2| \leq k$. By Lemma 1.5.2 we have $|S_1| \leq 2k - 2$. □

Lemma 1.5.9. $c_{k,3} \leq 5k - 2$.

Proof. Either there is a segment s of order one or not.

In the first case if s is a segment of order one, then there are at most $c_{1,k} \leq k$ segments in each half-plane. By lemma 1.5.2 the segment s is crossed by at most $3(k - 1)$ other segments. Altogether there are at most $5k - 2$ segments.

In the second case all segments are of order zero. Thus by Observation 1.5.5 there are at most $4k - 1$ segments. □

Lemma 1.5.10. $c_{k,4} \leq 8k - 4$.

Proof. If there exists a segment s of order one then it has at most $c_{1,k} \leq k$ segments in one half-plane and at most $c_{2,k} \leq 3k - 1$ in the other half-plane. By lemma 1.5.2 segment s is crossed by at most $4(k-1)$ other segments. Altogether there are at most $8k - 4$ segments.

If there is no segment of order one then every segment has order zero. By Observation 1.5.5 there are at most $5k - 1$ segments. □

We cannot improve the additive constant in $c_{k,5}$ and we don't know any better upper bound than $10k - 2$, which follows from Theorem 1.5.6.

Lemma 1.5.11. $c_{k,6} \leq 14k - 8$.

Proof. If there exists a segment s of order two then it has at most $c_{2,k} \leq 3k - 1$ segments in one half-plane and at most $c_{3,k} \leq 5k - 2$ segments in the other half-plane. By lemma 1.5.2 segment s is crossed by at most $6(k - 1)$ other segments. Altogether there are at most $14k - 8$ segments.

If there is no segment of order two then every segment has order zero or one. By Observations 1.5.4 we have $\alpha_0 = 6$ and $\alpha_1 = 3$ and therefore by Observation 1.5.5 we get $|V_0| \leq 7k - 1$ and $|V_1| \leq 4k - 1$. So in this case there are at most $11k - 2$ segments. □

1.5.3 Lower bounds

In this section we construct lower bounds for $c_{k,l}$. We only describe the construction. The proof that the constructed set of segments contains no more than k pairwise crossing segments or no more then l pairwise disjoint segments is a little technical verification and we omit it.

Every construction starts with points on the circle. We split the points into several sets and for each set we call procedure `LAYER`, which constructs the segments between given points.

procedure `LAYER(`P, k`)`**:**

The procedure gets $2m$-point set P and k and produces a set of segments with endpoints in P and with at most k pairwise crossing segments.

Put $n = 2m$ and denote the points by numbers from zero to $n - 1$ in the order as they appear on the circle. For every even p connect point p with point $p + 2k - 1 \bmod n$ by an segment. The constructed set of segments contains at most k pairwise crossing segments and at most $\lfloor m/k \rfloor$ independent segments. For illustration see figures in 1.7.1. Let us note that each call of `LAYER` constructs the segments of the same order.

Lemma 1.5.12. $c_{k,2} \geq 3k - 1$.

Take $n = 2 \cdot (3k - 1)$ points on a circle and apply procedure `LAYER`. See 1.7.1 for examples.

Lemma 1.5.13. $c_{k,3} \geq 5k - 2$.

Take $n = 2 \cdot (4k - 2)$ points on a circle and apply `LAYER`. Then add k halving segment so that they do not create a clique of size $k + 1$. See 1.7.2 for examples.

Lemma 1.5.14. *$c_{k,4} \geq 7.5k - 3$ for k even and $c_{k,4} \geq \lfloor 7.5k \rfloor - 4$ for k odd.*

If k is even then take $n = 3 \cdot (5k - 2)$. If k is odd, take $n = 3 \cdot (5k - 3)$. Take set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ of n points on a circle. Let $P_0 = \{ p_i \in P \,|\, i \bmod 3 = 0, 1 \}$ and $P_1 = P \setminus P_0$. Apply `LAYER` on P_i to get segments of order i for $i = 0, 1$. See 1.7.3 for examples.

Lemma 1.5.15. *$c_{k,5} \geq 10k - 6$ for $k > 0$.*

Take set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ of $n = 3 \cdot (6k - 2)$ points on a circle. Let $P_0 = \{ p_i \in P \,|\, i \mod 3 = 0, 1 \}$ and $P_1 = P \setminus P_0$. Apply `LAYER` on P_i to get segments of order i for $i = 0, 1$. At the end we add k segments of order two. They are halving segments such that they do not create a clique larger than k. See 1.7.4 for examples.

Note that for k odd, there is an asymmetric construction with $\geq 10k - 5$ segments. See 1.7.4 for examples.

1.5.4 Overview of values $c_{k,l}$ for small k, l

Here we summarize the bounds for $l \le 6$.

$$\begin{array}{ll} c_{k,1} = k & 7.5k - 3 \le c_{k,4} \le 8k - 2 \\ c_{k,2} = 3k - 1 & 10k - 6 \le c_{k,5} \le 10k - 2 \\ c_{k,3} = 5k - 2 & c_{k,6} \le 14k - 8 \end{array}$$

What abount $c_{2,l}$? Currently, the best upper bounds are $5l$ (Theorem 1.5.7 gives only $2^4(4-2-1)l = 16l$, but we can modify its proof and use the result of Ageev that $f(2) = 5$) and $2l \log l + l/2$. The simple lower bound is $\frac{19}{6}l$ for $l \bmod 6 = 0$ (place copies of the construction for $c_{2,6}$ next to each other).

From the upper and lower bounds in this section we get the range of values for small k, l. They are in following table. The values in brackets mean the truth found by a computer program, which uses brute force.

$c_{row,col}$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	5	8	12	15–18(15)	?–20(19)	?–29($\ge$ 22)
3	3	8	13	18–20(19)	24–28	?–34	?–45
4	4	11	18	27–28	35–38	?–48	?–61
5	5	14	23	33–36	44–48	?–62	?–77
6	6	17	28	42–44	55–58	?–76	?–93
7	7	20	33	48–52	64–68	?–90	?–109

Note: To improve the lower bound on Ramsey-type question on segments (Theorem **??** in the first chapter), we would need $c_{5,5} \ge 46$, or $c_{6,6} \ge 71$. That is the hope for a tiny improvement.

1.6 Open Problems

Open Problem: In Section 1.3 we presented a problem of coloring circle graphs. Theorem 1.3.1 shows that every circle graph with $\omega = \omega(G)$ has

$$\chi(G) \le 16(2^\omega - \omega - 1).$$

On the other hand there are circle graphs, which require $c \cdot \omega \log \omega$ colors, for some constant c (Theorem 1.4.2). Still, there is an exponential gap. Can you improve these bounds?

Ageev [1] showed that $\chi(G) \le 5$ for circle graphs with $\omega(G) = 2$. What can you show for $\chi(G)$ when $\omega(G) = 3$? Theorem 1.3.1 gives only $\chi(G) \le 64$.

Open Problem: In Section 1.4 we presented the proof of the lower bound on $f(k)$. May be it is possible to modify the construction and get a better bound. Interesting questions are:

- How much is $\chi(\overline{G}(k,t))$? We showed just the simple lower bound $\chi(G) \geq |V(G)|/\alpha(G) \geq c \cdot kt$.
- The graph $G(k,t)$ has $\chi(G(k,t)) \geq |V(G)|/\alpha(G) \geq c \cdot k^2 t/k = c \cdot kt$. Graph $\overline{G}(k,t) \subseteq G(k,t))$ is a subgraph with $\omega(\overline{G}(k,t)) \leq O(k+t)$. Is it possible to choose a better subgraph? We would like to find a subgraph G' with $\omega(G') = O(k + \log t)$ or with $\omega(G') = O(k + \sqrt{t})$, but on the other hand is must have sufficiently many vertices.

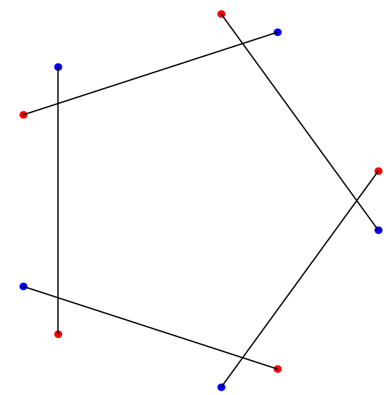
Open Problem: Section 1.5 brings some open problems too. We have $7.5k - const \leq c_{k,4} \leq 8k - const$. The gap brings us to the following question.

By Observation 1.5.5 the number of segments in first two layers is $|V_0 \cup V_1| \leq (\alpha+1)k - 1 + (\lfloor \alpha/2 + 1 \rfloor)k - 1 \leq (3\alpha/2 + 2)k - 2$. Can you show that $|V_0 \cup V_1| \leq (3\alpha/2 + 3/2)k - 2$?

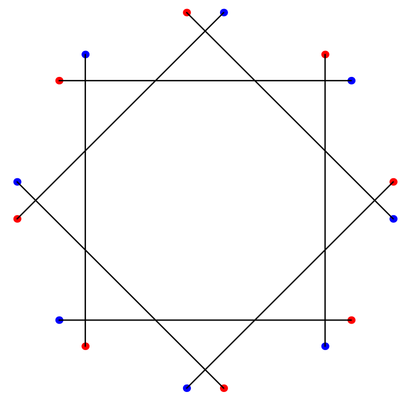
Open Problem: We had shown that $c_{2,l}$ is between $\frac{19}{6}l$ (for l mod $6 = 0$) and $min\{\,5l,\, 2l \log l + l/2\,\}$. What can you show for $c_{3,l}$? The upper bound from Theorem 1.5.7 gives only $2^4(8-3-1)l = 64l$.

1.7 Attachments

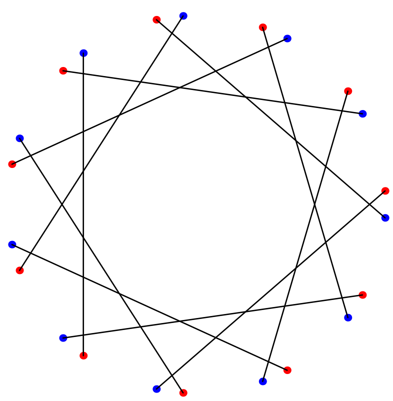
1.7.1 Lower bound for $c_{k,2}$



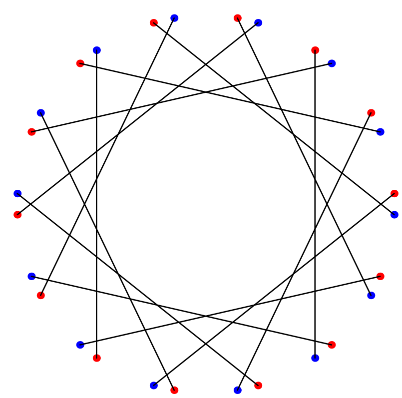
n=5 $k = 2$, $l = 2$



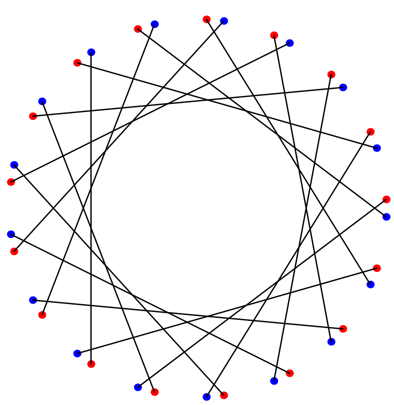
n=8 $k = 3$, $l = 2$



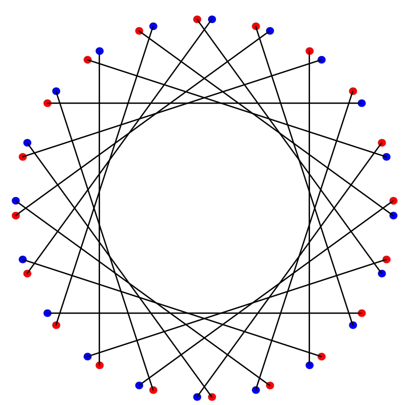
n=11 $k = 4$, $l = 2$



n=14 $k = 5$, $l = 2$

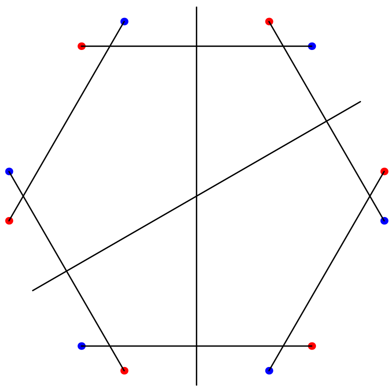


n=17 $k = 6$, $l = 2$

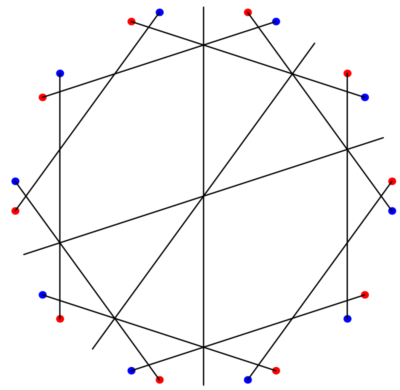


n=20 $k = 7$, $l = 2$

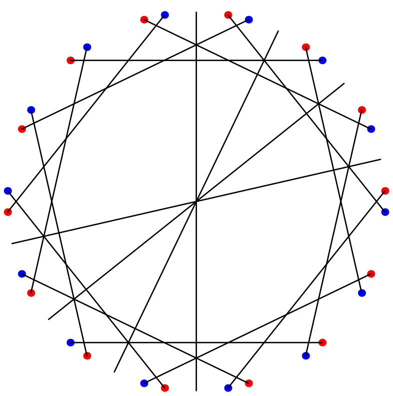
1.7.2 Lower bound for $c_{k,3}$



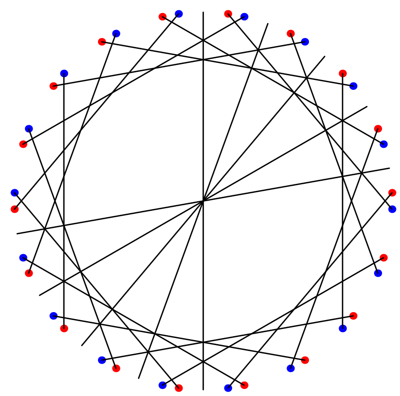
n=8 $k = 2,\ l = 3$



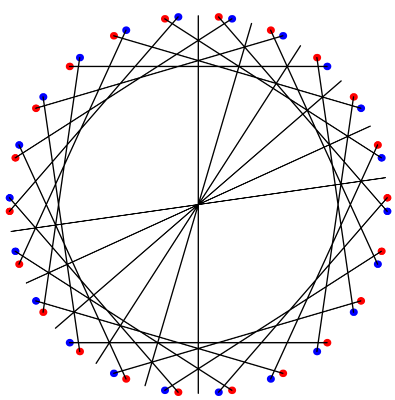
n=13 $k = 3,\ l = 3$



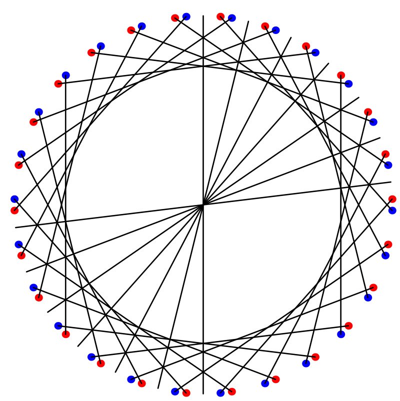
n=18 $k = 4,\ l = 3$



n=23 $k = 5,\ l = 3$

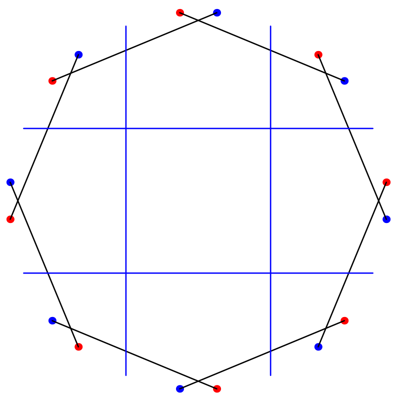


n=28 $k = 6,\ l = 3$

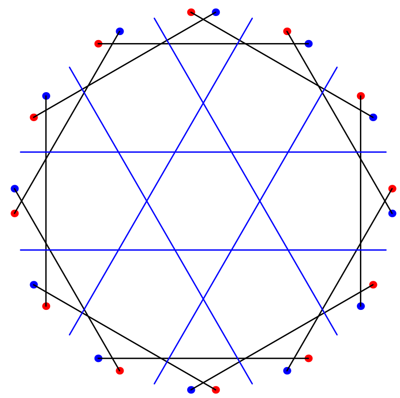


n=33 $k = 7,\ l = 3$

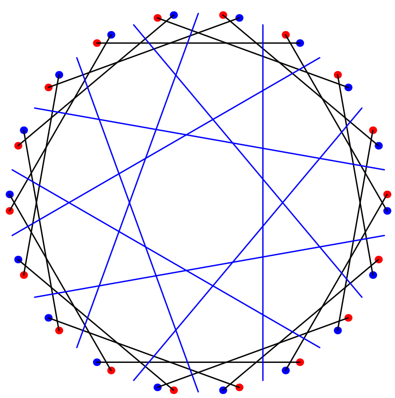
1.7.3 Lower bound for $c_{k,4}$



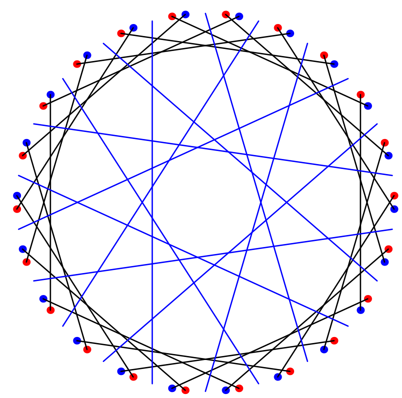
n=12 $k = 2$, $l = 4$



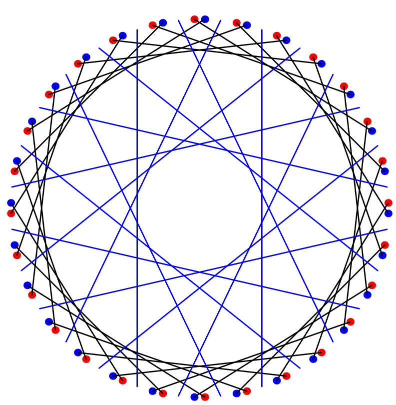
n=18 $k = 3$, $l = 4$



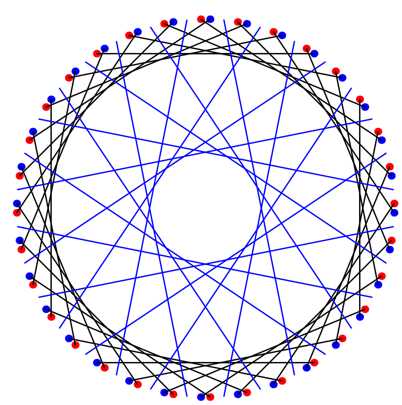
n=27 $k = 4$, $l = 4$



n=33 $k = 5$, $l = 4$



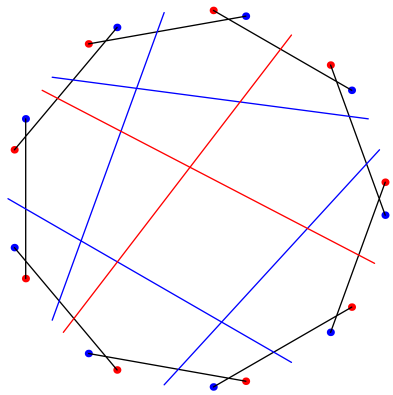
n=42 $k = 6$, $l = 4$



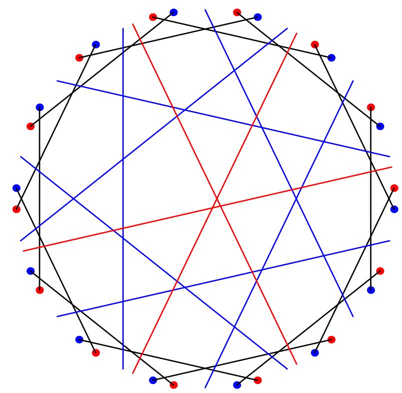
n=48 $k = 7$, $l = 4$

1.7.4 Lower bound for $c_{k,5}$

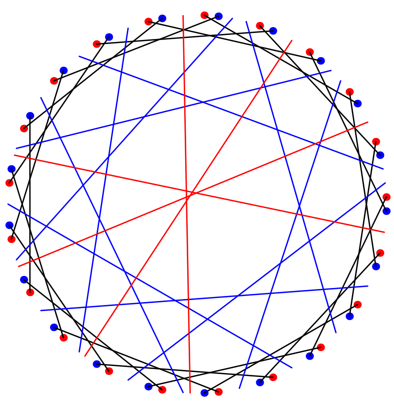
There is also symmetric construction for k even but it has one less segment.



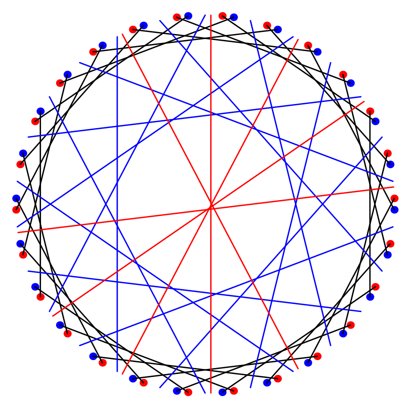
n=15 $k = 2$, $l = 5$



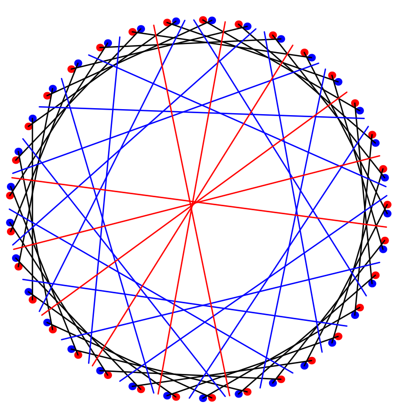
n=24 $k = 3$, $l = 5$



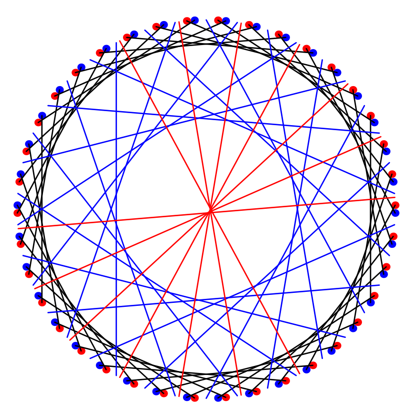
n=35 $k = 4$, $l = 5$



n=44 $k = 5$, $l = 5$

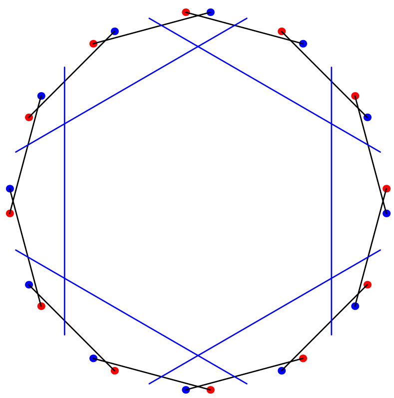


n=55 $k = 6$, $l = 5$

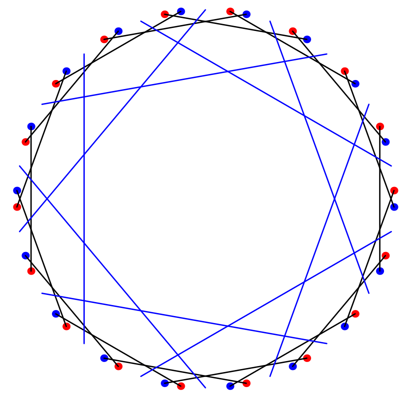


n=64 $k = 7$, $l = 5$

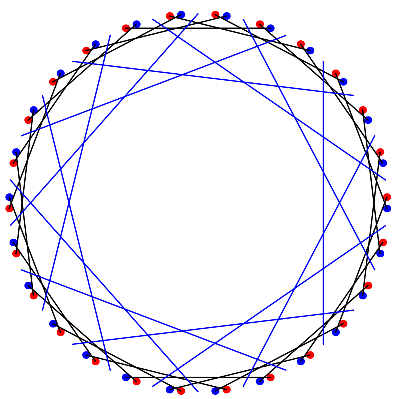
1.7.5 Lower bound for $c_{k,6}$



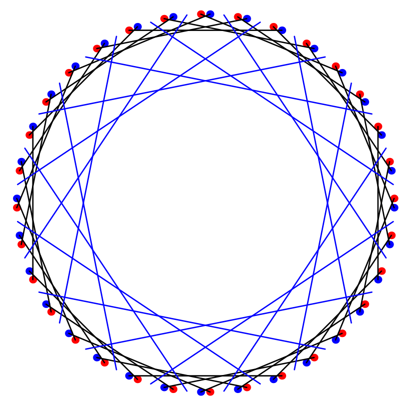
n=18 $k = 2$, $l = 6$



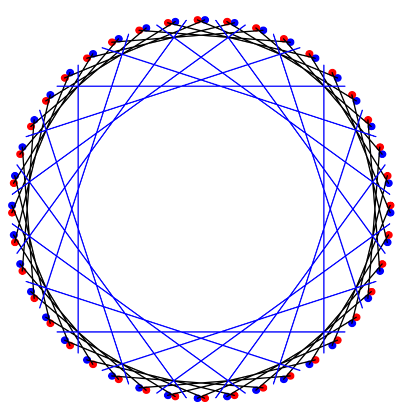
n=27 $k = 3$, $l = 6$



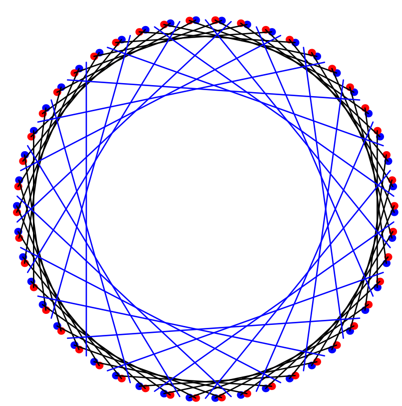
n=39 $k = 4$, $l = 6$



n=48 $k = 5$, $l = 6$



n=60 $k = 6$, $l = 6$



n=69 $k = 7$, $l = 6$

Bibliography

[1] A. A. Ageev. A triangle-free circle graph with chromatic number 5. *Discrete Math.*, 152(1-3):295–298, 1996.

[2] J. Chalopin, D. Gonçalves, and P. Ochem. Planar graphs are in 1-STRING. In *SODA '07: Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 609–617, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.

[3] L. Esperet and P. Ochem. On circle graphs with girth at least five. Research Report RR-1431-07, LaBRI, Université Bordeaux 1, may 2007.

[4] A. Gyárfás. On the chromatic number of multiple interval graphs and overlap graphs. *Discrete Mathematics*, 55(2):161–166, 1985.

[5] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastos Mathematics*, 19:413–441, 1987.

[6] A. Kostochka. On upper bounds on chromatic numbers of graphs. *Transaction of the Institute of mathematics (Siberian branch of the Academy of Science of USSR)*, 10:204–226, 1988.

[7] A. Kostochka and J. Kratochvíl. Covering and coloring polygon-circle graphs. *Discrete Math.*, 163(1-3):299–305, 1997.

[8] T. A. McKee and F. McMorris. *Topics in intersection graph theory*. SIAM Monographs on Discrete Mathematics and Applications 2, 1999.