Charles University in Prague
Faculty of Mathematics and Physics

Doctoral Thesis

Combinatorial and Computational Geometry

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Branch I4: Discrete Models and Algorithms

2008

Acknowledgement: I am very grateful to my advisor, Pavel Valtr, for his guidance, advice and other support during my years as a student at Charles University. I would like to thank to people at Department of Applied Mathematics for creating friendly working atmosphere.

Further, I am grateful for all support I got from Department of Applied Mathematics (KAM) and Institute of Theoretical Computer Science (ITI). It allowed me to visit several workshops and conferences and meet interesting people in the field of my research. Also, I would like to thank to my advisor and to Emo Welzl (ETH Zürich), Géza Tóth (Rényi Institute in Budapest), who helped me to arrange a stay at ETH and in Budapest. There I got known about other research centers, broaden my scope and learned more about the way, how people do research. I also want to thank to Jan Kratochvíl, who helped me to get an internship at Mitsubishi Electrics Reasearch Labs (MERL). I was not doing a research related to my thesis there, but it was a great experience, because MERL is a commercial research center focused on products with applications, which they can sell.

Finally I want to thank to my family for the support during my studies and to my girl friend Gabriela, who motivated me to finish this thesis.

I hereby declare that I have written the thesis on my own and using exclusively the cited sources. For any work in the thesis that has been co-published with other authors, I have the permission of all of them to include this work in my thesis. I authorise the Charles Univesity to lend this document to other institutions and individuals for academic or research purposes.

Jakub Černý
Prague, 20 January, 2008 ....

Preface

Before I started writing my thesis, I was thinking who will read it. My supervisor and two opponents? Yes, but I would like to write something for other people, too. Something useful. That is the reason, why the thesis also contains a short survey of topics, which I like, and which are somehow related to my thesis. Besides the survey, I included some nice proofs, pearls in geometry, which I met during my studies and during the work on this thesis, which I love, and which illustrate useful methods in the research area. Some of them can be called "the proofs from the Book"[1]. I spiced it up with several open problems, which I was also trying to solve. I apologize to my supervisor and opponents, who will have to read a little more pages with these beautiful proofs, but I hope, they will enjoy it.

What can you find in this thesis? How is the thesis related to my research and my results?

Ramsey type results

After explaining what is Ramsey type result, we show Dilworth's theorem and partial ordering on disjoint segments – the method which is very useful further in this thesis. We present Ramsey type result on segments in the plane, whose improvement is motivation for investigating circle graphs G with bounded $\alpha(G)$ and $\omega(G)$ (chapter 4 about circle graphs).

Geometric and topological graphs (the drawings of graphs)

First, we survey the basic results about plane graphs. Then we define crossing number $\text{CR}(G)$ of a graph G and show the basic results (Crossing lemma, Crossing lemmas for special classes of graphs). In fact, there are more definitions of crossing number. We survey them and show how they are related to each other. We finish the explanation on crossing numbers by a few results on decay of $\text{CR}(G)$. If we have graph G and we command to delete positive fraction of edges – let us say 2%, then what is the minimum decay of $\text{CR}(G)$? This is result of Černý, Kynčl and Tóth [12].

[1] Erdős believed, that in heaven, there is a book full of theorems containing their simplest proofs. Aigner and Ziegler wrote "Proofs from the BOOK", the book which contains some of these beautiful proofs.

At the end of the chapter we present extremal results on the drawings of graphs with no forbidden (topological) subgraphs. The most frequent question is how many edges can have a graph with no forbidden subgraph. The class of forbidden subgraphs often represents some geometric constraint (like there are no two crossing edges, which yields plane graphs. They have $|E(G)| \leq 3|V(G)| - 6$.). Here we present the sketch of my result [9], that a geometric graph on n vertices and with no three pairwise crossing edges has at most $2.5n$ edges.

Convex independent sets

In fact, it is about a well known Ramsey type result called Erdős–Szekeres theorem and its generalizations. The theorem says that for every integer k, if we have sufficiently many points in the plane, we can find a k-point subset, which lies in the vertices of a convex polygon.

At the beginning of the chapter, we survey the proofs of Erdős–Szekeres theorem and related results. One proof uses cups and caps. Most interesting generalization of Erdős–Szekeres theorem is about empty convex polygons. Valtr [81] generalized cups and caps to open cups and open caps. He used it in theorem providing a sufficient condition for a set of points containing large empty convex polygon.

The rest of the chapter contains improvements of the result on open cups and open caps. Most of it is based on my paper [11], but it also contains my unpublished results (the tight bounds for $k = 4$ or $l = 4$).

Circle graphs

First, we look at the coloring of circle graphs. The main result improves the result of Kostochka and Kratochvíl [43], that a circle graph G with $\omega(G) \leq k$ can be colored by $f(k)$ colors (for some exponential function f). Partial improvement was presented at Eurocomb'07, now I prepare full version for publication.

Then we reproduce the lower bound on previous coloring, which is by Kostochka in Russian and was never published in English.

Last third of the chapter tries to count the maximum number of vertices of circle graphs with $\alpha \leq k$ and $\omega \leq l$. That partially answers the question from first chapter, motivated by Ramsey type question on segments in the plane. Here we contribute with one new upper bound and exact or improved bounds for small k and l (based on computer program using brute force). These results were not published yet.

What exactly are my results and where were they published?

- *Černý, Kynčl, Tóth: Improvement on the decay of crossing number* [12] – Theorem 2.3.8 and its proof in subsection 2.3.3.
- *Černý: Geometric graphs with no 3 pairwise disjoint edges* [9] – Theorem 2.4.4 in subsection 2.4.1.
- *Černý: Simple proof for open cups and caps* [11] – first part of section 3.3. The rest of the section contains tight bounds for $k = 4$ or $l = 4$, which I did not published.
- *Černý: Coloring circle graphs* [10] – section 4.3. In fact the result was improved and the presented version was not published yet.
- Whole section 4.5 except for Theorem 4.5.6 and its proof (which is from [43] and includes 3 observations) contains my *unpublished results.*

Contents

Chapter 1

Ramsey-type questions

Motivation: What is a Ramsey-type question? One of the classical Ramsey-type results is Erdős–Szekeres theorem for sequences. In this chapter, we look at similar, but more geometric results.

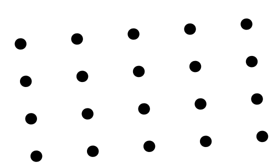
Theorem 1.0.1 (Erdős–Szekeres [21] for sequences). *Any sequence of at least $(r-1)(s-1)+1$ different real numbers contains either a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s.*

Proof. Let $n_1, n_2, \ldots, n_k \geq (r-1)(s-1)+1$ be the sequence of different real numbers. We label each number n_i, in the sequence, by a pair (a_i, b_i), where a_i is the length of the longest monotonically increasing subsequence ending with n_i and b_i is the length of the longest monotonically decreasing subsequence ending with n_i.

There are no two pairs $(a_i, b_i) = (a_j, b_j)$ with $i < j$, because either $n_i < n_j$ and thus $a_j > a_i$ or $n_i > n_j$ and thus $b_j > b_i$ (we extend the sequence witnessing a_i, resp. b_i, by element n_j).

Obviously $a_i \geq 1$, $b_i \geq 1$ for all i. Suppose for the contradiction that there is neither monotonically increasing subsequence of length r nor a monotonically decreasing subsequence of length s. Thus $a_i < r$, $b_i < s$ for all i. But then there are only $(r-1)(s-1)$ different pairs (a_i, b_i) for $(r-1)(s-1)+1$ numbers. That is a contradiction. □

This theorem is tight, as shown on the next figure for $r = 6$ and $s = 5$. Order the points by their x-coordinate. Then y-coordinate of the i-th point is the value n_i.



1.1 Dilworth's theorem and partial ordering for segments

A *partially ordered set (poset)* is a pair $(X, \preceq)$, where X is a set and the binary relation $\preceq$ on S is reflexive, antisymmetric and transitive.[1] Two elements a, $b \in X$ are *comparable* if $a \preceq b$ or $b \preceq a$. A *chain* is a totally ordered subset of X (any two elements are comparable). An *antichain* is a subset of X of pairwisely incomparable elements.

Theorem 1.1.1 (Dilworth's theorem [15]). *Let $P = (X, \preceq)$ be a finite poset. Let k be the maximum length of a chain and let l be the maximum length of an antichain. Then*

i) The S can be partitioned into at most k antichains.

ii) The S can be partitioned into at most l chains.

Proof. (i) For any $x \in X$, define $rank(x)$ as the length of the longest chain with maximal element in x. All elements with the same rank form an antichain and $1 \leq rank(x) \leq k$.

(ii) We use induction on $|X|$. The case $|X| = 1$ is clear. Let $A = \{a_1, \ldots, a_l\}$ be a maximal antichain of length l. All other elements are comparable with some element in A. Split X into $X_{\preceq} = \{x \in X : x \preceq a \text{ for some } a \in A\}$ and $X_{\succeq} = \{x \in X : x \succeq a \text{ for some } a \in A\}$. There is no $x \in X$ such that $a_i \preceq x$ and $x \preceq a_j$ for some $a_i, a_j \in A$, because than a_i, a_j would be comparable. Thus every $x \in X \setminus A$ is either in $X_{\preceq}$ or in $X_{\succeq}$. From induction hypothesis $X_{\succeq}$ is a union of disjoint chains $C^1_{\succeq}, C^2_{\succeq}, \ldots, C^l_{\succeq}$ and $X_{\preceq}$ is a union of disjoint chains $C^1_{\preceq}, C^2_{\preceq}, \ldots, C^l_{\preceq}$. Moreover we can assume that the numbering of chains is such that a_i is the minimal element of $C^i_{\succeq}$ and also a_i is the maximal element of $C^i_{\preceq}$. Put $C_i := C^i_{\succeq} \cup C^i_{\preceq}$. Chains C_i are disjoint and they cover X, what finishes the proof. □

Corollary 1.1.2 (variant of Dilworth's theorem). *Let P be a poset with no $(k+1)$-chain and no $(l+1)$-antichain. Then P has at most kl elements.*

Let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of segments in the plane. Assume without loss of generality that there is no vertical segment.[2] Denote the left and right endpoint of segment s_i by a_i and b_i, respectively. For a point p in the plane, $x(p)$ denotes the x-coordinate of p and $y(p)$ denotes the y-coordinate of p. A segment s_i *lies below* a segment s_j, if and only if every vertical line intersecting

[1] For all $a, b, c \in S$ holds

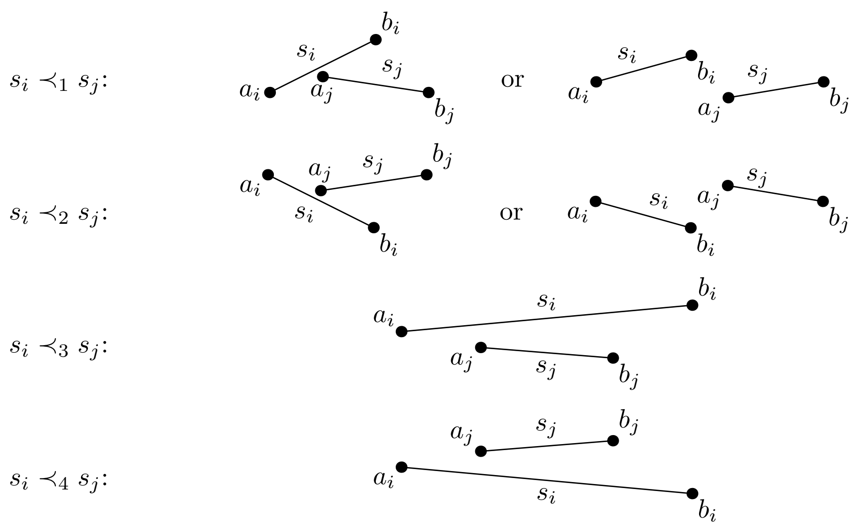
- $a \preceq a$ (reflexivity),
- if $a \preceq b$ and $b \preceq a$ then $a = b$ (antisymmetry),
- if $a \preceq b$ and $b \preceq c$ then $a \preceq c$ (transitivity).

[2] Otherwise rotate the coordinate system a little.

both s_i and s_j intersects s_i strictly below s_j. Equivalently, we say that the segment s_j *lies above* the segment s_i.

We define four binary relations $\prec_k$ on S for $k \in \{1,2,3,4\}$ as follows

$$\begin{aligned}
s_i \prec_1 s_j &\iff x(a_i) \le x(a_j),\ \ x(b_i) \le x(b_j),\ \ s_i \text{ lies above } s_j,\\
s_i \prec_2 s_j &\iff x(a_i) \le x(a_j),\ \ x(b_i) \le x(b_j),\ \ s_i \text{ lies below } s_j,\\
s_i \prec_3 s_j &\iff x(a_i) \le x(a_j),\ \ x(b_i) \ge x(b_j),\ \ s_i \text{ lies above } s_j,\\
s_i \prec_4 s_j &\iff x(a_i) \le x(a_j),\ \ x(b_i) \ge x(b_j),\ \ s_i \text{ lies below } s_j.
\end{aligned}$$



Observation 1.1.3. *Each of the relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$ is antisymmetric and transitive. Moreover, any two disjoint segments r, s in the plane are comparable by one of these relations.*

Open problem 1 (Géza Tóth). *Can you show, that there does not exist only three binary relations, which have the same properties as $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$?*

1.2 Ramsey-type question in segments

Let $r(n)$ be the smallest positive integer such that every set of at least $r(n)$ segments in the plane contains either n pairwise crossing or n pairwise disjoint segments.

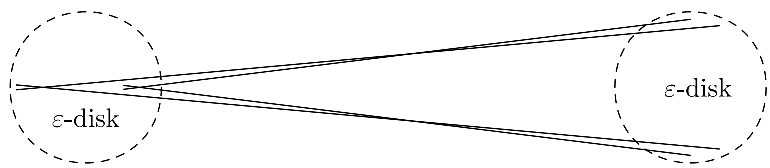
Theorem 1.2.1 (Upper bound [47]). $r(n) \le n^5$.

Proof. Let us start with n^5 segments. We want to find n pairwise disjoint segments or n pairwise crossing segments.

Any two segments, which are comparable by $\prec_i$ for $i \in \{1,2,3,4\}$, are necessarily disjoint (Observation 1.1.3). By Dilworth theorem there is a chain of size at least n in $\prec_1$ (its elements are pairwise disjoint segments) or there is an antichain of size at least n^4 in $\prec_1$ (the elements incomparable by $\prec_1$). In the second case take $\prec_2$ and repeat the same idea. We get either a chain of size at least n in $\prec_2$ or an antichain of size at least n^3. Now the antichain contains elements, which are incomparable by neither $\prec_1$ nor $\prec_2$. We continue similarly for $\prec_3$ and $\prec_4$. Always we either get a chain of size at least n in $\prec_i$, which is a subset of n pairwise disjoint segments, or we get a big set of incomparable elements. We finish with n elements, which are incomparable by all $\prec_i$. Thus by Observation 1.1.3 these segments must pairwisely intersect. □

For the construction of the lower bound we need a little more definitions.

A family of segments is in *general position* if no three of their endpoints are collinear. Let $S = \{s_1, s_2, \ldots, s_n\}$ be the set of segments in general position in the plane. We say that S can by *flattened* if for every $\varepsilon > 0$ there are two disks of radius ε at unit distance from each other, and there is another family of segments $S' = \{s'_1, s'_2, \ldots, s'_n\}$ in general position such that S and S' have the same crossing pattern (s'_i cross s'_j if and only if s_i cross s_j) and every s'_i has each endpoint in different disk.



Lemma 1.2.2 ([37]). *Any family S of segments in general position, whose endpoints are at the vertices of a convex polygon, can be flattened.*

Proof. Let $p_0, p_1, \ldots, p_{2n-1}$ denote the the endpoints of the segments S in counterclockwise order.

Moving the points on the convex curve so that we do not change their order, or placing the points on another convex curve in the same order, do not change the crossing pattern.

We place the points on a part of parabola $y = \sqrt{x}$, which lies in ε-disk centered in origin. More accurately, we take points $p'_i = (\varepsilon/4^i, \sqrt{\varepsilon}/2^i)$ for $0 \leq i \leq 2n-1$. By connecting the corresponding pairs of points by segments, we get a family S' with the same crossing pattern[3] as S.

For any two disjoint segments $p'_i p'_j$, $p'_k p'_l \in S'$ for some $i < k < l < j$, the slope of $p'_i p'_j$ is smaller than slope of $p'_k p'_l$. It can be shown by easy calculation. Thus we can extend the segments S' to the right until they touch the line $x = 1$ and the extension do not change the crossing pattern of S'.

Now we just flatten the family S' in direction of y-axis (affine transformation $(x, y) \to (x, \delta y)$ for suitable δ). Finally we perturb some points to get general position. □

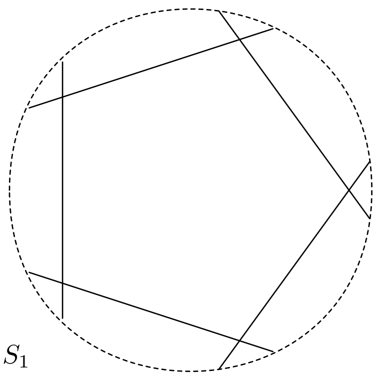
[3]Because of the exponential growth of the distance between two consecutive points, it is not possible to draw a nice figure.

Theorem 1.2.3 (Lower bound [47, 37]).

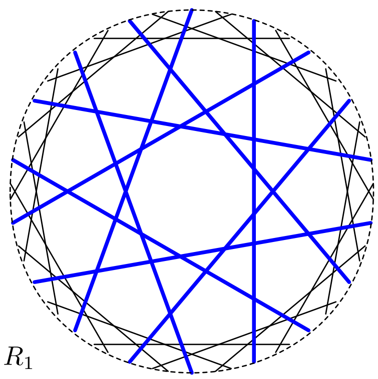
$$r(n) \geq n^{\log 5/\log 2} \geq n^{2.332},$$

$$r(n) \geq n^{\log 27/\log 4} \geq n^{2.377}.$$

Proof. The family S_1 on the left of the next figure has 5 segments with neither 3 pairwise disjoint nor 3 pairwise crossing segments. Similarly, the family R_2 on the right of the next figure has 27 segments with neither 5 pairwise disjoint nor 5 pairwise crossing segments. It is not difficult to verify this properties and we leave it to the reader.



S_1



R_1

Now consider the family S_1. Let ε be the minimum distance between the endpoints of S_1. Replace every segment $s \in S_1$ by a suitably flattened copy of S_1, whose endpoints are at the distance at most $\varepsilon/2$ from the endpoints of s. We get a family S_2 of 5^2 segments. By replacing every segment $s \in S_2$ by a very flattened copy of S_1, we get the family S_3 of 5^3 segments. Similarly, for every k, we construct a family S_k of 5^k segments in general position.

Family S_k has at most 2^k pairwise disjoint segments and at most 2^k pairwise crossing segments. Thus $r(2^k) \geq 5^k$ and the first part of the theorem follows.

Similarly, we can use family R_1 instead of S_1 in the construction. For every k we get family R_k of 27^k segments with neither 4^k pairwise disjoint segments nor 4^k pairwise crossing segments. From that the second part of the theorem follows. □

In the previous proof, if we find a better configuration of segments than R_1, we can further improve the lower bound. That is, we want to find a large set of segments with endpoints in convex position such that it contains only small subsets of pairwise crossing or pairwise disjoint segments. We tried to do that in the notion of circle graphs: a set of segments S with endpoints in convex position determines a circle graph G. A subset of crossing segments correspond to a clique in G and a subset of disjoint segments correspond to an independent set in G. The results are described in section 4.5. It shows, that the lower bound cannot be improved too much by using a better set of segments with endpoints in convex position.[4] On the other hand, there is still some hope for a tiny improvement.

The result can be easily generalized to convex sets (see Larman, Matoušek, Pach and Töröcsik [47]). For generalization of the method to unions of comparability graphs see Dumitrescu and Tóth [16].

1.3 Note on ramsey-type questions in geometric graphs

We will define geometric graphs in the next section. For now, it might suffice, that geometric graph is a graph drawn into the plane by straight line segments. There are several ramsey-type questions in geometric graphs. The most often is the following.

An enemy colors the edges of geometric K_n with 2 colors. Can we always find a monochromatic subgraph with certain properties? What are the monochromatic subgraphs, we can always find? In any 2-coloring of edges of a geometric graph, we can always find the following objects as monochromatic subgraphs.

- Non-crossing spanning tree. Károlyi, Path and Tóth [37].
- Non-crossing cycles of lengths 3, 4, ..., $\lfloor\sqrt{n/2}\rfloor$. Károlyi, Pach, Tóth, Valtr [39]. This result is tight up to a multiplicative constant.
- Non-crossing path of length $\Omega(n^{2/3})$. This is result of Károlyi, Pach, Tóth, Valtr [39]. They improved the bound of Károlyi, Path and Tóth [37] who showed $\Omega(\sqrt{n})$. It is believed that one can find a path of linear length. Can you find it?

Most of these problems can be generalized to more colors.

[4]Circle graph G with $\alpha(G) \leq a$ and $\omega(G) \leq a$ has $|V(G)| \leq a^2 \log a + a^2/2 - a/2$ (see the proof of Theorem 4.5.6). At the best, this method can give us $r(a^k) \geq (a^2(\log a + \frac{1}{2} - \frac{1}{2a}))^k$, which implies

$$r(n) \geq n^{2+\frac{\log(\log a + \frac{1}{2} - \frac{1}{2a})}{\log a}}.$$

Let $p(a) := 2 + \frac{\log(\log a + 1/2 - 1/2a)}{\log a}$ denote the power of n in $r(n)$. Function $p(a)$ is growing upto $a = 4$ and then is slowly decreasing. We have $p(2) \leq 2.32$, $p(3) \leq 2.59$, $p(4) \leq 2.623$, $p(5) \leq 2.621$, $p(10) \leq 2.58$, $p(100) \leq 2.43$. See section 4.5 for more precise results for small a (mainly the overview of results).

Chapter 2

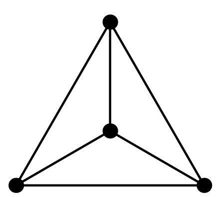
Geometric and Topological graphs

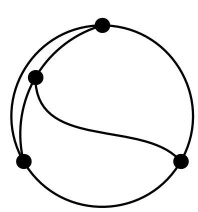
Here, I would like to show beautiful proofs, which I really like.
Some of them are real pearls in the theory.

2.1 Introduction

An *(abstract) graph* G is a pair $(V(G), E(G))$ where $V(G)$ is the set of *vertices* and $E(G)$ is the set of *edges* $\{u, v\}$ each joining two vertices $u, v \in V(G)$.

Drawing $D(G)$ of a graph G into the plane is a mapping f, which assigns to each vertex a distinct point in the plane and to each edge uv assigns a continues arc connecting points $f(u)$ and $f(v)$ (continues arc is an image of a closed interval under continues mapping). The point, which is image of a vertex, is called *vertex of drawing* $D(G)$ and the arc connecting $f(u)$ and $f(v)$ is called the *edge* uv *of* $D(G)$. The mapping f must also satisfy the following conditions: (1) No edge of $D(G)$ passes through a vertex (except for the endpoints of the edge). (2) Two edges of $D(G)$ have a finite number of intersection points. (3) Any intersection of two edges of $D(G)$ is a proper crossing (they cannot just touch). (4) No three edges of $D(G)$ have a common intersection point.





Because we use the drawing of G very often, we simply say "topological graph" instead of "drawing of a graph". Or we say "geometric graph" instead of "drawing of a graph by straight-line segments". More precise definitions are below.

A *geometric graph* is a graph G drawn in the plane by straight line segments. It is defined as a pair $(V(G), E(G))$, where $V(G)$ is a finite set of points in general position in the plane, i.e. no three points are collinear, and $E(G)$ is a set of line segments with endpoints in $V(G)$. $V(G)$ and $E(G)$ are the *vertex set* and the *edge set* of G, respectively. Let H and G be two geometric graphs, we say that H is a *(geometric) subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A *topological graph* is defined similarly. It is a graph drawn in the plane in such a way that edges are Jordan curves. Two of these curves share a finite number of points and no curve passes through a vertex. Obviously, geometric graphs are a subclass of topological graphs. Topological graph G is *simple* if every two edges of G share at most one point.

We say that two edges *cross* each other if they have an interior point in common. Two edges are *disjoint* if they have no point in common.

A *geometric thrackle* is a geometric graph, such that every two edges intersect. A *crossing family* is a set of edges of the geometric graph, such that every two edges cross each other.

2.2 Plane graphs

Planar graph is an abstract graph, which can be drawn in the plane with no crossing edges. *Plane graph* is a topological graph, the drawing of a graph in the plane.

Theorem 2.2.1 (Euler formula[1]). *Let $G = (V, E)$ be a connected planar graph and $D(G)$ be its plane drawing. Put $v := |V|$, $e = |E|$ and $f := |F|$, where F is the set of faces of $D(G)$ (number of connected regions, which we get after cutting the plane along the edges E). Then*

$$v + f = e + 2.$$

Corollary 2.2.2. *Let $G = (V, E)$ be a planar graph. Then $|E| \leq 3|V| - 6$. That further implies that G has a vertex v with $deg(v) \leq 5$ (in fact there are at least 4 such vertices).*

Proof. The proof of Euler formula and inequality $|E| \leq 3|V| - 6$ is for example in Matoušek, Nešetřil [51]. Here we show just the extension, that there are at least 4 vertices with degree at most 5 in G.

We can assume that G is a plane triangulation. Otherwise we add an edge to each face which is not triangle. Adding edges can only increase the degrees of vertices. Thus the minimum degree of a vertex in G is at least 3.

Suppose for a contradiction, that $\deg(v) \geq 6$ for every vertex except for at most 3 vertices, which have $3 \leq \deg(v) \leq 5$. Thus $\sum_{v \in V} \deg(v) \geq 6n - 9$. On

[1]This formula holds also for convex polytopes in $\mathbb{R}^3$. Vertices are vertices of the polytope, similarly edges and faces. In fact, that is where we took to notation for graphs. Euler also discovered this formula for polytopes. It is interesting, that the formula can be generalized for polytopes in higher dimensions ($\mathbb{R}^d$, $d \geq 4$). Then there are also higher dimensional faces and we get more than one equation.

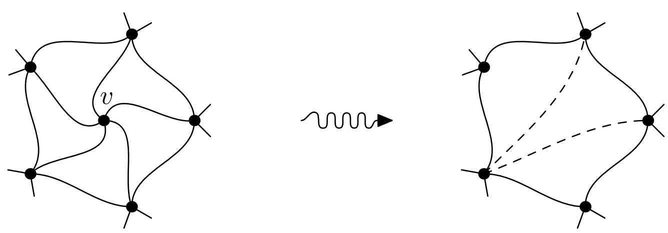
the other hand by the corollary of Euler formula $\sum_{v \in V} \deg(v) = 2|E| \leq 6n - 12$. That is a contradiction. □

Theorem 2.2.3 (Fáry's theorem [24]). *Any planar graph G admits a drawing into the plane such that the edges are straight line segments, which do not cross.*

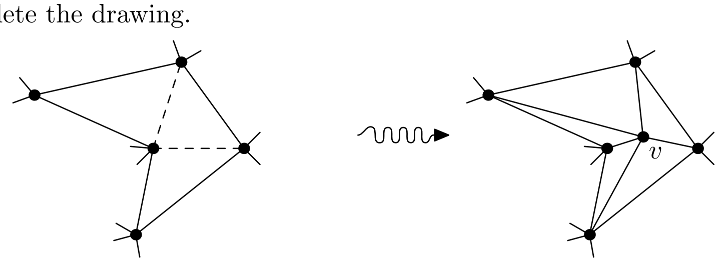
The original Fáry's proof uses a decomposition of G into 3 connected trees and then barycentric coordinates. Here we present much simpler proof.

Proof. [2] Let $G = (V, E)$ be a planar graph with n vertices. We may assume that G is maximal planar graph, thus its drawing $D(G)$ is a plane triangulation. Otherwise we add a new edge into each face which is not triangle.

Choose some three vertices $a, b, c \in V$ forming a triangular face. We prove by induction on n that there exists a straight-line drawing of G in which the triangle abc is the outer face. The result is trivial for $n = 3$, because a,b, and c are the only vertices in G. In other cases, all vertices in G have at least three neighbors.



Suppose that $n > 3$. By the corollary of Euler formula, G has a vertex v of degree at most 5. Moreover there are at least 4 such vertices, thus at least one is different from a, b, c. Let G' be planar graph, which we get by removing v from G and retriangulating the face formed by removing v. By induction hypothesis, G' has a straight line drawing with outer face abc. Remove the added edges in G', forming a polygon P with at most five sides into which v should be placed to complete the drawing.



Now, it is just a simple case analysis to show, that every polygon P with at most 5 sides contains an interior point p, which sees all vertices (imagine the polygon as walls). Then place v into point p. The edges incident with v do not cross any other edges because of visibility from p to all vertices of P. That completes the proof. □

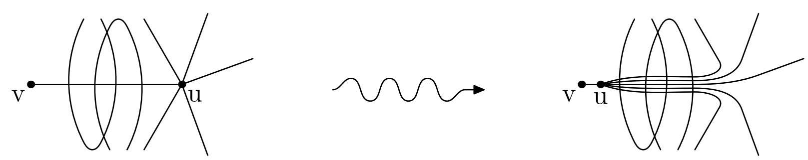
[2] The author of this proof is unknown. You can find this proof for example on Wikipedia.

Theorem 2.2.4 (Hanani-Tutte). *If graph G has drawing in the plane such that every pair of edges cross even number times then G is planar.*

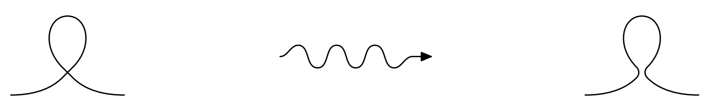
Proof (Pelsmajer, Schaefer, Štefankovič [67]). [3] [4]

Let $D(G)$ be the drawing of G where each pair of edges cross even number times. Assume that G has only one connected component otherwise we can show the theorem for each component separately.

Pull operation: To pull a vertex u towards a vertex v along the edge uv means to redraw the vertex u on the edge uv closer to the vertex v, shorten the edge uv to the part between new u and v and redraw all other edges incident with u in such a way that they follow their original drawing up to close neighborhood of u where they form an infinitesimally narrow beam together with the other edges incident with u and then they follow the original drawing of edge uv until they join vertex u.



We always pull u so close to v, that there is no edge crossing uv. If we create some self-crossing edges, we replace their self-crossings as shown on the next figure.



The property of pulling operation is that after the pulling, each pair of edges cross even number times. Indeed, the edge uv does not cross anything, which is an even number. To the other edges incident with u we have added all crossings of the original edge uv, which is again an even number.

Take arbitrary spanning tree T of G and choose one vertex r as a root. Pull the neighbors of r in T towards the root r. Then pull the neighbors of these neighbors towards the root and continue this way until the whole spanning tree T is in the neighborhood of r and is not crossed by any edge of G.

Now, we want to redraw the other edges to get plane drawing of G. For a tree T, we denote the unique path in T between x and y by xTy. Put

[3] Actually, the Hanani-Tutte theorem is much stronger: If graph G has a drawing in the plane such that every pair of *independent* edges cross even number times then G is planar. Two edges are independent, if they have no vertex in common. Pelsmajer, Schaefer, Štefankovič [67] prooved the stronger version. Here we show just the weak version.

[4] Once I was thinking about this proof. I discovered this simple proof. I got enthusiastic. I told several people and they were surprised, how easy it is to proof Hanani-Tutte theorem. But I got disappointed a little later, when I searched the web, because the authors of [67] were faster by a few months. The simple proof was not known for 35 years and I missed that by a few months. That's life. Here in this proof I present my version of the proof.

$F := E(G) \setminus E(T)$. All edges of F starts and ends on the spanning tree T. Consider an edge $e = xy \in F$. The edge ye together with the unique path xTy forms a cycle C_e. There is no edge $f \in F$ starting inside of C_e and ending outside of C_e, because e and f would have odd number of crossings (f does not cross xTy).

In other words, if you imagine an infinitisimaly thin cycle C_T around T, then the endpoints of edges F form a valid system of parenthesis.

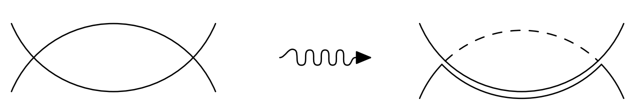
Let us start with $i = 0$ and $F_0 = F$. Take an edge $e_i = xy \in F_i$ such that there is no edge of F_i starting and ending inside C_{e_i} (all edges start and end on T). Such an edge exists because we can take e_i with shortest $C_{e_i} \cap T$. Redraw $e_i = xy$ along xTy so close to the the path xTy or to the already redrawn edges, that it does not intersect any other edge of F_i. Set $F_{i+1} = F_i \setminus \{e_i\}$ and repeat this step until $F_i \neq \emptyset$. The invariant is that at every step i all edges $E \setminus F_i$ are drawn without crossings. We finish with a plane drawing of the graph G. □

Alternative proof. [5]

First, we follow the first proof. Pull the vertices towards the root r until we get all vertices to a small neighborhood of r. Then the whole spanning tree T is not crossed by any other edge.

Now we proceed by induction on the number of crossings. Until there are two crossing edges e, f in $D(G)$, we repeat the following.

The edges e, f cross. Thus they have at least two crossings. Take two crossings $p, q \in e \cap f$ that are consecutive on e. Denote the part of e between p and q by $e_{p,q}$. Similarly, denote the part of f between p and q by $f_{p,q}$. Curves $e_{p,q}$ and $f_{p,q}$ form a Jordan cycle C, because p, q are consecutive on e. Since the whole spanning tree T (and all vertices V) is either inside C or outside C, every edge $h \in E$, which cross the cycle C, has to cross it even number times. Thus the parity of $|h \cap e_{p,q}|$ and $|h \cap f_{p,q}|$ is the same. Let us say that $f_{p,q}$ contains less or equal number of crossings with other edges than $e_{p,q}$. (Otherwise swap the names for e and f). Then redraw the part $e_{p,q}$ of edge e along $f_{p,q}$ and get rid of the crossings p and q.



If we create self-crossing edges, we replace their self-crossings as we did in the pulling phase (see second figure in the first proof of Hanani-Tutte Theorem).

All pairs of edges cross even number times and we decreased the number of crossings. □

[5] Joke: A professor at a lecture hall explains some theorem. Suddenly some student raises his arm and says: "Mr. professor, I have a counter example!". Professor answers: "It does not matter. I have two proofs."

2.3 Crossing numbers

If we get a drawing of a graph G, we can easily count the number of crossings. The number of crossings in a drawing $D(G)$ is the crossing number $\textsc{cr}(D(G))$ of the drawing $D(G)$. *Crossing number* of an abstract graph G, denoted $\textsc{cr}(G)$, is the smallest number of crossings in any drawing of G.

The first motivation for crossing number comes from 50s [44, 29] and the task was to determine $\textsc{cr}(K_n)$ and $\textsc{cr}(K_{n,n})$. There are some partial results and famous conjectures for $\textsc{cr}(K_n)$ and $\textsc{cr}(K_{n,n})$, but the exact values were not determined yet.

The breakthrough in applications came with the following theorem, which was discovered by Ajtai, Chvátal, Newborn, Szemeredi [5] and independently by Leighton [48]. This theorem is a basic ingredient in quite simple proofs for many geometric problems (for example Szemerédi-Trotter theorem for line-point incidences, upper bounds for unit distances in the plane). Author of the crossing number method is Székely [70]. For a survey of applications of crossing number method, see Matoušek [50]).

Theorem 2.3.1 (Crossing lemma). *Let G be a graph with n vertices and m edges. If $m \geq 4n$ then*

$$\textsc{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

Proof. Let G be a topological graph (drawing of an abstract graph G) with the minimum $\textsc{cr}(G)$.

Simple estimate on $\textsc{cr}(G)$: Graph G has $\textsc{cr}(G)$ crossings. We remove each crossing by deleting one edge involved in the crossing. After removing all crossings one by one, we end up with a graph with no crossings. That is a plane graph. This plane graph has at most $m - \textsc{cr}(G) \leq 3n - 6$ edges. Thus we get

$$\textsc{cr}(G) \geq m - 3n. \tag{2.1}$$

Probabilistic strengthening: Let $0 < p < 1$ be a fixed parameter. We will choose the suitable value of p later.

Choose a random subgraph $H \subseteq G$ by choosing each vertex of G at random, independently and uniformly, with probability p. Set of chosen vertices determines an induced subgraph H. Put $n_H := |V(H)|$, $m_H := |E(H)|$. The expected number of vertices in H is $\mathbb{E}[n_H] = pn$. The expected number of edges in H is $\mathbb{E}[m_H] = p^2 m$, because each edge of G stays in H iff both of its end-points stay in H. Similarly, the expected crossing number of topological subgraph H is $\mathbb{E}[\textsc{cr}(H)] \leq p^4 \textsc{cr}(G)$.[6] Each crossing of G remains in H is both crossing edges remains in H, that means end-vertices of these edges must stay in H.

Because every possible topological subgraph H satisfies the simple estimate (2.1), the expected values have to satisfy the same inequality. Thus $\mathbb{E}[\textsc{cr}(H)] \geq$

[6] Note that the crossing number of abstract subgraph H is smaller than or equal to the crossing number of the particular drawing of H, inherited from the topological graph G.

$\mathbb{E}[m_H] - 3\mathbb{E}[n_H]$, what is

$$p^4 \text{CR}(G) \geq p^2 m - 3pn.$$

Now we set $p := 4n/m$ and after some algebraic operations we get

$$\text{CR}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

Note that the choice $p = 4n/m < 1$, because $4n \leq m$. Thus it corresponds to a probability. □

One improvement of the theorem is due to Pach, Tóth [63]: If $m \geq 7.5n$ then $\text{CR}(G) \geq \frac{1}{33.75} \frac{m^3}{n^2}$. Currently the best constant in the theorem is due to Pach, Radoičić, Tardos, Tóth [59]: If $m \geq \frac{103}{6} n$ then $\text{CR}(G) \geq \frac{1024}{31827} \frac{m^3}{n^2}$.

All improvements of the constant in Crossing lemma use the same probabilistic proof. They just improve the simple estimate on $\text{CR}(G)$,[7] which immediately leads to a better constant.

The crossing lemma is asymptotically tight in general case. But for special classes of graphs, we can get better lower bounds.

Pach, Spencer, Tóth [61] showed a better bound for graphs with monotone graph properties. Graph property $\mathcal{P}$ is *monotone* if whenever a graph G satisfies $\mathcal{P}$, then every subgraph of G also satisfies $\mathcal{P}$, and whenever G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union satisfies $\mathcal{P}$, too.

Theorem 2.3.2 (Pach, Spencer, Tóth [61]). *Let $\mathcal{P}$ be monotone graph property such that every graph G with n vertices, satisfying property $\mathcal{P}$, has at most $O(n^\alpha)$ edges, for some fixed $\alpha > 0$.*

Then there exist constants $c, c' > 0$ such that every graph G with n vertices and $m \geq cn \log^2 n$ edges, satisfying property $\mathcal{P}$ has

$$\text{CR}(G) \geq c' \frac{m^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

Particularly, for graphs with $girth(G) > 2r$, we get the following theorem. Székely found a quite simple proof for this theorem.

Theorem 2.3.3 (Pach, Spencer, Tóth [61], Székely [69]). *Let G be a simple topological graph with n vertices and m edges. If $m \geq 4n$ and $girth(G) > 2r$ then*

$$\text{CR}(G) \geq c_r \cdot \frac{m^{r+2}}{n^{r+1}},$$

where $r > 0$ is a fixed integer and constant c_r depends on r.

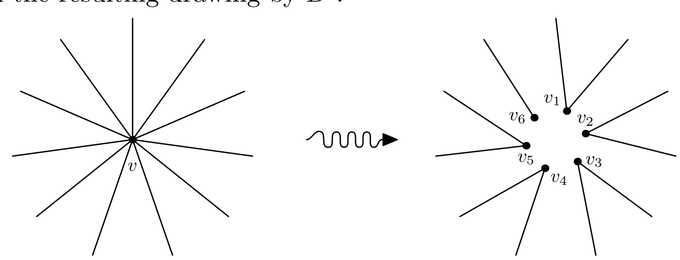
The proof of this theorem is a nice illustration of *embedding method.* The basic idea of embedding method is the following. Let G be a graph, for which we need an upper bound on $\text{CR}(G)$. The method starts with an optimal drawing

[7] Which stopped being simple and becomes more and more complicated.

of a connected subgraph $H \subseteq G$, which is has the same vertex set and known $\text{CR}(H)$. Then we embed each edge $e = uv \in E(G) \setminus E(H)$ "infinitesimally close" to a path in H connecting u and v. Now we can easily count the added crossings (they lie either close to an original crossing in the drawing of H or close to a vertex) and find an upper bound $\text{CR}(G) \leq F(\text{CR}(H), n, m)$ for some function F. For application of embedding method see for example Leighton [48], Richter and Thomassen [68], Székely [69], Fox and Tóth [26], Černý, Kynčl and Tóth [12].

Proof of the theorem 2.3.3. Let G be a graph with n vertices, m edges, crossing number $\text{CR}(G)$ and $girth(G) > 2r$. Let D be the drawing of G realizing $\text{CR}(G)$. Let ρ denote the average degree of G, that is $\rho := 2m/n$.

Auxiliary graph G' with maximum degree ρ: We split every vertex of G whose degree exceeds ρ into vertices of degree at most ρ, as follows. Let v be a vertex with degree $\deg(v) > \rho$, and let $vw_1, vw_2, \ldots, vw_{deg(v)}$ be edges incident with v, listed in the clockwise order. Replace v by $\lceil deg(v)/\rho \rceil$ new vertices $v_1, v_2, \ldots, v_{\lceil deg(v)/\rho \rceil}$, placed in clockwise order on a very small circle around v. Without introducing any new crossings, connect w_j with v_i, if and only if $\rho(i-1) < j \leq \rho i$, for $1 \leq j \leq \rho$ and $1 \leq i \leq \lceil deg(v)/\rho \rceil$. Repeat this procedure with every vertex whose degree exceeds ρ. Denote the resulting graph by G' and the resulting drawing by D'.



Observe, that we did not add any crossings. Thus $\text{CR}(G') \leq \text{CR}(G)$. The girth of G' still exceeds $2r$. Maximum degree of G' is at most $\Delta := \lfloor \rho \rfloor$. Further we have $|V(G')| \leq \sum_{v \in V} \lceil \deg(v)/\rho \rceil \leq n + \sum_{v \in V} \deg(v)/\rho = 2n$. Thus the average degree of G' is at least $\rho/2 = m/n$.

r-distance graph G'': Now, we define a graph G'' with $V(G'') := V(G')$, and $E(G'') =$ pairs of vertices from $V(G')$, whose distance in G' is exactly r.

Since $girth(G) > 2r$, the r-neighborhood of every vertex $v \in V(G)$ is a tree. (We will use this local property very often in this proof.) Thus maximum degree of a vertex in G'' is at most $\Delta(\Delta-1)^{r-1}$.

Embedding method: We construct the drawing D'' of G'' by embedding method. We use the drawing D' of G' as a plan for embedding. Every edge $e \in E(G'')$ correspond to a unique r-path P_e in G'. Draw e as a curve "infinitesimally close" to this unique path P_e. Repeat this for every edge in $E(G'')$ and finally obtain the drawing D''.

We can split crossing in D'' into two categories. A crossing of the *first category* lies very close to the original crossing of two edges e_1, e_2 of D', which are parts of two different r-paths. The number of r-paths containing a fixed edge

e is at most $r(\Delta-1)^{r-1} \leq r\Delta^{r-1}$ (there are r choices for the edge on r-path and for each vertex on a "free" end of the path we have at most $\Delta - 1$ choices). Therefore, every crossing of D' corresponds to at most $r^2\Delta^{2r-2}$ crossings of D'' of the first category.

A crossing of the *second category* lies very close to a vertex of D', where two different r-paths, passing through vertex, cross. The number of different r-paths passing through a fixed vertex v is at most $\Delta \cdot r\Delta^{r-1}$, because there are at most Δ edges incident with v and there are at most $r\Delta^{r-1}$ r-paths containing a fixed edge. Thus around every vertex of G', there are at most $\binom{r\Delta}{2}$ crossings of the second category.

Altogether we have

$$\mathrm{CR}(G'') \leq r^2\Delta^{2r-2}\mathrm{CR}(G) + 2n\binom{r\Delta^r}{2}.$$

That is

$$\mathrm{CR}(G) \geq \frac{\mathrm{CR}(G'')}{r^2\Delta^{2n-2}} - n\Delta^2. \tag{2.2}$$

Lower bound on $E(G'')$**:** Now we show that $E(G'') \geq c_r'' \cdot \frac{m^r}{n^{r-1}}$, for some constant c_r'' dependent on r.[8]

For any real $s > 0$, we construct a large subgraph $H' \subseteq G'$ with minimum degree at least $\rho/(4+s)$. Consequently we get $H'' \subseteq G''$ with many edges.

Delete all vertices of G' with degree smaller than $\rho/(4+s)$, together with edges incident with them. We iterate this for the resulting graph. More formally we construct a sequence $G' = G_0' \supseteq G_1' \supseteq \dots$ of induced subgraphs as follows. If G_i' has a vertex v_i of degree $\deg(v_i) < \rho/(4+s)$, we let $G_{i+1}' := G_i' \setminus v_i$. If not, we terminate the sequence and set $H' := G_i'$. The average degree of H' is still at least $\rho/2$, because with each vertex we had deleted at most $\rho/(4+s) < \rho/4$ edges. Since G' has at most $2n$ vertices, altogether we had deleted at most $4m/(4+s)$ edges. Therefore at least $sm/(4+s)$ edges are left in H'. Because the maximum degree in H' is at most $\Delta = \lfloor \rho \rfloor$, there are at least $sn/(4+s)$ vertices in H'.

Let H'' be the r-distance graph of H'. Clearly, $H'' \subseteq G''$. For every $v \in V(H')$, there are at least $(\frac{\rho}{(4+s)})^r$ paths of length r starting in v. Thus $E(G'') \geq E(H'') \geq \frac{sn}{4+s} \cdot \frac{1}{2}(\frac{\rho}{4+s})^r \geq s(\frac{1}{2(4+s)})^{r+1} \cdot \frac{m^r}{n^{r-1}}$.

Last step: By Applying the crossing lemma (Theorem 2.3.1) on G'', we obtain that $|E(G'')| \leq 8n$, or

$$\mathrm{CR}(G'') \geq \frac{c}{(2n)^2}\left(c_r'' \cdot \frac{m^r}{n^{r-1}}\right)^3. \tag{2.3}$$

[8] Erdős and Simonovits ([20], combination of formula (13) and Theorem 5) showed even that $|E(G'')| \geq (\frac{1}{2} - o(1)) \cdot \frac{m^r}{n^{r-1}}$. By using their lower bound, we can improve the constant c_r in the statement of this theorem.

Finally, by combination of inequalities (2.2) and (2.3) we get

$$\mathrm{CR}(G) \geq \frac{c c_r''^3}{2^{2r} r^2} \cdot \frac{m^{r+2}}{n^{r+1}} - \frac{4m^2}{n} \geq C(r) \cdot \frac{m^{r+2}}{n^{r+1}} - \frac{4m^2}{n},$$

where $C(r)$ is a constant depending on r. The term $\frac{4m^2}{n}$ is little-oh of the main term. Hence we finished the proof. □

2.3.1 Different definitions of a crossing number

It is clear, what is a crossing of two edges. But there are four definitions of crossing number. Let G be a simple graph.

- The *crossing number* of G, $\mathrm{CR}(G)$, is the minimum number of crossings over all drawings of G.
- The *rectilinear crossing number* of G, $\mathrm{LIN\text{-}CR}(G)$, is the minimum number of crossings in any drawing of G, where edges are represented by straight-line segments.
- The *pairwise crossing number* of G, $\mathrm{PAIR\text{-}CR}(G)$, is the minimum number of pairs of crossing edges over all drawings of G.
- The *odd crossing number* of G, $\mathrm{ODD\text{-}CR}(G)$, is the minimum number of pairs of edges with odd number of crossing over all drawings of G.

The following lemma is in fact a simple observation.

Lemma 2.3.4. *For any simple graph, we have*

$$\mathrm{ODD\text{-}CR}(G) \leq \mathrm{PAIR\text{-}CR}(G) \leq \mathrm{CR}(G) \leq \mathrm{LIN\text{-}CR}(G).$$

There is a difference between CR and LIN-CR, because Guy [30] has shown that $\mathrm{LIN\text{-}CR}(K_8) = 19 > 18 = \mathrm{CR}(K_8)$. Moreover, Bienstock and Dean [7] constructed graphs G_n with $\mathrm{CR}(G_n) = 4$, but arbitrary large $\mathrm{LIN\text{-}CR}(G_n)$.

There is also difference between PAIR-CR and ODD-CR. Pelsmajer, Schaefer, Štefankovič [66] have shown that there is a graph G with

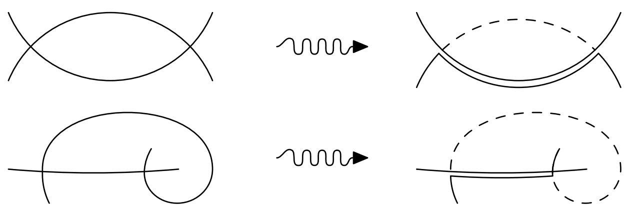
$$\mathrm{ODD\text{-}CR}(G) \leq \left(\frac{\sqrt{3}}{2} + o(1)\right) \mathrm{PAIR\text{-}CR}(G).$$

Tóth [73] improved this bound a little to $\mathrm{ODD\text{-}CR}(G) \leq 0.855 \cdot \mathrm{PAIR\text{-}CR}(G)$. Pach and Tóth [65] have shown for every graph G that

$$\mathrm{ODD\text{-}CR}(G) \leq \mathrm{CR}(G) \leq 2(\mathrm{ODD\text{-}CR}(G))^2.$$

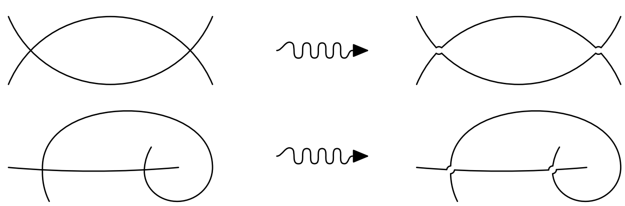
Lemma 2.3.5. *Let G be a graph and $D(G)$ be its drawing into a plane with minimum number of crossings (witnessing $\mathrm{CR}(G)$). Then each pair of edges e, f cross at most once.*

Proof. Suppose on the contrary, that e, f are two edges that cross at least twice. Take two crossing $p, q \in e \cap f$ which are consecutive on e. Denote the part of e between p and q by $e_{p,q}$. Similarly the part of f between p and q by $f_{p,q}$. Curves $e|_{p,q}$ and $f_{p,q}$ form a cycle C, because p, q are consecutive on e. Let us say that $f_{p,q}$ contains less or equal number of crossings with the other edges than $e_{p,q}$. (Otherwise swap the names for e and f). Then redraw the part $e_{p,q}$ of edge e along $f_{p,q}$ as shown on the next figures. There are two possibilities – either e crosses f in the points p, q from the same side of f or from different sides.



If we create self-crossings on edge e, we shorten the edge e by deleting loops. We reduced the total number of crossings at least by one, what is a contradiction. □

Note: Another option in the previous proof was to swap parts $e_{p,q}$, $f_{p,q}$ between edges e and f and make a small distance between e, f at points p, q, where e, f touch. As shown on the next figure.



That removes only crossings p, q, but it is sufficient.

One might think, that previous lemma shows, that $\mathrm{CR} = \mathrm{PAIR\text{-}CR}$, but it is not true. Why? We start with the optimal drawing for $\mathrm{CR}(G)$. Then we might redraw some edge e so, that the new edge e contains more than one crossing, but on the other hand the redrawing reduces the number of crossing pairs. In other words the optimal drawing of a graph G for $\mathrm{CR}(G)$ need not to be optimal drawing for $\mathrm{PAIR\text{-}CR}(G)$.

Open problem 2 (Pach, Tóth [64]). *Is it true, that for every graph G, $\mathrm{CR}(G) = \mathrm{PAIR\text{-}CR}(G)$?*

Tóth [73] improved the result of Valtr [80] and showed currently the best upper bound, that every graph G with $k := \mathrm{PAIR\text{-}CR}(G)$ has

$$\mathrm{CR}(G) \leq \frac{9k^2}{\log^2 k}.$$

2.3.2 Decay of crossing number

In this subsection we look what happens, if we delete some edges from a graph. How much does the crossing number decrease?

Richter and Thomassen [68] conjectured that there is a constant c such that every graph G has an edge e with $\text{CR}(G \setminus e) \geq \text{CR}(G) - c\sqrt{\text{CR}(G)}$. They only proved that G has an edge e with $\text{CR}(G \setminus e) \geq \frac{2}{5}\text{CR}(G) - O(1)$. The following theorem together with Crossing lemma shows, that the conjecture holds for graphs with $\Omega(n^2)$ edges.

Theorem 2.3.6 (Pach, Tóth [64]). *Let G be a connected graph on n vertices and $m \geq 1$ edges. For every edge $e \in E(G)$ we have*

$$\text{CR}(G \setminus e) \geq \text{CR}(G) - m + 1.$$

Proof. Take a drawing of G with the minimum number of crossings. The chosen edge e cross any other edge at most once by Lemma 2.3.5. Thus by removing e we loose at most $m - 1$ crossings. □

What if we want to delete more edges? Let us say some positive fraction. Some edges are more responsible for the crossing number, because some edge might contain more crossings than other edge. Thus Fox and Cs. Tóth [26] asked what is the minimum decrease of crossing number in the case, when the theorem can choose which edges will be deleted.

Theorem 2.3.7 (Fox, Cs. Tóth [26]). *For every $\varepsilon > 0$, there is an n_ε such that every graph G with $n(G) \geq n_\varepsilon$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon}{24}\right) m(G)$$

and

$$\text{CR}(G') \geq \left(\frac{1}{28} - o(1)\right) \text{CR}(G).$$

We improved their result to the following.

Theorem 2.3.8 (Černý, Kynčl, Tóth [12]). *For every $\varepsilon, \gamma > 0$, there is an $n_{\varepsilon,\gamma}$ such that every graph G with $n(G) \geq n_{\varepsilon,\gamma}$ vertices and $m(G) \geq n(G)^{1+\varepsilon}$ edges has a subgraph G' with*

$$m(G') \leq \left(1 - \frac{\varepsilon\gamma}{2394}\right) m(G)$$

and

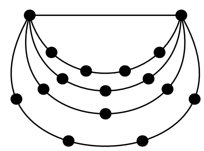
$$\text{CR}(G') \geq (1 - \gamma)\text{CR}(G).$$

2.3.3 Proof of the theorem on decay of $\mathrm{CR}(G)$

Because this is my thesis and I am co-author of the proof of Theorem 2.3.8, I show the proof here. The proof is taken from [12] almost without changes. It is quite simple proof and it also illustrate the use of embedding method.

The proof is based on the argument of Fox and Tóth [26], the only new ingredient is Lemma 2.3.10.

Definition 2.3.9. *Let $r \geq 2, p \geq 1$ be integers. A $2r$-earring of size p is a graph which is a union of an edge uv and p edge-disjoint paths between u and v, each of length at most $2r-1$. Edge uv is called the main edge of the $2r$-earring.*



5-earring of size 4

Lemma 2.3.10. *Let $r \geq 2, p \geq 1$ be integers. There exists n_0 such that every graph G with $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges contains at least $m/3pr$ edge-disjoint $2r$-earrings, each of size p.*

Proof. By the result of Alon, Hoory, and Linial [6], for some n_0, every graph with $n \geq n_0$ vertices and at least $n^{1+1/r}$ edges contains a cycle of length at most $2r$.

Suppose that G has $n \geq n_0$ vertices and $m \geq 6prn^{1+1/r}$ edges. Take a *maximal* edge-disjoint set $\{E_1, E_2, \ldots, E_x\}$ of $2r$-earrings, each of size p. Let $E = E_1 \cup E_2 \cup \cdots \cup E_x$, the set of all edges of the earrings and let $G' = G \setminus E$. Now let E'_1 be a $2r$-earring of G' of maximum size. Note that this size is less than p. Let $G'_1 = G' \setminus E'_1$. Similarly, let E'_2 be a $2r$-earring of G'_1 of maximum size and let $G'_2 = G'_1 \setminus E'_2$. Continue analogously, as long as there is a $2r$-earring in the remaining graph. We obtain the $2r$-earrings $E'_1, E'_2, \ldots, E'_y$, and the remaining graph $G'' = G'_y$ does not contain any $2r$-earring. Let $E' = E'_1 \cup E'_2 \cup \cdots \cup E'_y$.

We claim that $y < n^{1+1/r}$. Suppose on the contrary that $y \geq n^{1+1/r}$. Take the main edges of $E'_1, E'_2, \ldots, E'_y$. We have at least $n^{1+1/r}$ edges so by the result of Alon, Hoory, and Linial [6] some of them form a cycle C of length at most $2r$. Let i be the smallest index with the property that C contains the main edge of E'_i. Then C, together with E'_i would be a $2r$-earring of G'_{i-1} of greater size than E'_i, contradicting the maximality of E'_i.

Each of the earrings $E'_1, E'_2, \ldots, E'_y$ has at most $(p-1)(2r-1)+1$ edges so we have $|E'| \leq y(p-1)(2r-1) + y < (2pr-1)n^{1+1/r}$. The remaining graph, G'' does not contain any $2r$-earring, in particular, it does not contain any cycle of length at most $2r$, since it is a $2r$-earring of size one. Therefore, by [6], for the number of its edges we have $e(G'') < n^{1+1/r}$.

It follows that the set $E = \{E_1, E_2, \ldots, E_x\}$ contains at least $m - 2prn^{1+1/r} \geq \frac{2}{3}m$ edges. Each of $E_1, E_2, \ldots, E_x$ has at most $p(2r-1)+1 \leq 2pr$ edges, therefore, $x \geq m/3pr$. □

Lemma 2.3.11. Fox and Cs. Tóth [26] *Let G be a graph with n vertices, m edges, and degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Let ℓ be the integer such that $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. If n is large enough and $m = \Omega(n \log^2 n)$ then*

$$\text{CR}(G) \geq \frac{1}{65} \sum_{i=1}^{\ell} d_i^2.$$

Proof of the Theorem. Let $\varepsilon, \gamma \in (0,1)$ be fixed. Choose integers r, p such that $\frac{1}{r} < \varepsilon \leq \frac{2}{r}$ and $\frac{132}{p} < \gamma \leq \frac{133}{p}$. Then there is an $n_{\varepsilon,\gamma}$ with the following properties: (a) $n_{\varepsilon,\gamma} \geq n_0$ from Lemma 2.3.10, (b) $(n_{\varepsilon,\gamma})^{1+\varepsilon} > 6pr \cdot (n_{\varepsilon,\gamma})^{1+1/r}$,

Let G be a graph with $n \geq n_{\varepsilon,\gamma}$ vertices and $m \geq n^{1+\varepsilon}$ edges.

Let $v_1, \ldots, v_n$ be the vertices of G, of degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ and define ℓ as in Lemma 2.3.11, that is, $\sum_{i=1}^{\ell-1} d_i < 4m/3$ but $\sum_{i=1}^{\ell} d_i \geq 4m/3$. Let G_0 be the subgraph of G induced by $v_1, \ldots, v_\ell$. Observe that G_0 has at least $m/3$ edges. Therefore, by Lemma 2.3.10 G_0 contains at least $m/9pr$ edge-disjoint $2r$-earrings, each of size p.

Let M be the set of the main edges of these $2r$-earrings. We have $|M| \geq m/9pr \geq \frac{\varepsilon\gamma}{2394} m$. Let $G' = G \setminus M$ and $G'_0 = G_0 \setminus M$.

Take an optimal drawing $D(G')$ of the subgraph $G' \subset G$. We have to draw the missing edges to obtain a drawing of G. Our method is a randomized variation of the embedding method. For every missing edge $e_i = u_i v_i \in M \subset G_0$, e_i is the deleted main edge of a $2r$-earring $E_i \subset G_0$. So there are p vertex-disjoint paths in G_0 from u_i to v_i. For each of these paths, draw a curve from u_i to v_i infinitesimally close to that path. Call these p curves *potential $u_i v_i$–edges* and call the resulting drawing D.

To get a drawing of G, for each $e_i = u_i v_i \in M$, choose one of the p potential $u_i v_i$–edges at random, independently and uniformly, with probability $1/p$, and draw the edge $u_i v_i$ as that curve.

There are two types of new crossings in the obtained drawing of G. First category crossings are infinitesimally close to a crossing in $D(G')$, second category crossings are infinitesimally close to a vertex of G_0 in $D(G')$.

The expected number of first category crossings is at most

$$\left(1 + \frac{2}{p} + \frac{1}{p^2}\right) \text{CR}(G') = \left(1 + \frac{1}{p}\right)^2 \text{CR}(G').$$

Indeed, for each edge of G', there can be at most one new edge drawn next to it, and that is drawn with probability at most $1/p$. Therefore, in the close neighborhood of a crossing in $D(G')$, the expected number of crossings is at most $(1 + \frac{2}{p} + \frac{1}{p^2})$. See figure 2.1(a).

In order to estimate the expected number of second category crossings, consider the drawing D near a vertex v_i of G_0. In the neighborhood of vertex v_i we have at most d_i original edges. Since we draw at most one potential edge along each original edge, there can be at most d_i potential edges in the neighborhood. Each potential edge can cross each original edge at most once, and any two potential edges can cross at most twice. See figure 2.1(b). Therefore, the total

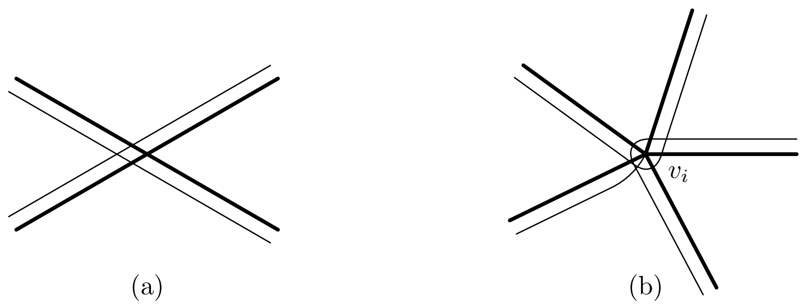


Figure 2.1: The thick edges are edges of G', the thin edges are the potential edges. Figure shows (a) a neighborhood of a crossing in $D(G')$ and (b) a neighborhood of a vertex v_i in G'.

number of first category crossings in D in the neighborhood of v_i is at most $2d_i^2$. (This bound can be substantially improved with a more careful argument, see e. g. [26], but we do not need anything better here.) To obtain the drawing of G, we keep each of the potential edges with probability $1/p$, so the expected number of crossings in the neighborhood of v_i is at most $\frac{1}{p}2d_i^2$.

Therefore, the total expected number of crossings in the random drawing of G is at most $(1+\frac{2}{p}+\frac{1}{p^2})\text{CR}(G')+\frac{2}{p}\sum_{i=1}^{\ell} d_i^2$.

There exists an embedding with at most this many crossings, therefore, by Lemma 2.3.11 we have

$$\text{CR}(G) \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G') + \frac{2}{p}\sum_{i=1}^{\ell} d_i^2 \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G') + \frac{130}{p}\text{CR}(G).$$

It follows that

$$\left(1-\frac{130}{p}\right)\text{CR}(G) \leq \left(1+\frac{1}{p}\right)^2 \text{CR}(G')$$

so

$$\left(1-\frac{130}{p}\right)\left(1-\frac{1}{p}\right)^2 \text{CR}(G) \leq \text{CR}(G')$$

consequently

$$\text{CR}(G') \geq \left(1-\frac{132}{p}\right)\text{CR}(G) \geq (1-\gamma)\text{CR}(G).$$

□

2.4 Geometric graphs with no forbidden subgraphs

We investigate properties of subclasses of geometric or topological graphs with some geometrical constrains. One of the simplest questions is how to characterize graphs with no crossing edges. These graph are known as *plane* graphs and have been studied for more than hundred years.

Kupitz, Erdős and Perles initiated and many others continued in the investigation of the following general problem. Given a class $\mathcal{H}$ of so-called forbidden geometric subgraphs, determine or estimate the maximum number $t(\mathcal{H}, n)$ of edges that a geometric graph with n vertices can have without containing a subgraph belonging to $\mathcal{H}$.

There are many nice results for various forbidden classes—k pairwise disjoint edges, k pairwise "parallel" edges, k pairwise crossing edges, self-crossing paths, even cycles and many others. For a survey of results on geometric graphs see Pach [57], Felsner [25].

In the following subsections, we look at some forbidden classes more in detail.

2.4.1 $k+1$ pairwise disjoint edges

We focus on geometric graphs with no $k+1$ pairwise disjoint edges. For $k \geq 1$, let $\mathcal{D}_k$ denote the class of all geometric graphs consisting of k pairwise disjoint edges. Denote $d_k(n) = t(\mathcal{D}_{k+1}, n)$ the maximum number of edges of a geometric graph on n vertices with no $k+1$ pairwise disjoint edges.

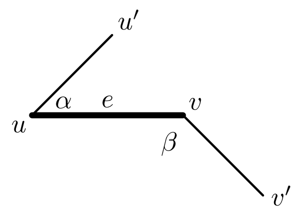
Let's look at the history of this problem. One of the first investigations on geometric graphs, besides planar graphs, was motivated by repeated distances in the plane. Erdős asked how many times can the maximum distance among n points in the plane be repeated. Connect each pair of points with the maximum distance by an edge. It's clear that the resulting graph cannot have two disjoint edges. Suppose on the contrary that there are two disjoint edges e, f. The convex hull of endpoints of e and f forms either a triangle or a quadrilateral. In both cases there is a distance longer than the length of the edge. That's a contradiction. The former question turns to the following: How many edges can have a geometric graph with no two disjoint edges? Erdős proved that at most n.

Theorem 2.4.1 (Erdős [18]). *$d_1(n) = n$.*

The proof is a beautiful illustration of a discharging method and it is very simple.

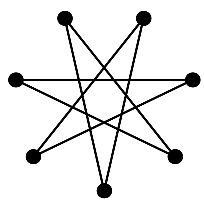
Proof (Perles). For each vertex, mark one edge incident to it. For vertices of degree one, there is no choice. At the other vertices, mark the left edge at the largest angle (For each such a vertex v, take a hen and place it close to v. The hen cannot sit in v nor on any edge incident to v, thus we place the hen to the wedge with largest angle. Then the hen looks from its place to v and marks the edge on the left).

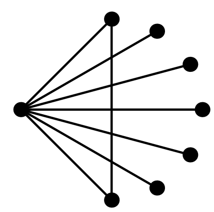
If there remain an unmarked edge $e = uv$, we have the situation as in the following figure:



There must be edges uu' and vv' because we marked the left edge at the largest angle at every vertex. Angles α and β are less than the largest angles at the vertices u, v, so they are less than π. But this implies that edges uu', vv' are disjoint and so there cannot be an unmarked edge in the graph. Thus, the number of edges is not more than the number of vertices.

On the other hand there exist graphs achieving this bound:





□

Definition 2.4.2.

- *A vertex v is pointed if all edges incident with it lie in a half-plane whose boundary contains the vertex v (see figure 2.2a).*

- *A vertex v which is not pointed is cyclic. This means that in every open half-plane determined by a line passing through the vertex v, there is an edge incident with v (see figure 2.2b).*

- *We say that an edge xy is to the left of an edge xz if the ray $\overrightarrow{xz}$ can be obtained from the ray $\overrightarrow{xy}$ by a clockwise turn of less than π. Similarly, we define that an edge is to the right of another edge (see figure 2.2c).*

The marking (deleting) idea from the proof of Theorem 2.4.1 can be generalized to marking in several rounds as in the proof of the following theorem.

Theorem 2.4.3. *$d_2(n) \leq 3n$.*

Proof (Goddard, Katchalski and Kleitman [28]). Let G be a geometric graph with no three pairwise disjoint edges. Set $G_0 = G$. We construct two subgraphs $G_i = (V, E_i)$ of graph G for $i \in \{1, 2\}$ as follows.

For each pointed vertex x in G_0 delete the leftmost edge at x. Denote the resulting graph by G_1. For each pointed vertex x in G_1 delete two rightmost edges at x (if any). Denote the resulting graph by G_2.

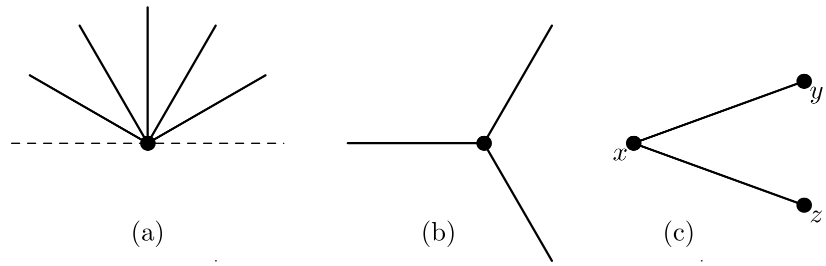
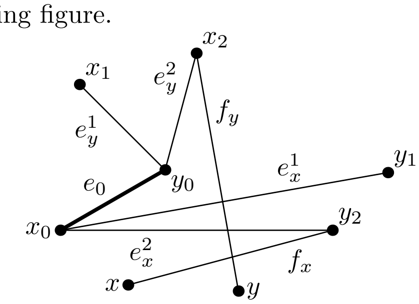


Figure 2.2: (a) an example of a pointed vertex, (b) an example of a cyclic vertex, (c) there are edges xy, xz where the edge xy is to the left of the edge xz.

Since we have deleted at most 3 edges at each vertex, we had deleted at most $3n$ edges. We want to show, that G_2 contains no edge. Assume for the contradiction that the edge $e_0 = x_0y_0$ was not deleted.

Then there exist two edges $e_x^1 = x_0y_1$, $e_x^2 = x_0y_2$ to the right of x_0y_0 and two edges $e_y^1 = y_0x_1$, $e_y^2 = y_0x_2$ to the right of y_0x_0. There is also an edge $f_x = x_2y$ to the left of x_2y_0 and there is also an edge $f_x = y_2x$ to the left of y_2x_0. See the following figure.



There are two cases. Either f_x, f_y are two disjoint edges, but then f_x, f_y, e_0 would be three disjoint edges, or f_x, f_y cross. In the second case, we may without loss of generality assume, that the supporting lines of f_x and f_y cross on the same side of e_0 as y_2. Then y_0x_2, x_0y_1, y_2x are three pairwise disjoint edges. Edges y_0x_2, y_2x are disjoint because they lie on opposite sides of x_0y_1. In both cases we got a contradiction. □

Theorem 2.4.4. *$d_2(n) \leq 2.5n$.*

This upper bound is tight up to an additive constant. It is a nice exercise to find a lower bound showing $d_2(n) \geq \lceil 2.5n \rceil - 3$. Take an inspiration from the next figure.

Sketch of the proof (Černý [9]). Let $G = (V, E)$ be a geometric graph with no three disjoint edges. Denote the number of cyclic vertices in G by γ. Using case analysis it is not difficult to show that $\gamma \leq 2$.

We construct two subgraphs $G_i = (V_i, E_i)$, $i = 1, 2$ as follows. For each pointed vertex in G delete the rightmost edge. Denote the resulting graph by

G_1. For each pointed vertex in G_1 delete the leftmost edge (if any). If there are two vertices c, d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of the segment cd. The vertex cyclic in G_1 must be one of the vertices c, d because $\gamma \leq 2$. Denote the resulting graph by G_2. Deleting in the second round is for each pointed vertex in G_1 (not in G)!

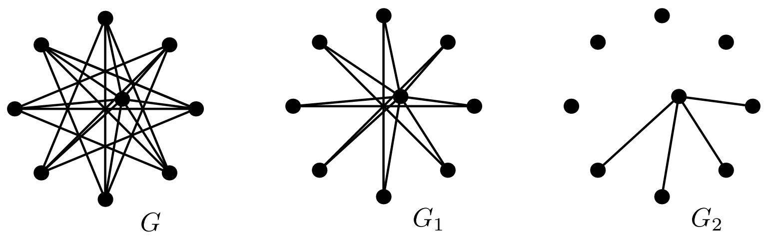


Figure 2.3: (Example) Graphs G, G_1 and G_2. In the first round delete the rightmost edge at each pointed vertex of G and obtain graph G_1. In the second round delete the leftmost edge at each pointed vertex of G_1 and if there are two vertices c, d cyclic in G then for each vertex cyclic in G_1 delete the edge to the left of the segment cd. We obtain graph G_2.

We have deleted at most $2n - \gamma$ edges to get the graph G_2. The deleting scheme reminds the one from the proof of Theorem 2.4.3, but one round is missing. Thus one can show that the graph G_2 contains no two disjoint edges (again by case analysis).

Finally, use a discharging method to show, that graph G_2 has at most $(n + 2\gamma)/2$ edges (as a surprise, again by case analysis). We then conclude that G has at most $(2n - \gamma) + (n + 2\gamma)/2 \leq 2.5n$ edges. The use of discharging method again (as a surprise) need several lemmas, which can be proved by case analysis.

We do not prove the technical steps (case analysis), the reader can find it in the paper [9]. □

Theorem 2.4.5 (Pach-Törőcsik). *$d_k(n) \leq k^4 n$.*

Proof. This proof is again a nice application of binary relations $\prec_1$, $\prec_2$, $\prec_3$, $\prec_4$ and Dilworth theorem (see chapter 1). Each relation compare some disjoint segments (edges of geometric graph). Each pair of disjoint edges is comparable by at least on of these binary relations.

Let G be a geometric graph. Edges $E(G)$ are segments in the plane. For fix $i \in \{1, 2, 3, 4\}$, there is no $k + 1$ chain in partial ordering $\prec_i$, because it would correspond to $k + 1$ disjoint edges. Further by Dilworth theorem, $E(G)$ can be partitioned into at most k classes so that two edges from the same class are not comparable by $\prec_i$.

By overlaying four partitions of $E(G)$ (for $i \in \{1, 2, 3, 4\}$) we obtain a decomposition of $E(G)$ into at most k^4 classes E_j $(1 \leq j \leq k^4)$. Two edges from the same class are comparable by none of $\prec_i$, thus they must cross. Hence by theorem 2.4.1 $|E_j| \leq n$. Therefore $|E(G)| = \sum_{j=1}^{k^4} |E_j| \leq k^4 n$. □

Historical notes and survey of result

Alon and Erdős (1989) proved $d_2(n) \leq 6n$. One year later O'Donnell and Perles (1990) improved it to $d_2(n) \leq 3.6n+c$. Later Goddard et al. [28] (1993) showed $d_2(n) \leq 3n$. At the end, Mészáros [53] improved that to $d_2 \leq 3n-1$. Combining some of the ideas of the proof of Goddard et al. [28] with a discharging method Černý [9] (2003) showed the upper bound $d_2(n) \leq \lfloor 2.5n \rfloor$. The best known lower bound $d_2(n) \geq \lceil 2.5n \rceil - 3$ is an unpublished result of Perles.

For $d_3(n)$ Goddard et al. [28] showed $3.5n \leq d_3(n) \leq 10n$. Later Tóth and Valtr [75] improved that to $4n-9 \leq d_3(n) \leq 8.5n$. The proof uses similar ideas as the presented proof of Theorem 2.4.3.

The first general upper bound $d_k(n) = O(n(\log n)^{k-3})$ was given again by Goddard et al. [28]. In 1993 Pach and Törőcsik [62] introduced the order relations on disjoint edges and as an application of Dilworth's Theorem they showed that $d_k(n) \leq k^4 n$. That was the first upper bound linear in n. Tóth and Valtr [75] added a concept of zig-zag and improved the bound to $d_k(n) \leq k^3(n+1)$. Later Tóth [72] further improved the bound to $d_k(n) \leq 256k^2 n$. Original constant in Tóth's proof was a bit bigger. This one is due to Felsner [25].

It is believed that the truth is about $d_k(n) \sim ckn$. It is also an interesting problem, if this is true for geometric graphs whose edges can be intersected by a line. That would give general upper bound $d_k(n) \leq c(k \log k)n$. Just bisect the vertex set of the graph and count edges in both parts recursively.

Kupitz [46] proved that for every k and $n \geq 2k+1$, geometric graph on n vertices in convex positions with no $k+1$ disjoint edges has at most kn edges. This bound is tight, because there are graphs with so many edges.

Open problem 3. *All previous results were for geometric graphs. What can you say for simple topological graphs with no $k+1$ pairwise disjoint edges?*

2.4.2 k pairwise crossing edges

A topological graph G is *k-quasi-planar* if G contains no k pairwise crossing edges. For $k=2$ we get planar graphs and for $k=3$ quasi-planar graphs. For $k \geq 1$, let $\mathcal{C}_k$ denote the class of all topological graphs consisting of k pairwise crossing edges.[9] Let $c_k = t(\mathcal{C}_k, n)$ be the maximum number of edges of a topological graph on n vertices with no k pairwise crossing edges.

From the subsection on planar graphs follows, that every topological graph on n vertices with no 2 crossing edges has at most $3n-6$ edges.

Theorem 2.4.6. *Every topological graph G on n vertices with no 3 pairwise crossing edges (quasi-planar graph) has at most $10n-20$ edges.*

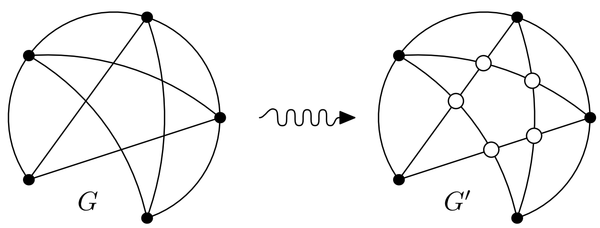
Proof (Ackerman and Tardos [2]). Let $G=(V,E)$ be a topological graph with n vertices and with no 3 pairwise crossing edges. Without loss of generality we might assume that G is a drawing with minimum $\textsc{cr}(G)$. Further we might assume that G is connected, otherwise we use the induction on components. We

[9] So $\mathcal{C}_k$ is a crossing family.

can assume that the minimum degree of a vertex is at least 5, otherwise delete such a vertex together with all edges incident with it, and use induction.

Making a plane graph G' from G: We start with the topological graph G. We add all crossing points X of edges E as a new vertices. Vertices X divides original edges of G into new edges of G'. Graph G' has two types of vertices – original vertices V and new vertices X. More precisely $G' = (V', E')$, where $V' = V \cup X$ and E' are parts of original edges E connecting 2 vertices of V' and not containing other vertex of V'.

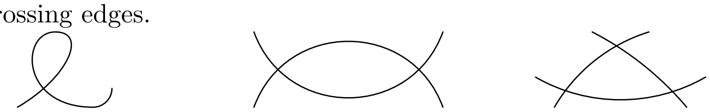
We use the notation that dashed variables correspond to the graph G' and undashed variables to the graph G. For example edge e' is an edge of G' and edge f is an edge of G.



There are no 3 edges crossing at one point, because they would be 3 pairwise crossing edges. Thus for every $x \in X$, $\deg_{G'}(x) = 4$. G' is plane graph, so we can denote its faces by F'. For a face $f \in F'$ define the size of f, denoted by $|f|$, as the number of edges on the boundary of f. (Let us note that some edge can appear on the boundary twice.) Further define $v(f)$ to be the number of original vertices on the boundary of f. (Similarly, some original vertex can be counted several times.)

We will need some more notation. Faces of size 3, 4, 5 are called triangle, quadrilateral, pentagon. As a shortcut for a face f with $v(f)$ original vertices on the boundary, we write simply $v(f)$-face f i.e. 1-triangle, 2-pentagon.

Graph G' contains no faces of size 1 or 2. Otherwise they correspond to a self-crossing edge or empty lens in G and thus we might reduce the number of crossings in G. G' also contains no 0-triangle, because it corresponds to 3 pairwise crossing edges.



self-crossing edge *empty lens* 0-*triangle*

Placing the charge: Each face $f \in F'$ receives a charge

$$ch(f) := |f| - v(f) - 4.$$

Now, we claim that the total charge on faces is $4n - 8$.

When we are adding the new vertices X to G one by one, then each new vertex (crossing point) adds exactly two "new" edges. Thus we have $|E'| = |E| + 2|X|$.

$$\sum_{f\in F'} ch(f) = \sum_{f\in F'} |f| - v(f) - 4 = 2|E'| - \sum_{v\in V} \deg(v) - 4|F'|.$$

By plugging $\sum_{v\in V} \deg(v) = 2|E| = 2|E'| - 4|X|$ into the previous formula we get $\sum_{f\in F'} ch(f) = 4(|E'|-|F'|-|X|) = 4(|V'|-|X|-2) = 4n-8$. The last but one equality holds by Euler formula, which is $|E'| - |F'| = |V'| - 2$. That finishes the proof of the claim.

Redistribution of the charge: We want to redistribute the charge so to get

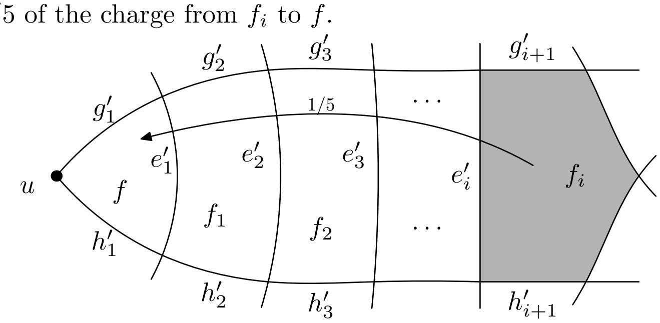
$$ch'(f) \geq v(f)/5 \tag{2.4}$$

for every $f \in F'$.

By definition of $ch(f)$, it already holds for every face f with $|f| \geq 4$. Moreover it holds for faces f which are k-triangles with $k \geq 2$, because $ch(f) = k-1$. There are no 0-triangles in G', thus the only problem is with 1-triangles.

Now we show how to recharge arbitrary 1-triangle f. 1-triangle f has one original vertex u and edges g_1', h_1', e_1', such that g_1', h_1' are incident with u (see the next figure). Edges g_1', h_1' are parts of original edges $g, h \in E$. Put $f_0 := f$. Look at the face f_1 on the other side of e_1'. If f_1 has positive charge, we stop. Otherwise f_1 is 0-quadrilateral with edges e_1', g_2', e_2', h_2' such that $g_2' \subset g$, $h_2' \subset h$. Note that edge e_1' has no original vertex as its end-point. Hence the face f_2 is not a triangle, because then f_2 would be necessary 1-triangle and therefore there would be multiple edges between two vertices of G.

But back to the case of 0-quadrilateral. Consider edge e_2' instead of e_1' and repeat the same idea. We repeat the idea until we find a face f_i with positive charge. We say that we recharge the 1-triangle f through edges e_1', e_2',… e_i' by sending $1/5$ of the charge from f_i to f.



After recharging all 1-triangles, we get new charges $ch'(f_j)$ on faces. We have to check, that $ch'(f) \geq v(f)/5$ for all faces $f \in F'$. If f is 1-triangle, then $ch'(f) = 1/5$, as we wanted. If f is quadrilateral, then $v(f) \geq 1$, otherwise we did not change its charge and inequality (2.4) still holds. Such a quadrilateral can be charged at most twice, because we are recharging 1-triangles through edges with no original vertex. Thus $ch'(f) \geq 3/5 \cdot v(f)$. If $|f| \geq 5$ then f has lost at most $|f|/5$ of its charge. Thus $ch'(f) \geq 4/5 \cdot |f| + v(f) - 4 \geq v(f)$. Therefore inequality (2.4) holds.

Charging original vertices from faces: For every original vertex v we do the final recharging. Every face incident with original vertex v gives 1/5 of its charge to v. Thus $ch(v) = 1/5 \cdot \deg(v)$ for every $v \in V$ and $ch''(f) \geq 0$ for every $f \in F'$ by inequality (2.4). Hence we get

$$4n - 8 \geq \sum_{f \in F'} ch''(f) + \sum_{v \in V} \deg(v)/5 \geq 2|E|/5.$$

That is $|E| \leq 10n - 20$. □

By further redistribution of charges we might get a better bound (as in [2]). Ackerman [1] used the same method to show $c_4 = O(n)$. But his proof contains more technical parts based on case analysis.

Open problem 4 (Pach [58]). *Is it true, that for any fixed k, $c_k = O(n)$?*

Historical notes and survey of result

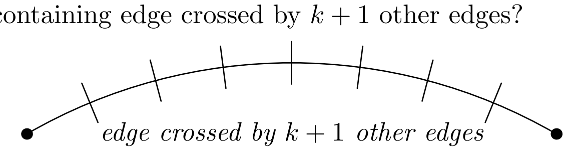
Pach [58] conjectured, that for every fixed k, $c_k = O(n)$. For $k = 2$ the conjecture is trivial, because in planar graphs $c_2 = |E| = 3n - 6$. Agarwal et. al [3] were first who proved that $c_3 = O(n)$. Later, Pach et. al. [60] simplified the proof and got better bound of $c_2 \leq 65n$. Recently Ackerman and Tardos [2] used discharging technique and showed $7n - O(1) \leq c_3 \leq 8n - 20$ and for simple topological graphs[10] they showed $c_3 = 6.5n - O(1)$. The last improvement was motivated by Ackerman [1], who showed $c_4 \leq 36(n-2) = O(n)$ using the same technique.

For general $k > 4$, the best upper bound is $O(n \log^{4k-16} n)$ as shown in [1]. For any fixed k, Valtr [78] obtained the upper bound $O(n \log n)$ for topological graphs with x-monotone edges.

Capoyleas and Pach [8] showed that every k-quasi-planar geometric graph on n vertices in convex position has at most $2(k-1)n - \binom{2k-1}{2}$ edges, which is tight.

2.4.3 $k + 1$ crossings on an edge

What is the maximum number of edges that a simple topological graph can have with no containing edge crossed by $k + 1$ other edges?



Gärtner, Thiele, and Ziegler showed a bound for $k = 1$. Pach and Tóth [63] generalized their result and showed the following theorem.

[10]Remark that the result distinguish between topological graphs and simple topological graphs. In other words, releasing the condition, that a pair of edges can cross at most once, allows us to get a graph with the same forbidden pattern, but with more edges.

Theorem 2.4.7 (Pach, Tóth [63]). *Let $G = (V, E)$ be a simple topological graph on n vertices with every edge crossed by at most k other edges. For $0 \leq k \leq 4$ we have $|E| \leq (k+3)(n-2)$.*

Moreover they showed

Theorem 2.4.8 (Pach, Tóth [63]). *Let $G = (V, E)$ be a simple topological graph on n vertices with every edge crossed by at most k other edges. For $k \geq 1$ we have $|E| \leq 4.108\sqrt{k}n$.*

These upper bounds leads to the improvement of the crossing lemma. In the folklore probabilistic proof of crossing lemma, we just make another simple estimates on $\text{CR}(G)$ using Theorem 2.4.8.

It can happen that many of the edges crossing one edge are emerging from the same vertex. Two edges are *independent* if they do not have a common end-vertex. Tóth and Pach asked what changes if the constrain is that G contains no edge crossed by $k+1$ independent edges.

Theorem 2.4.9 (Pinchasi). *Topological graph $G = (V, E)$ on n vertices with every edge crossed by at most k independent edges have at most $2.208 \cdot k^2 n$ edges.*

Proof. Take a maximum bipartite subgraph $H \subseteq G$. It is easy to show, that H has at least $|E(G)|/2$ edges.[11]

For every $e \in E(H)$, denote by $E_e \subseteq E(H)$ the set of edges crossing an edge e. Every edge $e \in E(H)$ is crossed by at most k independent edges. Thus E_e can be covered by at most k vertices (some of their endpoints) by classical König theorem.

We choose a random induced subgraph $H' \subseteq H$ by taking every vertex with probability p. We say that an edge e is *good* if it appears in H', but all of the edges E_e do not. An edge e is good with probability at least $p^2(1-p)^k$, because it suffice that both end-points of e are chosen and the k vertices covering E_e are not. That gives the first estimate

$$|E(H)| \cdot p^2(1-p)^k \leq \mathbb{E}[\,\#\text{good edges}\,] \leq 3n - 6$$

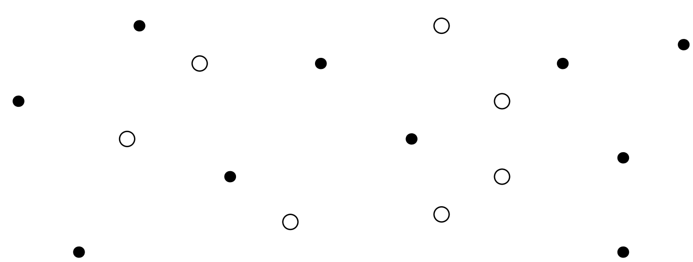
The second estimate holds because the subgraph of good edges is a planar graph. By taking $p = 1/k$ in the inequality we get $|E(H)| \leq 3/e \cdot k^2 n$. □

[11] We start with sets A, B, which are empty at the beginning. Then in succesive steps, we place each vertex $v \in V$ either in A or B. For each vertex v, we choose such a set, that the number of edges leading from v to already placed veritices in the other set is maximal. Thus at least half of the edges from v to already placed vertices appears in the constructed bipartite graph.

Chapter 3

Convex Independent Sets

Let X be a set of points in the general position in the plane. By *general position* we mean that no three points lie on a line and no two points have the same x-coordinate.[1] The points $Y \subseteq X$ are in *convex position* if Y lie in the vertices of convex polygon. Sometimes we say that "Y is *convex independent set*" synonymously to the phrase "points Y are in convex position".



3.1 Erdős–Szekeres theorem

Esther Klein observed, that any 5-point set in the plane in general position contains 4 points in convex position. Indeed, if the convex hull of these points contains 4 or 5 points, we are done. In the other case the convex hull is a triangle with two points inside. These two interior points together with one of the sides of the triangle forms convex quadrilateral.

Klein also asked for generalization. That attracted Erdős and Szekeres, who joined Klein and in 1935 they together proved [21] "Erdős–Szekeres theorem". Because later, Klein became Mrs. Szekeres, the theorem is sometimes called "Happy end theorem".

Theorem 3.1.1 (Erdős–Szekeres theorem). *For every positive integer n there exists a positive integer $N = N(n)$ such that any N-point set in the plane in general position contains n points in convex position.*

[1] We can get rid of the second condition by a little rotation of coordinate system.

There are several proofs for this theorem. Short proofs are using Ramsey theorem and we will show them here below. Longer proof, but providing a better bound on $N(n)$, is using cups and caps and we present it in the next section.

Ramsey theorem for k-tuples says that for every positive integers k, l_1, l_2 there exists integer $R_k(l_1, l_2)$ satisfying following condition. For any integer $m \geq R_k(l_1, l_2)$, if the k-element subsets of $\{1, 2, \ldots, m\}$ are colored by colors 1 and 2, then there exists a color $i \in \{1, 2\}$ and l_i-element subset $T \subseteq \{1, 2, \ldots, m\}$ such that every k-element subset of T is colored by color i. The set T is called monochromatic subset.

Proof 1. Let X be the set of at least $R_4(n, 5)$ points in the plane in general position. Color every 4-point set red if its points are in convex position and blue otherwise.

By the observation of Klein, there is no 5-point set $Y \subseteq X$ with all 4-point subsets colored blue. Hence by Ramsey theorem there must be an n-point set $Z \subseteq X$ in which all 4-point subsets are in colored red (in convex position).

From that follows that Z is convex independent. Otherwise consider triangulation of conv(Z). If there is point $z \in Z$ inside conv(Z) then z is also inside of some triangle $\triangle t_1 t_2 t_3$ for $\{t_1, t_2, t_3\} \subseteq Z$. But then z, t_1, t_2, t_3 should be colored blue. A contradiction. $\square$

Proof 2. Lewin [49] reported that his undergraduate student, Tarsy, had come up with the following proof while taking an exam in a combinatorics course.

Let X be a set of at least $R_3(n, n)$ points in general position in the plane. Denote the points by $x_1, x_2, \ldots, x_m$ according to the increasing x-coordinate. Color every triple $\{x_i, x_j, x_k\}$, $i < j < k$ by red, if x_j lies above line $x_i x_k$, and blue otherwise. Ramsey theorem provides monochromatic n-point set T.

Points T are in convex position. Otherwise consider triangulation of conv(T) and get a triangle $\{x_i, x_j, x_k\} \subseteq T$, $i < j < k$ with a point $x_q \in T$ inside. WLOG we may assume that $i < q < j$. The triple $\{x_i, x_q, x_j\}$ has different color than $\{x_i, x_q, x_k\}$. That is a contradiction. $\square$

Proof 3. This proof is by Johnson [35]. Let X be a set of at least $R_3(n, n)$ points in general position in the plane. Color every triple $Y \subset X$ by red if the interior of conv(Y) contains even number of points and blue otherwise.

An easy observation says that 4-point set is in convex position if and only if all of its 3-point subsets have the same color. The rest of the proof is similar to Proof 1. □

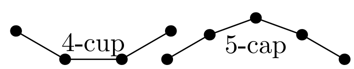
Therefore the bound on $N(n)$ derived from the proofs using Ramsey theorem is

$$N(n) \leq \min\{R_4(n,5), R_3(n,n)\}$$

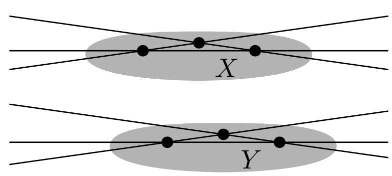
For details and historical notes see survey by Morris and Soltan [54] or the book by Matoušek [50].

3.1.1 Cups and caps

Let X be a set of n points and denote its points by p_1, p_2, $\ldots p_n$ according to the increasing x-coordinate. Let $Y \subseteq X$ be a set of points q_1, q_2, $\ldots q_k$ again ordered by the x-coordinate. For i = 1,2, $\ldots k-1$, let s_i be the slope of the line $q_i q_{i+1}$. The set $Y = \{q_1, \ldots, q_k\}$ is a *k-cup* or a *k-cap* if the sequence s_1, s_2, $\ldots$, s_k is increasing or decreasing, respectively. In other words if the points lie on the graph of a convex, resp. concave function. The point q_1 is *the left endpoint* of Y and the point q_k is *the right endpoint* of Y.



Let X and Y be two finite sets in the plane. We say that X is deep above Y (Y is deep below X) if the following conditions holds: (i) No line determined by points of $X \cup Y$ is vertical.[2] (ii) Each line determined by two points of X lies above all points of Y. (iii) Each line determined by two points of Y lies below all points of X.



Define $f(k,l)$ to be the smallest positive integer for which every $f(k,l)$-point set X contains k-cup or l-cap.

Theorem 3.1.2 (about cups and caps [21]). $f(k,l) = \binom{k+l-4}{k-2} + 1$.

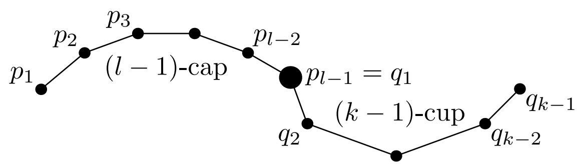
Proof. First we show the upper bound $f(k,l) \leq \binom{k+l-4}{k-2} + 1$. The inequality follows from the boundary conditions $f(3,l) = l$, $f(k,3) = k$ and the recurrence $f(k,l) \leq f(k-1,l) + f(k,l-1) - 1$. It is just counting with binomial coefficients.

Now we prove the recurrence. Suppose $k, l \geq 3$ and that X contains $f(k-1,l) + f(k,l-1) - 1$ points in general position. We want to show that there is always a k-cup or an l-cap in X.

[2] We can without loss of generality assume that condition (i) holds. Otherwise rotate the coordination system a little.

Let $Y \subseteq X$ be the set of left endpoints of $(k-1)$-cups of X. The set $X \setminus Y$ contains no $(k-1)$-cup. If $X \setminus Y$ contains l-cap then we are done. Hence $|X \setminus Y| < f(k-1,l)$ and therefore $|Y| \geq f(k,l-1)$.

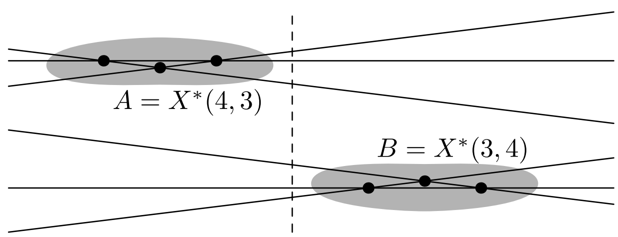
If there is k-cup in Y then we are done. So the remaining case is when Y contains $(l-1)$-cap $\{p_1, \dots, p_{l-1}\}$. By the definition of Y there is a $(k-1)$-cup $\{q_1, \dots, q_{k-1}\}$ with $q_1 = p_{l-1}$.



The figure show that either $(k-1)$-cup can be extended to a k-cup by adding point p_{l-2} or $(l-1)$-cap can be extended to an l-cup by adding point q_2. That finishes the proof of upper bound.

Let us proceed to the proof of lower bound. Let $f^*(k,l) = f(k,l) - 1$. Now we construct $f(k,l)^*$-point set $X^*(k,l)$ with no k-cup and no l-cap. The construction is easy for $k = 3$ or $l = 3$ because we just take $X^*(k,l)$ to be an $(l-1)$-cap or $(k-1)$-cup, respectively. So it remains to show the recurrence $f^*(k,l) \geq f^*(k-1,l) + f^*(k,l-1)$ for $k,l \geq 3$. The recurrence is equivalent to the lower bound $f(k,l) \geq f(k-1,l) + f(k,l-1) - 1$.

Let $A := X^*(k,l-1)$ and $B := X^*(k-1,l)$. Place A deep above and to the left of B. See following figure. Then $X^*(k,l) := A \cup B$.



Any cup C either lies entirely in A or it has one point in A (left endpoint) and the other points lie in B. Thus there is no k-cup in $X^*(k,l)$. Any cap C either lies entirely in B or it has one point in B (right endpoint) and the other points lie in A. Thus there is no l-cap in $X^*(k,l)$. $\square$

Because $N(n) \leq f(n,n)$, we get $N(n) \leq \binom{2n-4}{n-2} + 1$. This upper bound was not improved for 63 years, but in 1998 there were thee improvements one after the other (Chung and Graham [14], Kleitman and Pachter [40], Tóth and Valtr [74]). The last authors gave the upper bound

$$N(n) \leq \binom{2n-5}{n-2} + 2.$$

Here we sketch their proof.

Sketch of proof. Let P be the set of points in general position with no n points in convex position. Let ℓ be a horizontal line passing through the highest vertex of $\text{conv}(P)$. We can assume that there is only one such a vertex, otherwise rotate the coordinate system a little to make it so.

The whole trick of the proof is to use projective mapping f sending the line ℓ to infinity.[3] Projective mapping preserves convexity. Each k-cup in $f(P)$ corresponds to $(k+1)$-cup in P (one point is in infinity). Each cap in $f(P)$ corresponds to cap of the same size in P. Thus $f(P)$ contains no $(n-1)$-cup and no n-cap. Hence $|P| = |f(P)| + 1 \leq f(n-1, n) + 1 = \binom{2n-5}{n-2} + 1$. □

In 2005 Tóth and Valtr [76]) further improved the upper bound by one. Thus the currently best upper bound is

$$N(n) \leq \binom{2n-5}{n-2} + 1.$$

3.1.2 Note on projective mappings

Points of *real projective plane* $\mathbb{P}^2$ can be represented by 3 dimensional vectors of $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(\lambda x_1, \lambda x_2, \lambda x_3) \in \mathbb{R}^3$ for $\lambda \neq 0$ correspond to the same point of projective plane. Vectors $(x_1, x_2, x_3) \in \mathbb{R}^3$ are called *homogeneous coordinates.*

Mapping $(x_1, x_2) \to (x_1, x_2, 1)$ is called *canonical embedding* of $\mathbb{R}^2$ into projective plane $\mathbb{P}^2$. We choose one representative in each equivalence class, usually the *canonical representative* of (x_1, x_2) is $(x_1, x_2, 1)$.

A projective point with $x_3 = 0$ corresponds to a "point at infinity" in the direction (x_1, x_2). The set of all such "infinite" points satisfies the homogeneous linear constraint $x_3 = 0$ and thus behaves like a line, called the *line at infinity.*

Projective mapping $f : \mathbb{P}^2 \to \mathbb{P}^2$ is mapping of the form $f(x) = Ax$, where x is projective point in homogeneous coordinates and A is full rank matrix.

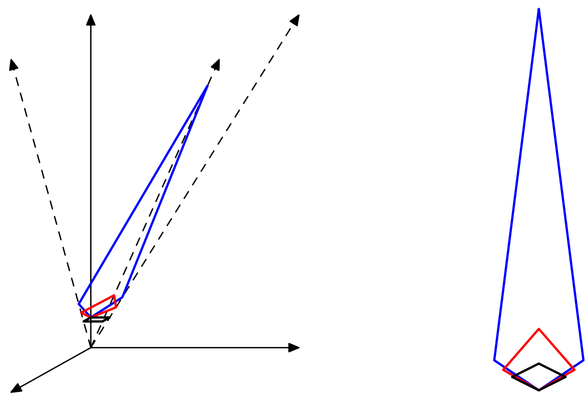
Projective mapping is useful in computer graphics, because besides usual transformation like rotation, shrink, shear it includes more difficult transformations like shift or perspective.

How does projective mapping work? Imagine, that you draw your points on a sheet of paper, take a camera and take a picture of the sheet. The points on the photo are image of a projective mapping applied to your original points. For different angles of the camera you get images of different projective mappings.

Another example is on the following figures. Canonically embed your points from $\mathbb{R}^2$ into $\mathbb{R}^3$. Each point (x_1, x_2) correspond to a "line" passing through the origin and $(x_1, x_2, 1)$. The hyper-plane $h = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_3 = 1\} \subseteq \mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$ and intersection of h with the lines corresponding to the points gives you the points in h, which are the image of projective mapping. Similarly, if you take a different hyper-plane h', then h' is isomorphic to $\mathbb{R}^2$ and

[3] For more explanation on projective mappings see the note on projective mappings (subsection 3.1.2) and the figures in the proof of theorem about k-convex sets and l-holes (Theorem 3.2.1).

the intersection of h' with the lines corresponding to the points gives you the image of a projective mapping.[4]



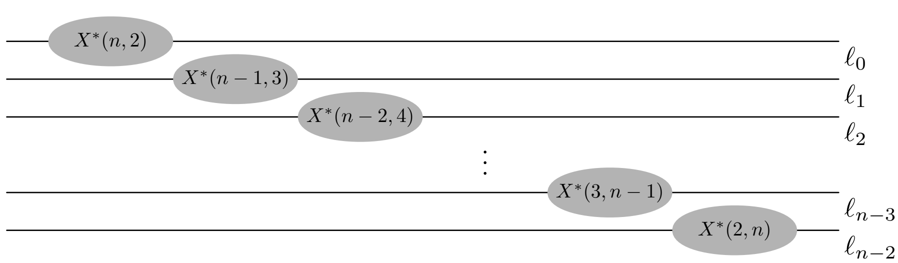
Fact: projective mapping preserves convexity.

3.1.3 Lower bound for $N(n)$

The exact values of $N(n)$ are known only for $n \leq 6$. $N(3) = 3$, $N(4) = 5$, $N(5) = 9$, and recently was shown [71] that $N(6) = 17$.

Theorem 3.1.3 (Lower bound for $N(n)$ [22]). $N(n) \geq 2^{n-2} + 1$.

Proof. [17][5] We construct the set X with 2^{n-2} points and no n-point subset in convex position. For $i \in \{0, \ldots, n-2\}$, let T_i be the $\binom{n-2}{i}$-point set with no $(n-i)$-cup and no $(i+2)$-cap. Such a set exists, see Theorem 3.1.2 for construction (we take $T_i = X^*(n-i, i+2)$). We can further assume that no line determined by two points of T_i is vertical. Then shrink each T_i in vertical direction so that every line determined by two points of T_i is almost horizontal.



Place each set T_i to the neighborhood of a horizontal line $\ell_i = \{(x, y) \in \mathbb{R}^2 \,|\, y = -i\}$ and to the right of already placed sets $T_0, \ldots, T_{i-1}$. Since we

[4]For simplicity we omit the discussion which hyper-planes are admissible. Not all hyper-planes are admissible.

[5]The proof from [17] is a based on the original proof from [22], but moreover it shows that the shrunken sets T_i do not need to lie on a convex arc. They can lie almost arbitrarily.

shrunk each T_i enough, the set T_i lies deep above T_j for $i < j$. Let X be the union of T_i for any $0 \le i \le n-2$. Then

$$|X| = \sum_{i=0}^{n-2} \binom{n-2}{i} = 2^{n-2}.$$

Suppose that $Y \subseteq X$ is in convex position. Let k and l be the smallest and the largest values of i so that $Y \cap T_i \neq \emptyset$. If $k = l$ then Y contains no $(k+2)$-cap and no $(n-k)$-cup. Further, from the construction we have:

- $Y \cap T_k$ is a cap of at most $k+1$ points.
- $Y \cap T_l$ is a cup of at most $n-l-1$ points.
- $|Y \cap T_i| = 1$ for $k < i < l$.

Thus $|Y| \le (k+1) + (n-l-1) + n - l - 1 = n-1$. Hence X contains no n-point subset in convex position. □

Conjecture 1 (Erdős, Szekeres [22]). $N(n) = 2^{n-2} + 1$.

The conjecture holds for $n \le 6$.

Let $m = m(n, k, l)$ be the smallest number such that every m-point set in general position in the plane contains either n points in convex position, or a k-cup, or an l-cap.[6] Erdős, Tuza and Valtr [17] proved that[7]

$$m(n, k, l) \ge 1 + \sum_{i=n-l}^{k-2} \binom{n-2}{i}.$$

The idea of the proof is simple. We construct the set $X_{n,k,l}$ with no n points in convex position, nor a k-cup, nor an l-cap Take $X_{n,k,l}$ as the union of T_i for $n-l \le i \le k-2$ (It is similar to the construction of lower bound for $N(n)$). It can be shown that $X_{n,k,l}$ has required properties.

Conjecture 2 (Erdős, Tuza and Valtr [17]). $m(n, k, l) = 1 + \sum_{i=n-l}^{k-2} \binom{n-2}{i}$.

The authors proved that Conjecture 2 is equivalent to Conjecture 1. With present power of computers it is still quite difficult find a configuration of points in the plane, which would disprove Conjecture 1. Conjecture 2 is some kind of parameterization of the lower bound on $N(n)$. Set $X_{n,k,l}$ has less points than the lower bound for $N(n)$ and thus $m(n, k, l)$ is more suitable for looking for counter examples.

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[6] It is interesting only for $n > k, l$. Otherwise the k-cup or l-cap contains n points in convex position.

[7] There is a mistake in survey [54]. We recommend to see the original paper [17].

3.2 Empty convex independent sets

Erdős [19] also asked if for every k there exists N such that any N-point set X in the plane contains k vertices of an empty convex polygon. An empty polygon is a polygon with no point of X in its interior. We say that $Y \subseteq X$ is a k-hole if Y lies in the vertices of an empty convex k-gon. His conjecture holds up to $k = 6$. First it was proved only for $k \leq 5$ [33]. In 1983 Horton [34] showed a construction of a set with no 7-hole. It was a little surprise because it disproves Erdős's conjecture for any $k \geq 7$. The question for $k = 6$ was open for a long time. Using a computer Overmars [56] found a configuration of 29 points without empty hexagons. Finally in 2005, Gerken [27] and independently Nicolás [55] showed that the conjecture holds also for $k = 6$. They were followed by Valtr [77], who simplified Gerken's proof, and recently by Koshelev [41], who showed better estimates on the minimum number of points containing empty hexagon.

What is the sufficient condition for the existence of a k-hole? The set X is *l-convex* if and only if every triangle determined by points of X contains at most l points of X in its interior. The l-convex sets were introduced by Valtr [79] and he also showed the following theorem:

Theorem 3.2.1 (about l-convex sets and k-holes). *For every positive integers k and l there exists a positive integer N such that any l-convex N-point set X in the plane contains a k-hole.*

Denote by $n(k, l)$ the smallest positive integer N such that any l-convex N-point set contains a k-hole. In 2001 Károlyi, Pach and Tóth [38] proved this theorem for $l = 1$. Later Károlyi, Lippner and Valtr [36] determined the exact value of $n(k, 1)$. The first proof for general l was given by Valtr [79]. He was followed by Kun and Lippner [45] who improved the bound to $n(k, l) \leq (l + 2)^{(l+2)^k - 1}$. Finally Valtr [81] again improved the bound to $n(k, l) \leq 2^{\binom{k+l}{l+2} - 1} + 1$. Valtr's last proof generalizes the result on cups and caps used in the proof of Erdős-Szekeres theorem to open cups and open caps.

We explain the result for open cups and caps in the next section, where we also proof the Theorem 3.2.1.

As a consequence of the theorem about open cups and caps we get a better bound $n(k, l) \leq \min\{2^{\binom{k+l-3}{l}}, 2^{\binom{\lfloor (k-1)/2 \rfloor + l}{l+1} - 1}\}$. This bound is doubly-exponential in $k + l$. The lower bound [45] is only exponential in $k + l$.

Open problem 5 (Valtr). *What is the order of growth of $n(k, l)$? Is it doubly-exponential or only exponential?*

The reduction to open cups and caps cannot be used, because the lower bound for open cups and caps is doubly-exponential. The reduction to open cups and caps is also too strong, because it finds a k-hole incident with a chosen point on $conv(X)$ or it provides a special configurations of points, which contradicts the l-convexity.

3.3 Open cups and open caps

Let X be a set of points in general position in the plane. Let $Y \subseteq X$ be a set of points q_1, q_2,..., q_k ordered by the increasing x-coordinate. Remind that in 3.1.1 we defined that Y is a k-cup, resp. k-cap if the points q_1,..., q_k lie on the graph of a convex, resp. concave function. The set Y is *open* in X if there is no point $p \in X$ with $x(q_1) < x(p) < x(q_k)$ lying above the polygonal line $q_1 q_2 \dots q_k$. See figure 3.1.

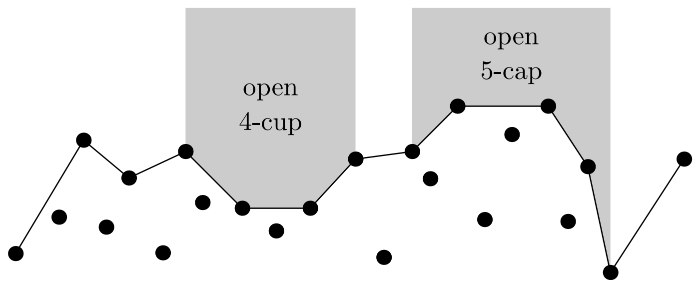


Figure 3.1: The set of points on the polygonal line is open. There is also an empty 4-cup and an empty 5-cap in the figure.

The upper envelope of Y is the polygonal line $q_{i_1}, q_{i_2}, \dots, q_{i_t}$ where $1 = i_1 < i_2 < \dots < i_t = k$ such that it is the graph of a concave function and there is no point of Y above this line (see figure 3.2). The point p_L is *the left neighbor* of the point p in set the Y, if $p_L \in Y$ and there is no point $q \in Y$ such that $x(p_L) < x(q) < x(p)$. The *right neighbor* is defined similarly.

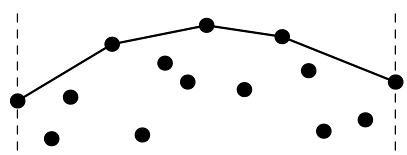


Figure 3.2: The black polygonal line is the upper envelope of the points in the figure.

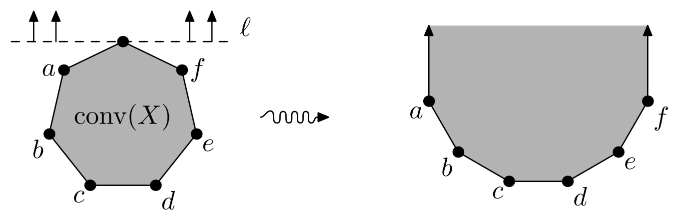
Theorem 3.3.1 (about open cups and open caps). *For any positive integers k and l there exits a positive integer N such that any N-point set in the plane contains an open k-cup or an open l-cap.*

We show a simple proof of theorem 3.3.1 later in subsection 3.3.1.

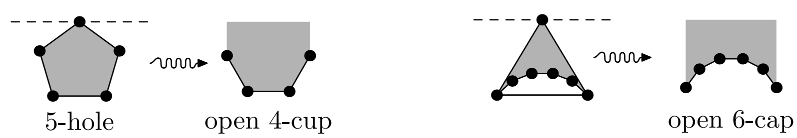
Theorem 3.2.1 about l-convex sets and k-holes is quite easy corrolary of theorem 3.3.1.

Proof of the theorem 3.2.1 (about l-convex sets and k-holes). We have an l-convex N-point set X and we want to find a k-hole. We use the projective transformation, which sends the horizontal line ℓ passing through the highest point of X

to the line at infinity. We can assume that there is exactly one point of X on the line ℓ, otherwise we can rotate the point set X a little.



We obtain an $(N-1)$-point set $\overline{X}$. Apply theorem 3.3.1 for $k-1$ and $l+3$ on $\overline{X}$ to find either an open $(k-1)$-cup or an open $(l+3)$-cap. In the backward projective transformation the open $(k-1)$-cup corresponds to a k-hole and the open $(l+3)$-cap corresponds to a triangle containing at least $(l+1)$-points, but that contradicts the l-convexity of the set X. See Valtr [81] for the details.



□

We define $g(k,l)$ as the smallest number N such that any N-point set in general position in the plane contains an open k-cup or an open l-cap. Valtr [81] showed first bounds on $g(k,l)$. We improved the upper bounds in [11], so currently the best bounds are

$$2^{\binom{\lfloor k/2\rfloor+\lfloor l/2\rfloor-2}{\lfloor k/2\rfloor-1}} \leq g(k,l) \leq \min\{2^{\binom{k+l-5}{l-3}}, 2^{\binom{\lfloor k/2\rfloor+l-3}{\lfloor k/2\rfloor-1}-1}\}.$$

The lower bound is by Valtr [81]. The upper bounds hold for $k,l \geq 4$ and they are proved in Lemmata 3.3.8 and 3.3.9. For $k \leq 4$ or $l \leq 4$, we know the exact values of $g(k,l)$. On the other hand these bounds are not as good as the recurrences themselves. The next theorem summarizes the best bounds.

Theorem 3.3.2 (recurrences). *$g(k,2)=2$, $g(2,l)=2$, $g(k,3)=k$, $g(3,l)=l$, $g(k,4)=2^{k-1}$, $g(4,l)=2^{l-1}$ and for $k,l \geq 5$ we have*

$$g(k,l) \leq g(k-2,l)\cdot[2g(k,l-1)-3]+2$$
$$g(k,l) \geq g(k-2,l)\cdot[g(k,l-2)-2]+2g(k-1,l).$$

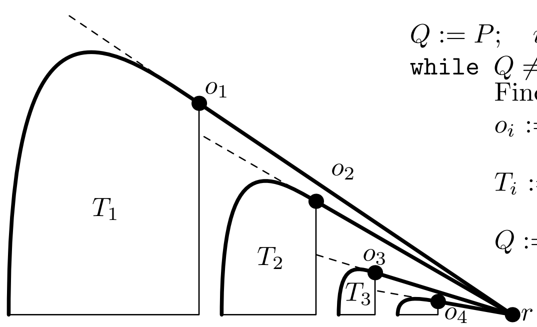
The tight bounds for $k=4$ or $l=4$ are proved in Lemmata 3.3.12, 3.3.13 and 3.3.17. The upper bound recurrence is proved in Lemma 3.3.4. The lower bound recurrence is proved in Lemma 3.3.10. We also remark, that the last recurrence is not tight and show the idea, how it can be improved.

3.3.1 Simple proof of the upper bound

Define $h(k,l,m)$ to be the largest number N such that there is an N-point set in general position which contains neither an open k-cup nor an open l-cap nor an open m-cap ending in the rightmost point. It is easy to see that $h(k,l,2) = 1$ and $h(k,l,l) = g(k,l) - 1$.

Lemma 3.3.3. $h(k,l,m) \leq h(k-1,l,l) \cdot h(k,l,m-1) + 1$ *for* $k,l \geq 3$.

Proof. Let P be the set of points in general position maximizing $h(k,l,m)$. That means that P is $h(k,l,m)$-point set which contains neither an open k-cup nor an open l-cap nor an open m-cap ending in the rightmost point. Denote the rightmost point of P by r. We construct sets T_i by the following algorithm. The construction is illustrated in the following figure.



$Q := P$; $i := 1$;
`while` $Q \neq \emptyset$ `do`
  Find the upper envelope of Q;
  $o_i :=$ the left neighbor of r
    on the upper envelope of Q;
  $T_i :=$ the points of Q lying to
    the left of o_i including o_i;
  $Q := Q - T_i$; $i := i + 1$;

Any open cap in T_i ending in the point o_i can be extended by the point r and becomes an open cap in P ending in the rightmost point of P. Thus $|T_i| \leq h(k,l,m-1)$.

Let $O = \{o_1, o_2, \dots\}$. The set O is open in P. O contains neither an open k-cup nor an open l-cap, because it would be the open cup or the open cap in P. Moreover O does not contain an open $(k-1)$-cup, because this cup can be extended by the point r. It is because the point r lies above every line determined by two points of O. Hence $|O| \leq h(k-1,l,l)$.

There are at most $|O| \leq h(k-1,l,l)$ sets T_i each containing at most $h(k,l,m-1)$ points and the rightmost point r. That gives us the recurrence. $\square$

Now it is easy to solve the recurrence. We know that $h(k,l,2) = 1$ and $h(k-1,l,l) = g(k-1,l) - 1$. Denote $g(k-1,l) - 1$ by a. By applying the recurrence from lemma 3.3.3 $(l-2)$-times we get $g(k,l) - 1 = h(k,l,l) \leq a(a(a \dots (a \cdot 1 + 1) \dots + 1) + 1) + 1 = a^{l-2} + a^{l-3} + a^{l-4} + \dots + a + 1 = (a^{l-1} - 1)/(a-1)$. Assuming that $a \geq 2$, which means $k \geq 4$ and $l \geq 3$ we get $g(k,l) - 1 < a^{l-1} - 1 < g(k-1,l)^{l-1} - 1$. Because $g(3,l) = l$ we get $g(k,l) \leq l^{(l-1)^{k-3}}$. That finishes the first proof.

We can further improve the previous proof and get a better recurrence.

Every T_i contains an open $(m-2)$-cap ending in o_i. Otherwise double the point o_i and shift one copy by ε along the line, which passes through o_i and which is tangent to the upper envelope of $T_i \cup \{r\}$. That increase the size of P, which is a contradiction with the maximality of P.

There is no open $(l-m+3)$-cap in O because every open cap in O can be extended by $m-3$ points at its left end. For $k, l \geq 3$ we get recurrence

$$h(k,l,m) \leq h(k-1, l-m+3, l-m+3) \cdot h(k,l,m-1) + 1,$$

but its solution gives worse upper bound than the one in the next subsection.

3.3.2 Better upper bounds

In the first part (lemma 3.3.4 and claim 3.3.7) we show the recurrences for $g(k,l)$ and in the second part we solve the recurrences (lemma 3.3.8 and lemma 3.3.9).

Lemma 3.3.4. *$g(k,l) \leq g(k-2,l) \cdot [2g(k,l-1)-3] + 2$ for $k, l \geq 3$.*

Proof. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of $n = g(k,l) - 1$ points in general position with neither an open k-cup nor an open l-cap.

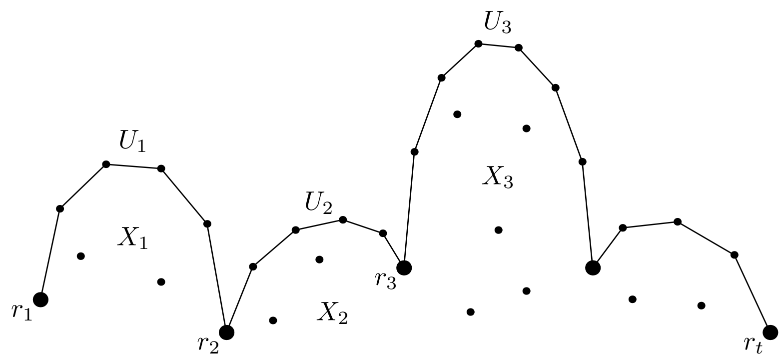
A maximal open cup is the open cup which cannot be extended to a larger cup in X. Let L be the set of left endpoints of maximal open cups with at least 2 points. So for every open cup with the left endpoint $x \notin L$, there is a point in X to the left of x, which extends the open cup. The leftmost point of X is in L, because the two leftmost points of X form an open 2-cup.

Denote the size of L by t and the points of L by $r_1, r_2, \ldots, r_t$. The points of L divide the set X into $t+1$ vertical strips. Denote the sets of points strictly contained in each strip by X_i for $i = 0, 1, 2, \ldots, t$. The leftmost strip is empty, because r_1 is the leftmost point of X.

Claim 3.3.5. *Every open cup in X_i can be extended in X by one point to the left and therefore $|X_i| \leq g(k-1,l) - 1$.*

The left endpoint of an open cup in X_i is not in L and thus there is a point in X extending the open cup. Hence there is no open $(k-1)$-cup in X_i and we get $|X_i| \leq g(k-1,l) - 1$.

For $i = 1, 2, \ldots, t-1$ denote the set of points on the upper envelope of $X_i \cup \{r_i, r_{i+1}\}$ by U_i (see the following figure). Let Y be the union of U_i for $i = 1, 2, \ldots, t-1$. The set Y is open in X. Thus if there is an open k-cup resp. an open l-cap in the set Y, then there is the same open k-cup, resp. l-cap in the whole set X (see figure 3.1).

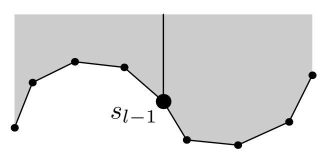


Claim 3.3.6. *The set Y does not contain an open $(l-1)$-cap. Thus $|Y| \leq g(k, l-1) - 1$.*

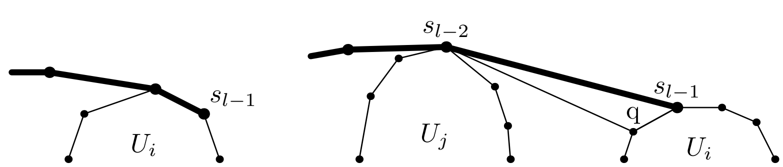
Let us prove this by contradiction. Assume that there is an open $(l-1)$-cap $C = \{s_1, \ldots, s_{l-1}\}$ in Y. By the width of the cap we mean $|x(s_1) - x(s_{l-1})|$. From all the open $(l-1)$-caps in Y choose the one whose width is the smallest.

Now we show that either there is a narrower open $(l-1)$-cap in Y or the open cap C can be extended in X to the right by one point so we have an open l-cap in X.

Where does the point s_{l-1} lie? If it lies in L then s_{l-1} is also the left endpoint of maximal open cup in X. See the following figure. Either the open cup or the open cap can be extended by one point.



Thus $s_{l-1} \in U_i - L$ for some i (it is an interior point of U_i). If there are at least two points of C in U_i then the cap can be extended by the right neighbor of s_{l-1} in U_i. See the following figure on the left. There must be some right neighbor because $s_{l-1} \notin L$.



In the remaining case s_{l-1} is the only point of C in U_i. The point s_{l-2} lies in U_j for some $j < i$. See previous figure on the right. Denote the left neighbor of s_{l-1} in Y by q. If the triangle $s_{l-2}s_{l-1}q$ is empty, then we have the open $(l-1)$-cap $s_1, \ldots, s_{l-2}, q$, which is narrower than C. Otherwise choose $w \in X$ to be the point in the triangle $qs_{l-2}s_{l-1}$ with the largest angle $\angle qs_{l-2}w$. The open l-cap $s_1, \ldots, s_{l-2}, w$ is again narrower than C. This finishes the proof of the claim.

Claim 3.3.7. *$g(k,l) \leq g(k-1,l) \cdot g(k,l-1)$ for k, $l \geq 2$.*

By the previous claims there are $t \leq |Y| \leq g(k,l-1)-1$ vertical strips each containing at most $g(k-1,l)-1$ points plus one for the point r_i. The leftmost strip is empty. We get $g(k,l)-1 \leq [g(k-1,l)-1+1] \cdot [g(k,l-1)-1]$ and the claim follows.

Using another trick we can get a better recurrence. Similarly as we defined L to be the set of all left endpoints of maximal open cups with at least two points, we can define R to be the set of all right endpoints of maximal open cups with at least two points. For the set R, we have similar claims as for the set L because of symmetry.

Denote the points of $R \cup L$ by $P = \{p_1, \ldots, p_{\bar{t}}\}$. The points of P split the plane into $\bar{t}+1$ vertical strips. Denote the set of points strictly contained in each strip by $Z_0, Z_1, \ldots, Z_{\bar{t}}$. Since the leftmost point of X is in L and the rightmost point of X in R, the outer strips are empty. For every set Z_i the first claim holds, because $Z_i \subseteq X_j$ for some j. So every open cup in Z_i can be extended in X by one point to the left. From the symmetric arguments it can also be extended by one point to the right. Thus the set Z_i does not contain an open $(k-2)$-cup and we have $|Z_i| \leq g(k-2,l)-1$.

The number of strips is $\bar{t}+1 = |L|+|R|+1$. The outer strips are empty and the others contain at most $g(k-2,l)-1$ points. By claim 2 the size of L, resp. R is at most $|Y| \leq g(k,l-1)-1$. Altogether we get the recurrence: $g(k,l)-1 \leq [2g(k,l-1)-3] \cdot [g(k-2,l)-1+1]+1$. Don't forget to count the points of P. □

Lemma 3.3.8. *$g(k,l) \leq 2^{\binom{k+l-5}{k-3}}$ for k, $l \geq 4$.*

Proof. We prove the formula by induction on k and l. For $k=4$ or $l=4$ we have $g(k,l) = 2^{k-1}$ or $g(k,l) = 2^{l-1}$, respectively, and the formula holds. From the recurrence in claim 3.3.7 we get

$$g(k,l) \leq g(k-1,l) \cdot g(k,l-1) \leq 2^{\binom{k-1+l-5}{k-4}} \cdot 2^{\binom{k+l-1-5}{k-3}} = 2^{\binom{k+l-5}{k-3}}.$$

□

Lemma 3.3.9. *$g(k,l) \leq 2^{\binom{\lfloor k/2 \rfloor + l-2}{\lfloor k/2 \rfloor -1}-1}$ for k even and k, $l \geq 2$.*

Proof. We prove the formula by induction on k and l. For $k=2$ or $l=2$ we have $g(k,l)=2$ and the formula holds. For k, $l \geq 3$ apply recurrence from lemma 3.3.4 and get $g(k,l) \leq g(k-2,l) \cdot [2g(k,l-1)-3]+2 \leq 2 \cdot g(k-2,l) \cdot g(k,l-1)$. Now apply the induction hypothesis and get

$$g(k,l) \leq 2 \cdot 2^{\binom{\lfloor k/2 \rfloor -1+l-2}{\lfloor k/2 \rfloor -2}-1} \cdot 2^{\binom{\lfloor k/2 \rfloor + l-1-2}{\lfloor k/2 \rfloor -1}-1}$$

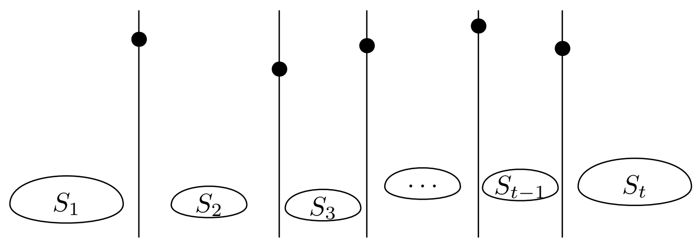
from which the lemma follows. □

3.3.3 Lower bound

Lemma 3.3.10. *$g(k,l) \geq g(k-2,l)\cdot\big(g(k,l-2)-2\big)+2g(k-1,l)$ for $k,l \geq 3$.*

Sketch of the proof. Valtr [81] shows the construction proving the recurrence $g(k,l) \geq g(k,l-2)\cdot g(k-2,l)$. This construction can be slightly improved.

We inductively construct sets $Y_{k,l}$ with no open k-cup and no open l-cap. For $k=3$ or $l=3$, the set $Y_{k,3}$ is an $(k-1)$-cup and $Y_{3,l}$ is a $(l-1)$-cap. The sets $Y_{k,4}$, $Y_{4,l}$ are constructed in the subsection 3.3.4. For $k,l \geq 5$, the set $Y_{k,l}$ can be constructed from the sets $L = Y_{k,l-2}$, $S = Y_{k-2,l}$ as follows.



The points of L divide the plane into $t = |L|+1$ vertical strips. For $i = 1,\ldots,t$ place a tiny copy S_i of S into the strip i in such a way that all lines determined by a pair of points in S_i go below L and all lines determined by a pair of points in L go above S_i. See Valtr [81] for details. The modification is such that we can place tiny copies of $Y_{k-1,l}$ instead of the outer sets S_1 and S_t. □

The lower bound $g(k,l) \geq 2^{\binom{\lfloor k/2\rfloor+\lfloor l/2\rfloor-2}{\lfloor k/2\rfloor-1}}$ for k, l even can be proved by induction from the recurrence $g(k,l) \geq g(k,l-2)\cdot g(k-2,l)$. See Valtr[81].

Other improvements

Let $X_{k,l}$ be the maximal set of points with neither open k-cup nor open l-cap.

Lemma 3.3.11. *Every point $p \in X_{k,l}$ is either the left endpoint of an open $(k-1)$-cup or the right endpoint of an open $(l-1)$-cap.*

Let us note, that p cannot be both the left endpoint of an open $(k-1)$-cup and the right endpoint of an open $(l-1)$-cap, because we would have an open k-cup or open l-cap. See the second figure in the proof of lemma 3.3.4.

Proof. Assume that there is a point p for which none of the conditions hold. Then we can double the point p to the points p and p'. Consider the vertical line passing thought p and rotate it very slightly counter clockwise. Denote this line by l. Line l is much steeper than any other line determined by two points in $X_{k,l}$. Now shift p' for very small ε along l. Denote the set by $X'_{k,l}$.

The set $X'_{k,l}$ contains neither an open k-cup nor an open l-cap. If there will be such a cup, resp. cap then it has to contain both points p and p', otherwise it would correspond to an open k-cup, resp. open l-cap in $X_{k,l}$. Denote this cup, resp. cap by C. Since p and p' are neighbors in $X'_{k,l}$, they must be neighbors also in C. The line pp' is much steeper than any other line in $X'_{k,l}$ so the points p, p' can be only on the left end of the open cup or on the right end of the open cap. So C corresponds to an open $(k-1)$-cup, resp. open $(l-1)$-cap in the original set $X_{k,l}$, which ends in p. That would be a contradiction with our assumption.

By this construction we got the set $X'_{k,l}$ with neither an open k-cup nor an open l-cap and with more points than $X_{k,l}$. That contradicts its maximality. □

There is a symmetric version of this lemma where you change the words "left" to "right" and vice-versa. The lower bound on $g(k,l)$ might be further improved by an application of lemma 3.3.11 or by its symmetric version. We cannot enumerate the number of points of this construction to improve the bound of lemma 3.3.10. On the other hand we can check, that it really improves the original construction for small cases, i.e. $k = 5$, $l = 5$.

3.3.4 Tight bounds for $k = 4$ or $l = 4$

We start with the simpler case $l = 4$.

Lemma 3.3.12. *$g(k,4) \leq 2^{k-1}$ for $k \geq 2$.*

Proof. We prove it by induction on k. For $k = 2$, the maximal set with neither an open 2-cup nor an open 4-cap is just one point. So $g(2,4) = 2$.

Let $X_{k,4}$ be a maximal set with neither an open k-cup nor an open 4-cap. The upper envelope of $X_{k,4}$ must have three points. If it has more points then we have an open 4-cap. If it has only two points then we can place one new point to the left of $X_{k,4}$ and deep below. This set also contain neither an open k-cup nor an open 4-cap and is larger. That contradicts with the maximality of $X_{k,4}$.

Let p be the middle point of the upper envelope of $X_{k,4}$. Denote the set of points to the left of p by L and the set of points to the right of p by R. Every line determined by two points of L goes below p. Otherwise we have an open 4-cap. Thus every open cup in L can be extended by the point p. Hence L contains neither an open $(k-1)$-cup nor an open 4-cap. The size of L is at most $2^{k-2} - 1$ from the induction hypothesis. Similarly the size of R is at most $2^{k-2} - 1$. Altogether with the point p we have $2^{k-1} - 1$ points, that is what we wanted to prove. □

The lower bound obtained from lemma 3.3.10 is $g(k,4) \geq 2^{k-1}$ and hence the bound is tight.

Now we look at the case $k = 4$.

Point x is to the left of y if $x(x) < x(y)$, point y is to the right of x if $x(x) < x(y)$, point z is between points x, y if $x(x) < x(z) < x(y)$. Deep below the set X means below every line determined by points of X. Below the point r means below the horizontal line passing through the point r.

Lemma 3.3.13. *There is a set $X_{4,l}$ containing no open 4-cup and no open l-cap having $2^{l-1} - 1$ points.*

Let X be a set of points in the plane. X is *spiky* set determined by two points u, v if it either contains only the points u, v or it can be constructed from a spiky set determined by u, v by spike operation.

Spike operation can be applied on two neighboring points x, y of a set X (assume $x(x) < x(y)$). Imagine that every point of X is a light placed on the top of a very thin skyscraper. We say that the point *see* a light of another skyscraper if there is no skyscraper obstructing in the view. The idea of spike operation is to place a new light between two skyscrapers in such a way, that it does not see any other light besides the light of its neighbors. Spike operation adds a new point z between x and y and place it below[8] every line determined by x and a point $x' \in X$ to the left of x, below every line determined by y and a point $y' \in X$ to the right of y and also below points x, y.

Let x, y, $z \in X$ and z was constructed by a spike operation applied on x and y. We say that z is *son* of x, y and the points x, y are *parents* of z.

Let Y be a spiky set determined by u, v. To *embed* Y into the set X between neighboring points $u', v' \in X$ means to apply the same sequence of spike operations, which were used to construct Y from $\{u, v\}$, but this time on points u', v' of X. By embedding of spiky set into a spiky set we get again a spiky set.

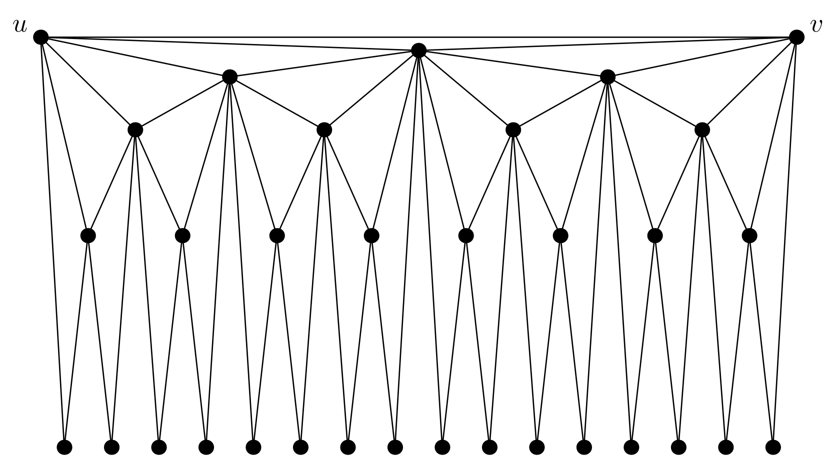


Figure 3.3: The spiky set X. Parents are connected with their sons by a segment.

Claim 3.3.14. *Spiky set has the following properties:*

[8] To simplify the placement, we can place the new point z deep below all existing points. But it is not necessary and it would make the figures more complicated.

i) Any point u cannot see any higher point besides of its parents. More precisely the cone with apex u and determined by parents of u contains no point of the spiky set.

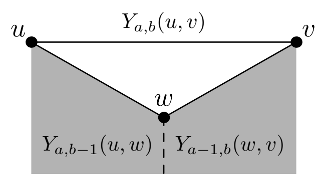
ii) Spiky set contains no open 4-cup.

Proof of the claim. i) follows from the definition of the spike operation. When u was constructed by spike operation, it did not see any other point. All its sons are placed below u.

ii) Suppose on the contrary that there is an open 4-cup and let y be its lowest point. But by the property i), the largest possible open cup containing y as the lowest point is $\{x, y, z\}$, where x and z are parents of y. A contradiction. □

Let $a, b \geq 1$. We construct spiky set $Y_{a,b}(u, v)$ determined by u, v, which contains neither an open $(a+1)$-cap ending in the leftmost point u nor an open $(b+1)$-cap ending in the rightmost point v, except for the 2-cup $\{u, v\}$. Denote the size of $Y_{a,b}(u, v)$ by $P_{a,b}$.

Obviously $Y_{a,b}(u, v) = \{u, v\}$ for $a = 1$ or $b = 1$. For $a, b > 1$ is $Y_{a,b}(u, v)$ defined recursively. First use spike operation on u, v to get a new point w, then embed $Y_{a,b-1}(u, w)$ between u and w and embed $Y_{a-1,b}(w, v)$ between w and v.



The set is obviously a spiky set and contains no open $(a+1)$-cap ending in u. If there would be such an open cap C then either C contains w and the part of C in $Y_{a-1,b}(w, v)$ is an open a-cap ending in w or C does not contain w and hence C is an open $(a+1)$-cap in $Y_{a,b-1}(u, w)$. In both cases we get a contradiction with properties of $Y_{a,b-1}(u, w)$ and $Y_{a-1,b}(w, v)$. Similarly, the constructed set contains no open $(b+1)$-cap ending in v. So the set has required properties.

From the construction we get a recurrence

$$\begin{aligned} P_{a,b} &= 2 && \text{for } a = 1 \text{ or } b = 1. \\ P_{a,b} &= P_{a,b-1} + P_{a-1,b} - 1 && \text{for } a, b > 1. \end{aligned} \tag{3.1}$$

Claim 3.3.15. *The set $Y_{a,b}(u, v)$ contains no open $(a + b - 1)$-cap.*

Proof of the claim. Take the highest point h of the maximal open cap in $Y_{a,b}(u, v)$. Denote the length of this cap by m. If h is one of the points u, v then $m = a$ or $m = b$, and the claim follows (remind that $a, b \geq 1$). Otherwise h is a point in the middle. In this case follow the decomposition of the set $Y_{a,b}(u, v)$ until we get to the points u', v' having h as their only common son. Every open cap having h as its rightmost and highest point lies in $Y_{a',b'}(u', h)$ and has length at most b'. Similarly, every open cap having h as its leftmost and highest point lies in $Y_{a'',b''}(h,v')$ and has length at most a''. Since $b' < b$ and $a'' < a$, the length of the maximal cap is $m \leq b' + a'' \leq a + b - 2$. □

Now we are ready to construct $X_{4,l}$. Take convex $(l-1)$-cap A with points a_1, a_2,… a_{l-1} ordered according to the x-axis and for $j = 1$ to $l-2$ embed the spiky set $Y_{l-j,j+1}(a_j, a_{j+1})$ between points a_j and a_{j+1} .

The set contains no open 4-cup because of the properties of spiky set. Let us show that it contains no open l-cap. Take the maximum open l-cap C. Cap C cannot lie inside any set $Y_{l-j,j+1}(a_j, a_{j+1})$, because of Claim 3.3.15. Hence it must contain some points of A. Let a_r and a_s be the leftmost and the rightmost point of $A \cap C$, respectively. All points of C to the left of a_r have to lie in $Y_{l-r+1,r}(a_{r-1}, a_r)$. Further, C necessarily contains all points of A between a_r and a_s and all points of C to the right of a_s have to lie in $Y_{l-s,s+1}(a_s, a_{s+1})$. Counting the length of C we get $r + (s-r-1) + l - s = l - 1$. So there is no open l-cap in $X_{4,l}$.

The size of the set $X_{4,l}$ is given by the following formula

$$|X_{4,l}| = P_{l-1,2} + P_{l-2,3} + \cdots + P_{3,l-2} + P_{2,l-1} - (l-3). \tag{3.2}$$

In the formula we just added the sizes of sets $Y_{l-j,j+1}(a_j, a_{j+1})$ for $1 \leq j \leq l-2$ and subtracted the number of points counted twice.

Claim 3.3.16. $|X_{4,l}| = 2^{l-1} - 1$ *for* $l \geq 1$.

Proof of the claim. The proof is by induction on l. Claim holds for $l \leq 3$ because $X_{4,3}$ is a 3-cup. Denote the $\sum_{i=1}^{l-2} P_{l-i,i+1}$ by S_l. From the equation (3.2) we get that the Claim 3.3.16 is equivalent to showing $S_l = 2^{l-1} + l - 4$ for $l \geq 1$. Now it suffice to show the inductive step which is $S_{l+1} = 2^l + (l+1) - 4$. By the identity (3.1) we have that every term $P_{l-i,i+1}$ of S_l contribute to the sum S_{l+1} twice. So $S_{l+1} = 2 \cdot S_l + P_{l,1} + P_{1,l} - (l-1) = 2 \cdot (2^{l-1} + l - 4) + 2 + 2 - l + 1 = 2^l + l - 3$. □

That finishes the proof of Lemma 3.3.13.

Lemma 3.3.17. *Every set with no open* 4*-cup and no open* l*-cap has at most* $2^{l-1} - 1$ *points.*

We start with a definition of property Q.

Definition 3.3.18 (property Q). *Let* $a, b \geq 1$ *and* $a + b < l$. *Set of points* X *has property* $Q_{a,b}(u,v)$ *if it satisfies the following:*

i) u *is the leftmost point of* X, v *is the rightmost point of* X *and all points of* X *lie below the segment* uv.

ii) X *contains no open* l*-cap and no open* 4*-cup*

iii) X *contains no open* $(a+1)$*-cap ending in* u *(except for the cap* $\{u,v\}$*)*

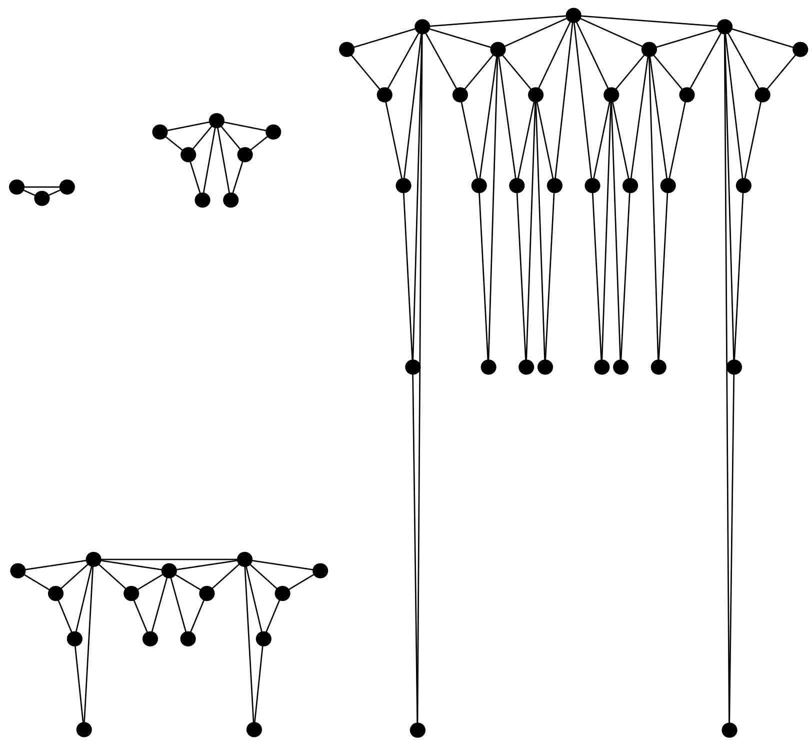


Figure 3.4: The sets $X_{4,3}$, $X_{4,4}$, $X_{4,5}$ and $X_{4,6}$.

iv) X contains no open $(b+1)$-cap ending in v (except for the cap $\{u,v\}$)

Denote by $Y^*_{a,b}(u,v)$ the maximum set having the property $Q_{a,b}(u,v)$ and by $P^*_{a,b}$ the number of points in $Y^*_{a,b}(u,v)$.

Claim 3.3.19. *$Y_{a,b}(u,v) = Y^*_{a,b}(u,v)$ and hence also $P_{a,b} = P^*_{a,b}$.*

Proof of the claim. We prove it by induction on $a+b$. Clearly $Y^*_{a,b}(u,v) = Y_{a,b}(u,v) = \{u,v\}$ for $a=1$ or $b=1$. Let $a,b>1$. We show that $Y^*_{a,b}(u,v)$ satisfies the same decomposition as spiky set $Y_{a,b}(u,v)$. We want to find a point w such that the vertical line passing through w splits $Y^*_{a,b}(u,v)$ into the sets Y_L, Y_R (put w into both sets).

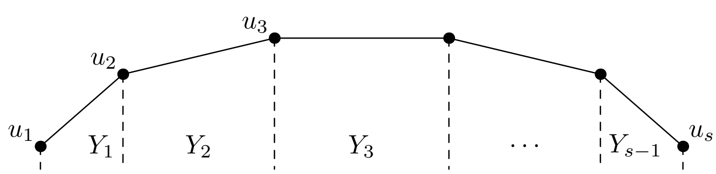
Choose w to be the point of $Y^*_{a,b}(u,v)\setminus\{u,v\}$ with the smallest angle $\angle uvw$. Because of the choice of w there is no point of $Y^*_{a,b}(u,v)\setminus\{u\}$ above line wv. Moreover there is no point of $Y^*_{a,b}(u,v)\setminus\{v\}$ above line uw, because we would get an open 4-cup.

We need to show that the sets Y_L, Y_R satisfies properties $Q_{a,b-1}(u,w)$ and $Q_{a-1,b}(w,v)$, respectively. Many items of property Q are satisfied because they were satisfied in the original set $Y^*_{a,b}(u,v)$ or because of the choice of w. It remains to show that Y_L contains no open b-cap ending in w, which holds because every such a cap can be extended by v to an open $(b+1)$-cap in $Y^*_{a,b}(u,v)$. Similarly, Y_R contains no open a-cap ending in w.

Y_L, Y_R might not be the maximal sets with property Q, but from the decomposition we get the inequality $P^*_{a,b} \leq P^*_{a,b-1} + P^*_{a-1,b} - 1$. From the induction hypothesis $P^*_{a,b} \leq P_{a,b-1} + P_{a-1,b} - 1 = P_{a,b}$. The opposite inequality holds because $P_{a,b}$ satisfies property $Q_{a,b}$. That finishes the proof of the claim. □

We have shown a little more during the proof. We have shown that $Y^*_{a,b}(u,v)$ has unique "spike" decomposition and hence all maximal sets with property $Q_{a,b}(u,v)$ are equivalent.

Proof of the Lemma 3.3.17. Let $X^*_{4,l}$ be the maximal set of points containing no open 4-cup and no open l-cap. Moreover we choose the one with maximum points on the upper envelope. Denote the points lying on the upper envelope by u_1, u_2,... u_s according to the x-axis. Clearly $s \leq l-1$. The vertical lines passing through the points u_i split the set $X^*_{4,l} - \{u_i | 1 \leq i \leq s\}$ into the sets Y'_1, ..., Y'_{s-1} again ordered by x-coordinate. Let $Y_j = Y'_j \cup \{u_j, u_{j+1}\}$ and let a_j, b_j be minimal integers such that the set Y_j has the property $Q_{a_j,b_j}(u_j, u_{j+1})$.



Each set Y_j must be a maximal set with the property $Q_{a_j,b_j}(u_j, u_{j+1})$, otherwise we replace Y_j by $Y_{a_j,b_j}(u_j, u_{j+1})$ and enlarge $X^*_{4,l}$. That would be a contradiction. Hence $|Y_j| \leq P_{a_j,b_j}$ (Claim 3.3.19).

For every $j \in \{1, \ldots, s-1\}$, let $C_L(j)$ be the largest open cap to the left of u_j and ending in u_j. Similarly, let $C_R(j)$ be the largest open cap to the right of u_{j+1} and ending in u_{j+1}. Denote the lengths of the open caps $C_L(j)$, $C_R(j)$ by $t_L(j)$, $t_R(j)$, respectively. We claim that $t_L(j) = l - a_j$ and $t_R(j) = l - b_j$. (In other words the union of the open cap $C_L(j)$ and an open a_j-cap having left endpoint in u_j has length $l-1$). If one of the caps $C_L(j)$, $C_R(j)$ were shorter, then we replace the set Y_j by the larger set $Y_{l-t_L(j),l-t_R(j)}(u_j, u_{j+1})$ and get a contradiction with the maximality of $X^*_{4,l}$.

For every pair of indices $j_1 < j_2$ is $a_{j_1} > a_{j_2}$. Otherwise the open $(l-a_{j_1})$-cap $C_L(j_1)$, points $u_{j_1}, \ldots, u_{j_2}$ on the upper envelope and the open a_{j_2}-cap would form an open cap of size at least l. Similarly, $b_{j_1} < b_{j_2}$ for every pair of indices $j_1 < j_2$.

We claim that $a_j + b_j = l+1$ for every $j \in \{1, \ldots, s-1\}$. The maximal open cap containing u_j and u_{j+1} is equal to the union of $C_L(j)$ and $C_R(j)$ and has size $(l-a_j) + (l-b_j)$. This size has to be at most $l-1$, what gives $a_j + b_j \leq l+1$. If $a_j + b_j < l+1$ then take the common son of u_j and u_{j+1} in the "spike" decomposition of Y_j (Claim 3.3.19) and lift it to the upper envelope.

The new set contains no open 4-cup and no open l-cap, but has one more point on the upper envelope. That's a contradiction with the choice of $X^*_{4,l}$.

Putting the observations together we get

$$|X^*_{4,l}| \leq \left(\sum_{i=1}^{s-1} |Y_j|\right) - (s-2) \leq \left(\sum_{i=1}^{l-2} P_{i,l-i+1}\right) - (l-3) = 2^{l-1} - 1.$$

We got the same sum as in the proof of Lemma 3.3.13. That finishes the proof of the lemma. □

Remark: It is possible to follow the ideas of the bounds for $l = 4$ and generalize them to $l > 4$. But the description of the structure is too complicated.

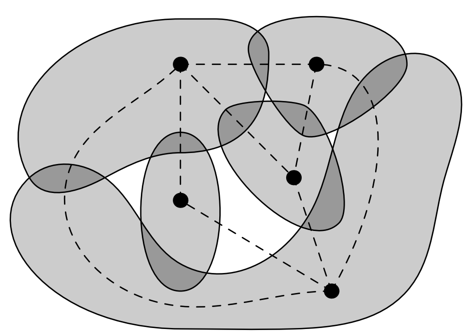
Chapter 4

Circle graphs

4.1 Intersection graphs

Intersection graph represents the pattern of intersection of a family of sets. The vertices of intersection graph are sets and two vertices are joint by an edge if and only if the sets intersects.

For example look at the following figure.



On the other hand every graph G can be represented as an intersection graph of some family of set. For every edge $e = \{u, v\}$ take witness $w_{\{u,v\}}$. As a set representing vertex v take $S_v := \{w_{\{x,v\}} \mid \{x, v\} \in E(G)\}$ for every $v \in V(G)$.

It is more interesting to investigate intersection graphs of some geometric objects. For various geometric objects we get various classes of intersection graphs.

Interval graphs (denoted by INT) are intersection graphs of intervals on a real line. *Circle-arc graphs* (CA) are intersection graphs of arcs on a circle. *Circle graphs* (CIR) are intersection graphs of chords of a circle. *Polygon-circle graphs* (POLYGON-CIR) are intersection graphs of convex polygons inscribed in a circle. *Chordal graphs* (CHOR) are intersection graphs of subtrees in a tree. *Function graphs* (FUN) are intersection graphs of functions on closed interval $[0, 1]$. *Permutation graphs* (PER) are intersection graphs of segments, which are linear functions on closed interval $[0,1]$. *Segment graphs* (SEG) are intersection graphs of segments in the plane. *String graphs* (STRING) are intersection graphs of strings (connected curves) in the plane.[1] Graph has *boxicity* k if it is an intersection graph of k-dimensional boxes. *Line graph* $L(G)$ is an intersection graph of edges of a graph G, where by edge we mean set of two vertices. To close the list of graph classes we also mention *comparability graphs* (denoted by CO). Vertices of comparability graph are elements of poset and two elements are joint by an edge if and only if they are comparable.

If $\mathcal{A}$ is some class of graphs, then $\overline{\mathcal{A}} := \{\overline{G} \,|\, G \in \mathcal{A}\}$ is class of complements of graphs from $\mathcal{A}$.

A graph G is *perfect* if and only if $\chi(H) = \omega(H)$ for every induced subgraph $H \subseteq G$. Perfect graphs are important because many problems like coloring, finding maximum clique, finding maximum independent set can be solved in polynomial time in perfect graphs. Many classes of intersection graphs are perfect graphs (INT, CHOR, CA, PER, line graphs for example).

There are two famous theorems about perfect graphs:

- **Perfect graph theorem** (Lovász 1972): graph is perfect if and only if its complement is perfect.
- **Strong perfect graph theorem** (Chudnovsky, Robertson, Seymour, Thomas 2002): An induced cycle of odd length at least 5 is called an odd hole. An induced subgraph that is complement of an odd hole is called an odd anti-hole. Graph is perfect if and only if it contains neither odd hole nor odd anti-hole.

The second theorem was called Berge conjecture, before it was proved.

Many classes of intersection graphs are well characterized. Many classes of intersection graphs (INT, CIR, CHOR) are recognizable in polynomial time. That means we can find the representation of the intersection graph in polynomial time. It is NP-complete for some other classes like STRING.

We list some basic results. Some of them are clear, other need some work. PER $=$ $\overline{\text{PER}}$, INT $=$ CHOR $\cap$ $\overline{\text{CO}}$, INT $\subset$ CA, CIR $\subset$ POLYGON-CIR, PER $\subset$ FUN, PER $\subset$ CIR $\subset$ SEG $\subset$ STRING.

Interesting recent result is PLANAR $\subseteq$ SEG. In other words, every planar graph can be represented as an intersection graph of segments in the plane. For a proof see [13].

For an overview of results on intersection graphs see for example [52].

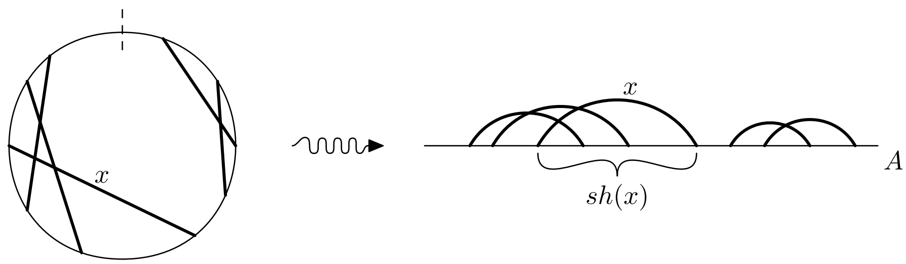
4.2 Circle graphs

Circle graphs (denoted by CIR) are intersection graphs of chords of a circle. We assume that no two chords have a common endpoint.

[1]String graphs are sometimes called “spaghetti” graphs, because they look like spaghetties on the plane (or on the plate?).

Circle graphs can be equivalently defined as *overlap graphs* of intervals on a line. Two intervals overlap if they intersect, but none of them is subinterval of the other. Overlap graph is a graph whose vertices are intervals and two vertices are joined by an edge if the corresponding intervals overlap.

Another representation is somewhere in between. We start with the chords of a circle. Cut the supporting circle at one point to get the arc A. The arc A is called the *supporting arc.* Then imagine, that the circle and chords are from a rubber. Unwrap the cut circle to a line as shown on the following figure.



We will often use this *unwrapped representation*, because it is suitable for drawing figures and also it enable us to speak about things like on the left, on the right, under a chord. We number the endpoints of chords along the arc A. The direction from smaller numbers to larger numbers is to the right, the opposite direction is to the left.

Let $G \in$ CIR and $V = V(G)$ be its set of chords. *Shadow* $sh(s)$ of chord $s \in V$ is the part of the arc A lying between the endpoints of s. We say that $B \subseteq A$ *lies under* the chords $W \subset V$ if $B \subseteq \bigcap_{s \in W} sh(s)$. Particularly a point p lies under the chord s if $p \in sh(s)$. Sometimes we say B *pierces* W instead of B lies under W. It has the same meaning, but it might be more expressible.

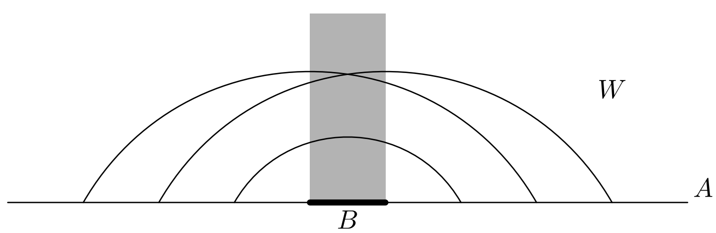


Figure 4.1: Arc $B \subseteq A$ pierces chords W (all chords in the figure). Conversely we say, that W covers B.

Conversely we say that $W \subset V$ *covers* an arc $B \subseteq A$ if and only if B lies under W. Denote by $V|B$ the chords induced by arc B. That are chords with both endpoints in B. Chord v lies under the chord w if $sh(v) \subset sh(w)$. Remind that the chords cannot have a common endpoint.

If A is an arc and $P \subset A$ then $conv\{P\}$ is the smallest arc $B \subseteq A$ containing P (in unwrapped representation).

4.3 Coloring circle graphs

We denote the clique and the chromatic number of the graph G by $\omega(G)$, $\chi(G)$, respectively. Obviously, $\chi(G) \geq \omega(G)$. Gyarfás [31],[32] defined the notion of binding function: f is a binding function for χ and for the class of graphs $\mathcal{A}$ iff $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{A}$. The main question of this paper is to determine or estimate the following binding function

$$f(k) = \max\{\, \chi(G) \mid G \in \text{CIR} \;\&\; \omega(G) \leq k\}.$$

Clearly $f(1) = 1$. Ageev [4] showed that $f(2) = 5$. Kostochka and Kratochvíl [43] showed $f(k) \leq 50 \cdot 2^k - 32k - 64$ for general k. The best lower bound is $f(\omega) = \Omega(\omega \log \omega)$ due to Kostochka [42]. (See Section 4.4 of this thesis for the proof of lower bound). We were told by several people that many scientists tried to improve the upper bound, but they were not able to decrease it even by an additive constant. In this paper we modify the proof of Kostochka and Kratochvíl and get a slightly better upper bound

$$f(k) \leq 16 \cdot 2^k - 16k - 16.$$

In the discussion with Jan Kára, we observed $\chi(G) \leq \omega\lceil \log n \rceil$, where n is the number of vertices of the graph. This upper bound shows, that the circle graphs with small maximum clique and large chromatic number must be very large.

Esperet and Ochem [23] have proven an interesting observation: every circle graph G with girth $g \geq 5$ contains either a vertex of degree at most one, or a chain of $g - 4$ vertices of degree 2. That implies the result of Ageev for $g = 5$.

4.3.1 The proof of the upper bound

This section presents the proof of the following theorem.

Theorem 4.3.1. $f(k) \leq 16 \cdot 2^k - 16k - 16$ *for* $k \geq 1$.

In all proofs in this section, we use the unwrapped representation of circle graphs. First we prove the following lemma, which is one of the basic tools for coloring chords.

Lemma 4.3.2. *Let* $G \in \text{CIR}$*,* $p \in A$ *point on the supporting arc. If* $W \subseteq V(G)$ *is a set of chords pierced by* p *then* W *can be colored by* $\omega(G)$ *colors.*

Proof. For any two disjoint chords, one must be under the other. That is a partial ordering on disjoint chords. Any two disjoint chords are comparable, thus incomparable chords form a clique. By Dilworth theorem all chords can be covered by at most $\omega(G)$ chains, which are the color classes. □

The trick of the main proof is to add something what looks useless at first sight. During the coloring of a circle graph, we work with sub-arcs of the supporting arc A.

Let $G = (V, E)$ be the circle graph and A supporting arc of its representation. Let $\mathcal{C}$ be a family $\mathcal{C}$ of sub-arcs of A. The family $\mathcal{C}$ *pierces* the set of chords $W \subseteq V$ if and only if

- *i)* $C_1 \cap C_2 = \emptyset$ for every $C_1, C_2 \in \mathcal{C}$ and
- *ii)* for every chord $w \in W$ there is an arc $C_w \in \mathcal{C}$ piercing w.

In other words, the arcs must be disjoint and every chord must be pierced by some arc. We denote by $\overline{\omega}(V, \mathcal{C})$ the maximum size of a clique $Q \subseteq V$ pierced by some arc $C \in \mathcal{C}$.

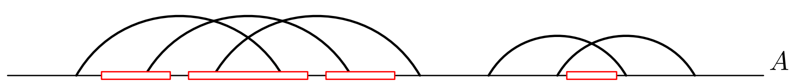


Figure 4.2: Here is an example. The set V contains 5 chords and the family $\mathcal{C}$ contains four disjoint arcs. Family $\mathcal{C}$ pierces V, $\overline{\omega}(V, \mathcal{C}) = 2$ but $\omega(G) = 3$.

Note that $\omega(G)$ can be three times larger than $\overline{\omega}(V, \mathcal{C})$. For example look at the left component in the figure 4.2.

It is easy to find a family of arcs piercing V. We can for example choose $\mathcal{C}$ as the endpoints of all chords.[2] Another option is the set of points of A lying between two consecutive endpoints of chords.

We define an auxiliary binding function

$$g(k) = \max\{\, \chi(G) \mid \quad G \in \textsc{cir} \text{ and there exists } \mathcal{C} \text{ piercing } V(G) \text{ such that } \overline{\omega}(V(G), \mathcal{C}) \leq k\},$$

where $\mathcal{C}$ is set of sub-arcs of the supporting arc.

Clearly $f(k) \leq g(k)$. If there is no $k+1$ clique in the graph, then there is no $k+1$ clique pierced by an arc. Moreover we show $f(k) \leq 2g(k-1)$ in Lemma 4.3.8. Hence it is sufficient to show the upper bound on $g(k)$.

Theorem 4.3.3. *$g(k) \leq 16 \cdot 2^k - 8k - 16$ for $k \geq 1$.*

For the proof of Theorem 4.3.3 we first show the recurrence $g(k) \leq 2 \cdot (g(k-1) + 4k)$ (Lemma 4.3.4). The theorem follows from the recurrence by induction.

Let us look at Theorem 4.3.3 from algorithmic point of view. We want to color a circle graph $G = (V, E)$ such that there are arcs $\mathcal{C}$ piercing G with $\overline{\omega}(V, \mathcal{C})$. We find the coloring by recursive procedure `color(`$V, \mathcal{C}$`)`. Parameter V is a subset of chords and $\mathcal{C}$ is a set of sub-arcs of the supporting arc A, which is piercing V. Following Lemma 4.3.4 corresponds to one call of the procedure. Recursive calls corresponds to the proof by induction. At the beginning of the algorithm $\overline{\omega}(V, \mathcal{C})$ is at most $\omega(G)$. For each recursive call `Color(`$W, \mathcal{D}$`)`, where $W \subseteq V$, we decrease $\overline{\omega}$. Thus the depth of the recursion is $\overline{\omega}(V, \mathcal{C})$.

[2] Remind that according to the definition the endpoints of a chord *ch* lie under *ch*. It is just technical detail which comes useful in implementation of the coloring.

Lemma 4.3.4. *$g(k) \leq 2 \cdot (g(k-1) + 4k)$ for $k \geq 1$.*

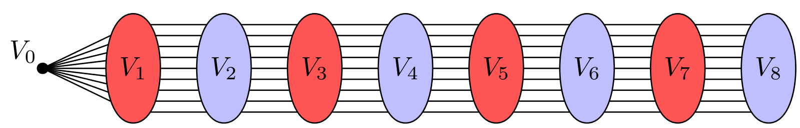
Proof. Let $G = (V, E)$ be the circle graph and $\mathcal{C}$ be the set of sub-arc of A witnessing that $\overline{\omega}(V, \mathcal{C}) \leq k$. We want to color the vertices V.

We find the coloring by recursive procedure `color(`V`, `$\mathcal{C}$`)`. Parameter V is a subset of chords and $\mathcal{C}$ is a set of sub-arcs of the supporting arc A, which is piercing V. Now we are ready to sketch the idea of the coloring.

`Color(`V`,`$\mathcal{C}$`):`

- We can color all connected components by the same colors so assume that V are the vertices of one connected component.

- **BFS layers:** Take the chord $s \in V$ incident with the leftmost endpoint on A and run breadth first search (BFS) starting at s to get BFS layers $V_0, V_1, \ldots, V_t$. That is $V_0 = \{s\}$ and chords in layer V_i are the chords in distance i from s in G. Chords V_i cross only chords from V_{i-1} and V_{i+1}. So we can alternate two color sets, one for even layers and the other for odd layers.



- **Coloring layer V_i:** Color all connected components of V_i independently one by one. So we can assume that V_i is one connected component.

  First, we construct a new set $\mathcal{D}$ of disjoint sub-arcs of A. We split V_i into U_1 and U_2. The chords U_1 are chords pierced by some arc of $\mathcal{D}$ and chords U_2 are not.

  The chords U_1 can be colored recursively by calling `color(`U_1`,`$\mathcal{D}$`)`, because $\mathcal{D}$ pierces U_1. The recursion is possible because $\overline{\omega}(U_1, \mathcal{D}) < \overline{\omega}(V, \mathcal{C})$ by the construction of $\mathcal{D}$ (details are in Claim C1).

  The chords U_2 can be colored directly by $4 \cdot \overline{\omega}(V, \mathcal{C})$ colors (details are in Claim C2).

Now, we can advance to the **detailed proof.** First two steps of the procedure `Color` are clear. Let us look at the third point. Coloring of layers V_0 and V_1 is easy because V_0 contains only one chord and V_1 can be colored by Lemma 4.3.2. We want to color the layer V_i for $i \geq 2$ (assume that it has one component). Take the smallest arc $B \subseteq A$ containing all endpoints of chords V_i and let $P_B = B \cap \{$ endpoints of $V_{i-1}\}$.

Claim 4.3.5. *Every chord $x \in V_i$ crosses a chord $y \in V_{i-1}$ for $i \geq 2$. Hence x covers a point $b \in P_B$, the endpoint of y. Moreover the second endpoint of y lies outside B.*

Every chord $x \in V_i$ has a neighbor $y \in V_{i-1}$ in the graph G. Being neighbors means that x crosses y and thus one endpoint of y must lie under x. No chord $x \in V_i$ can cross a chord $z \in V_0 \cup V_1 \cup \cdots \cup V_{i-2}$, because then x would be in distance $j < i$ from $s \in V_0$ and x would be in layer V_j.

Chords $V_0 \cup V_1 \cup \cdots \cup V_{i-2}$ are cutting the supporting circle into convex regions $\mathcal{R}$. Chords V_i lie fully in the regions $\mathcal{R}$. A chord $y \in V_{i-1}$ cannot lie completely in a convex region $R \in \mathcal{R}$, because it would not cross any chord of V_{i-2}. Thus the endpoints of chords V_{i-1} lies in different regions.

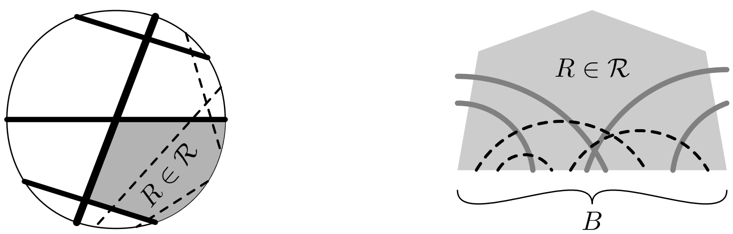
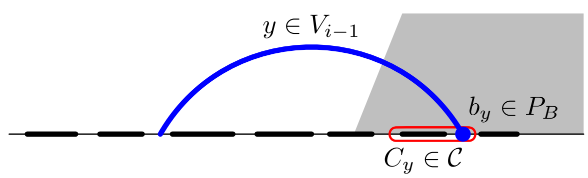


Figure 4.3: *On the left:* The black segments $V_0 \cup V_1 \cup \cdots \cup V_{i-2}$ cut the circle into regions $\mathcal{R}$. Dashed segments V_{i-1} must cross some black segment. *On the right:* Dashed segments of a component of V_i lie fully in region $R \in \mathcal{R}$. Grey segments V_{i-1} have at most one point in R.

Definition of new family of disjoint arcs: For every chord $y \in V_{i-1}$ with an endpoint $b_y \in P_B$, there is an arc $C_y \in \mathcal{C}$ lying under y. It holds because $\mathcal{C}$ pierces V. From all possible arcs lying under y we choose C_y to be the one, which is closest to b_y.



For every chord $y \in V_{i-1}$ with endpoint $b_y \in P_B$ and arc C_y define a new arc D as $conv\{C_y \cup \{b_y\}\}$ i.e. the shortest arc containing both C and b_y. Denote the set of all new arcs by $\mathcal{D}^*$. Choose $\mathcal{D} \subseteq \mathcal{D}^*$ to be the maximum subset of disjoint arcs such that their right ends are leftmost possible.

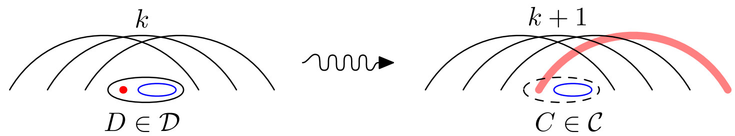
Further in this proof, $\mathcal{C}$ denotes the original arcs and $\mathcal{D}$ denotes the newly defined arcs.

Now we partition the set V_i into two sets U_1 and U_2. U_1 are the chords which are pierced by an arc from $\mathcal{D}$ and U_2 are the chords which are not.

We claim that $\chi(U_1) \leq g(k-1)$ (Claim C1) and $\chi(U_2) \leq 4k$ (Claim C2). It remains to prove the claims and that will finish the proof. $\square$

Claim 4.3.6 (C1). *Arcs $\mathcal{D}$ pierce U_1 and $\overline{\omega}(U_1, \mathcal{D}) = k-1$. Hence $\chi(U_1) \leq g(k-1)$.*

Proof. Set $\mathcal{D}$ pierces U_1 by the choice of U_1. Suppose on the contrary that $Q \subset U_1$ is a clique of size k covering $D \in \mathcal{D}$. By the definition of new arcs, D consists of a point b_x and an original arc $C \in \mathcal{C}$. Point b_x is an endpoint of chord $x \in V_{i-1}$. Chord x crosses all chords in the clique Q because its other endpoint lies outside B. Arc C lies under all chords of Q and also under x. So $Q \cup \{x\}$ is a clique in V covering C and its size is $k+1$. That is a contradiction.



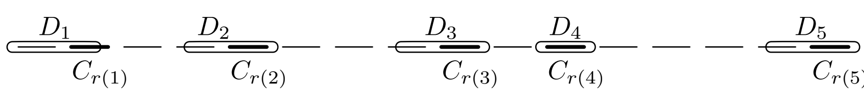
□

Claim 4.3.7 (C2). $\chi(U_2) \leq 4k$.

Proof. This proof is easy but technical. We want to color chords U_2, which are pierced by arcs $\mathcal{C}$. There is no chord of U_2 covering a new arc from $\mathcal{D}$. That brings us to the idea that there is no "long" chord in U_2 and thus all chords lie in the "gaps" between arcs $\mathcal{D}$. In fact their endpoints can lie on arcs $\mathcal{D}$ too. That makes the coloring difficult because such "gaps" overlap and chords from different gaps can cross above the arc of $\mathcal{D}$.

For $C \in \mathcal{C}$ define $S(C) = \{x \in U_2 \mid C \subseteq sh(x)\}$, what are the chords of U_2 pierced by the arc C. Similarly, we can define $S(p)$ for a point $p \in A$. Remind that by Lemma 4.3.2 the chords $S(C)$ can be colored by $\overline{\omega}(V, \mathcal{C})$ colors, which is equal to k. That will be our coloring tool in the proof of the claim.

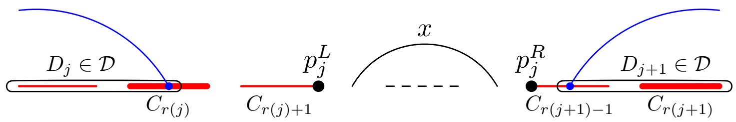
Every new arc $D \in \mathcal{D}$ is a convex hull of one original arc $C' \in \mathcal{C}$ and a point $b \in P_B$. The point b can lie on another arc $C'' \in \mathcal{C}$. There is no arc of $\mathcal{C}$ lying between C' and C'' because of the definition of new arcs $\mathcal{D}$. We conclude that every new arc D intersects at most two original arcs from $\mathcal{C}$. Denote the arcs of $\mathcal{C}$ by C_1, C_2,..., C_q as they appear from the left to the right. Similarly denote the new arcs of $\mathcal{D}$ by D_1, D_2,..., D_m. For $j \in \{1, \dots, m\}$, $r(j)$ is the index of the rightmost arc of $\mathcal{C}$ intersecting D_j. So $C_{r(j)}$ is that arc. See an example on the next figure. From technical reasons, we define $r(0) = 0$ and $r(m+1) = q+1$. You can imagine that as adding virtual arcs C_0 and C_{q+1} to the left and to the right of arcs $\mathcal{C}$.



Put $W := U_2 \setminus (\bigcup_{i=1}^{m} S(C_{r(i)}))$. That are all chords of U_2, which are not pierced by any arc $C_{r(j)}$ for $j \in \{1, \dots, m\}$. Now, we split W into sets W_j for $j \in \{0, \dots, m\}$. The set $W_j \subseteq U_2$ is induced by the arc $conv\{C_{r(j)}, C_{r(j+1)}\}$.

For every j, define p_j^L as the right endpoint of $C_{r(j)+1}$ and p_j^R as the left endpoint of $C_{r(j+1)-1}$ (see the next figure). If $C_{r(j)+1} = C_{r(j+1)-1}$ (there is exactly one arc of $\mathcal{C}$ between $C_{r(j)}$, $C_{r(j+1)}$), then define $p_j^R := p_j^L$. Moreover in case $C_{r(j)+1} = C_{r(j+1)}$ (there is no arc of $\mathcal{C}$ between $C_{r(j)}$, $C_{r(j+1)}$), define the points $p_j^L = p_j^R$ as arbitrary point on A lying between $C_{r(j)}$, $C_{r(j+1)}$.

Now we claim that each chord of W_j is pierced by at least one of the points p_j^L, p_j^R.



Suppose on the contrary that there is a chord $x \in W_j$ which is pierced neither by p_j^L nor by p_j^R. Chord x must be pierced by some arc of $\mathcal{C}$. It cannot lie to the left of p_j^L, because it is not pierced neither by $C_{r(j)}$ (otherwise $x \notin W$) nor by $C_{r(j)+1}$ (otherwise it is pierced by p_j^L). Symmetricaly, it cannot lie to the right of p_j^R. Thus it must lie between p_j^L, p_j^R. There is a point $b_x \in P_B$ under x. Point b_x together with either $C_{r(j)+1}$ or $C_{r(j+1)-1}$ forms a new arc D', which had to be considered during the choice of $\mathcal{D}$. The arc D' can be either added to $\mathcal{D}$ to enlarge it or it can replace arc D_{j+1}, because its right endpoint is more to the left. That is the contradiction with the definition of $\mathcal{D}$.

By now, we have shown, that each chord of U_2 is piereced by $C_{r(j)}$ or p_j^L or p_j^R for some $j \in \{0, 1, \ldots, m, m+1\}$. Now we proceed to the coloring step.

We color $\bigcup_{j=0}^{m+1} S(p_j^L)$ by $2k$ colors. We alternate two color sets. One for $S(p_0^L)$, second for $S(p_1^L)$, again the first one for $S(p_2^L)$, and so on.

Chord $x \in S(p_i^L)$ has the endpoints on $conv\{D_i, D_{i+1}\}$, otherwise it would be pierced by D_i or D_{i+1}. Hence two chords of the same color cannot cross.

Points p_i^L for $i \in \{0, 1, \ldots, m, m+1\}$ split the remaining uncolored chords into strips. Strips are well separated by the points p_i^L. Each strip contains only chords which are pierced by p_i^R or $C_{r(i+1)}$. Hence chords in each strip can be colored by $2k$ colors. Therefore we have colored all chords U_2 by $4k$ colors. That finishes the proof. □

Lemma 4.3.8. $f(k) \leq 2g(k-1)$.

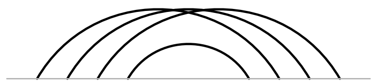
Proof. Let $G = (V, E)$ be a circle graph with $\omega(G) \leq k$. Put $\mathcal{C}$ to be a set of endpoints of all chords V. Now we follow the proof of Lemma 4.3.4, what you can imagine as a call of `Color(`$V, \mathcal{C}$`)`. For every layer V_i we find points P_B such that every chord $x \in V_i$ has a point $b \in P_B$ under it. Each new arc will consist of only one point $b \in P_B$. Thus $U_2 = \emptyset$ in this case. □

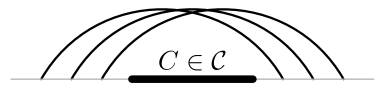
Proof of Theorem 4.3.3. For $k = 0$ is $g(0) = 0$ and the Lemma holds. For $k \geq 1$ we apply the recurrence from Lemma 4.3.4 and get $g(k) \leq 2 \cdot (16 \cdot 2^{k-1} - 8(k-1) - 16) + 8k = 16 \cdot 2^k - 8k - 16$. □

Proof of Theorem 4.3.1. By combination of Lemma 4.3.8 and Theorem 4.3.3 we get $f(k) \leq 2g(k-1) = 16 \cdot 2^k - 16k - 16$. □

If we find a better estimate on $g(k)$ for some small k, we can use it in the proof of Theorem 4.3.3 and further improve the upper bound.[3]

Note: What is the main difference between this proof and the proof from [43]? In fact the basic idea is the same. Kratochvíl and Kostochka were coloring circle graphs $\mathcal{H}(k)$ with no chord lying under a k-clique. We are coloring circle graph $G \in \mathcal{G}(k)$ for which there exists a set of arcs $\mathcal{C}$ piercing $V(G)$ with no arc $C \in \mathcal{C}$ lying under a k-clique. The forbidden configurations are on the following figures.



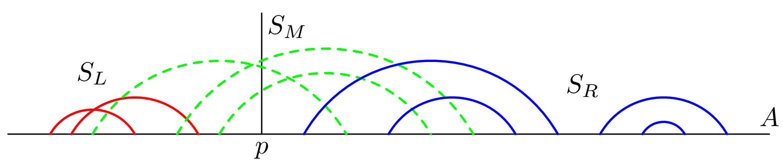


4.3.2 Second upper bound for $\chi(G)$

During the discussion with Jan Kára we found the following bound.

Lemma 4.3.9. *Any graph $G \in$ CIR can be colored by $\chi(G) \leq \omega\lceil \log n \rceil$ colors, where $\omega = \omega(G)$.*

Proof. We use the method divide et empera. First we split the problem into subproblems and then solve them recursively. The input of our problem is an arc A and n chords S with endpoints on A. For any point $p \in A$, we can split S into S_L (the chords to the left of p), S_M (chords whose shadow contains p) and S_R (the chords to the right of p).



By starting on the left end of A and going to the right we find a point $p \in A$ such that $S_L \leq n/2$ and $S_R \leq n/2$. That's easy because during the movement we can only add chords to S_L or remove chords from S_R. By Lemma 4.3.2 we color chords S_M by ω colors and we color S_L, S_R recursively. Note that S_L can be colored by the same colors as S_R. So in the second iteration we will need only another ω colors. The sets S_L, S_R become empty after at most $\lceil \log n \rceil$ iterations and the algorithm stops. Thus we had colored all chords with at most $\omega\lceil \log n \rceil$ colors. □

[3] Theorem 4.3.3 gives us $g(1) \leq 8$. The lower bound is 3, because of 5-cycle C_5. We cannot show anything better.

4.3.3 Generalization to polygon-circle graphs

All previous results can be generalized for polygon-circle graphs. The same proofs works. It is not a surprise because our proof is just a modification of former proof by Kratochvíl and Kostochka [43], which is for polygon-circle graphs. We decided to present the proofs just for circle graphs to make it more simple.

4.4 The lower bounds

The aim of this section is to prove the following two theorems (Theorem 4.4.1 and Theorem 4.4.2). They were originally proven by Kostochka in [42]. His paper is in Russian, it was not reproduced in English and thus almost no one knows the proof. That is why we decided to reproduce the proofs in English.[4]

Theorem 4.4.1. *There is a circle graph G with $\alpha(G) \leq k$ and $\omega(G) \leq k$ having $\Omega(k^2 \log k)$ vertices.*

We use the simple bound $\chi(G) \geq |V(G)|/\alpha(G)$ to get the following theorem.

Theorem 4.4.2. *There is a circle graph G with $\chi(G) = \Omega(\omega \log \omega)$.*

Let C be a circle in the plane. Take $kt+2$ points P on the circle and denote them by numbers $0, \ldots, kt+1$ in the order as they appear on C. Later, we will be counting with the points P modulo $kt+2$.

By $[i, j]$ we denote the segment connecting points $i, j \in P$. Segment $[i, j]$ splits the circle C into two arcs γ_1, γ_2. Let l_1, resp. l_2, be the number of points of P on γ_1, resp. γ_2. Then define the length of the segment $[i, j]$ as $\min\{l_1, l_2\}$.

Consider the following set of segments (chords of C):

$$H_i(k,t) := \{\, [j, j+it+1] \mid j \in \{0, \ldots, kt+1\} \,\}$$
$$H(k,t) := \bigcup_{i=1}^{\lfloor k/2 \rfloor} H_i(k,t).$$

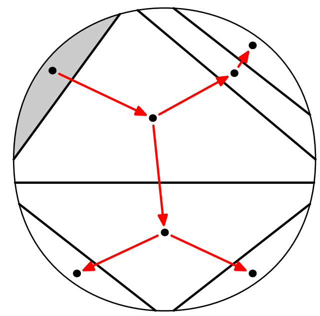
We say that $H_i(k,t)$ is the i-th layer of $H(k,t)$. All segments in one layer have the same length. For $i < k/2$, $|H_i(k,t)| = kt+2$, but for even k, $|H_{k/2}(k,t)| = (kt+2)/2$. The layers are disjoint. Hence $H(k,t)$ has at least $\lfloor (k-1)/2 \rfloor (kt+2)$ segments.

Let $G(k,t)$ be the intersection graph of $H(k,t)$. For a set R in the plane, ∂R denotes the boundary of R.

[4] The proofs might be different from original proofs, because I do not speak Russian enough to understand them.

Lemma 4.4.3. *Every subgraph $G' \subseteq G(k,t))$ has $\alpha(G') \leq k-1$*

Proof. Let $I \subseteq H(k,t)$ be an independent set in G'. All segments I cuts the circle into regions $\mathcal{R}$. Consider a graph G, whose vertices are the regions $\mathcal{R}$ and two regions are joint by an edge if they are neighbors. Graph G is a tree. Choose one leaf of the tree as a root and orient all edges away from root.



Every segment $s \in I$ cuts the circle C into two parts. Denote the part containing root region by $R(s)^+$ and the other part by $R(s)^-$.

Claim 4.4.4. *Boundary of every region $R \in \mathcal{R}$ contains at least $t+2$ points from P.*

First, let $R_r \in \mathcal{R}$ be the root region and r be the segment separating R_r. From the definition of $H(t,k)$ segment r has length at least t. Thus ∂R_r contains at least t points plus two endpoints of r.

Now consider $R \in \mathcal{R} \setminus \{R_r\}$. Let $S \subseteq I$ are the segments contained in ∂R. Let $r \in S$ be the segment separating R from the root region. Part $R(r)^-$ contains pt points for some natural number p. Regions $R(s)^-$ for $s \in S \setminus \{r\}$ are uncutting qt points from $R(r)^-$, for some natural number q. Since there are also the endpoints of $S \setminus \{r\}$ in $R(r)^-$ or because the minimum length of segment is t, we have $p > q$. Hence there remain at least t points of P in ∂R plus the endpoints of r.

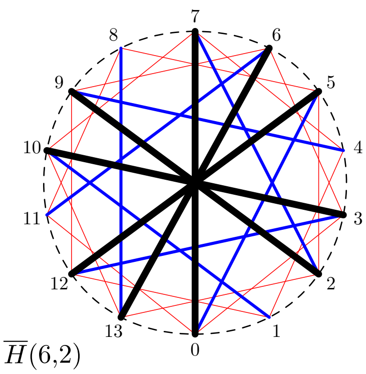
Let n be the number of regions in $\mathcal{R}$. Now we show that the union of ∂R for $R \in \mathcal{R}$ contains at least $nt+2$ points of P. We start with the root region, which has $t+2$ points of P, and we are adding neighboring regions one by one. Always when we add a neighboring region of already added regions, we add at least t new points of P. (Two points, endpoints of separating segment, were already counted). P has $kt+2$ points, thus $n \leq k$. Therefore there are at most $k-1$ cutting segments. □

We like the property $\alpha(G') \leq k$ for every subgraph $G' \subseteq G(k,t)$. On the other hand $G(k,t)$ has a large maximum clique. Thus we want to find a subgraph of $G(k,t)$, which has the size of maximum clique $O(k+t)$.

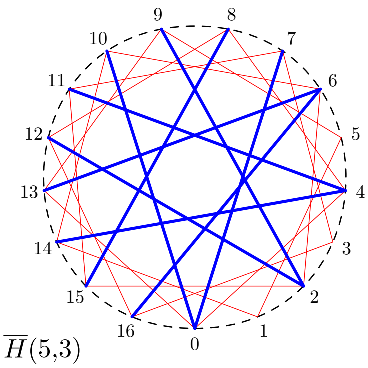
$$\overline{H}_i(k,t) := \{\, [j, j+it+1] \,|\, j \in \{0, i, 2i, \ldots, \lfloor (kt+1)/i \rfloor i\} \,\}$$

$$\overline{H}(k,t) := \bigcup_{i=1}^{\lfloor k/2 \rfloor} \overline{H}_i(k,t)$$

Look at the examples on the next figures.



$\overline{H}(6,2)$



$\overline{H}(5,3)$

In each layer $\overline{H}_i(k,t)$, we took only every i-th segment of $H_i(k,t)$. Every layer $\overline{H}_i(k,t)$ with $i < k/2$ contains $\lfloor (kt+1)/i \rfloor + 1\}$ segments. The last layer $\overline{H}_{k/2}(k,t)$ for k even contains a little less than $\lfloor (kt+1)/(k/2) \rfloor + 1$ segments. For simplicity, we just estimate the number of segments in $\overline{H}_i(k,t)$.

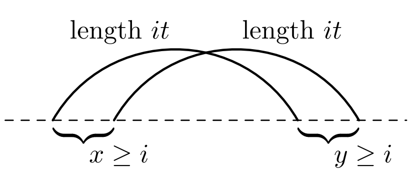
$\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \lfloor (kt+1)/i \rfloor + 1 \leq |\overline{H}(k,t)| \leq \sum_{i=1}^{\lfloor k/2 \rfloor} \lfloor (kt+1)/i \rfloor + 1$. Therefore, there are constants c_1 and c_2 such that $c_1 kt \log k \leq |\overline{H}(k,t)| \leq c_2 kt \log k$.

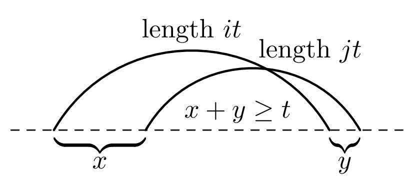
Denote the intersection graph of $\overline{H}(k,t)$ by $\overline{G}(k,t)$.

Lemma 4.4.5. $\omega(\overline{G}(k,t)) \leq 3k + 2t - 1$

Proof. In this proof, we will draw the segments of the circle graph in unwrapped representation.[5]

First, we make an easy observation. Let r and s be two crossing segments. Let $a < b < c < d$ be their endpoints on the circle C. Define $x := b - a$ and $y := d - c$. If r and s has the same length it, for some i, then both $x \geq i$ and $y \geq i$. It is because $\overline{H}_i(k,t)$ contains only every i-th segment of $H_i(k,t)$. If r and s have different lengths, then $x + y \geq t$, because the difference between lengths is at least t. Note that r, s can share an endpoint.



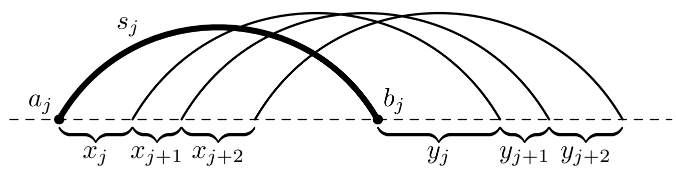


Now we are ready to start the proof. Let $Q \subseteq \overline{H}(k,t)$ be a clique. Let $s_1, s_2, \ldots, s_d$ be the segments of Q in the order as they appear from left to right. For $i \in \{1, 2, \ldots, d\}$ let a_i and b_i be the left and right endpoint of s_i,

[5]Imagine that the circle with segments is from a rubber. Cut it between points 0 and $kt+1$ and unwrap the circle to a line. All points $0, \ldots, kt+1$ will lie on a line and segments will become "arcs" above the line.

respectively. Necessarily $a_1 < a_2 < \cdots < a_d < b_1 < b_2 < \cdots < b_d$. Further define $x_i := a_{i+1} - a_i$ and $y_i := b_{i+1} - b_i$ for $i \in \{1, \ldots, d-1\}$.

Consider s_j, the shortest segment in Q. Remind that s_j splits the supporting circle into two arcs. The length of s_j is the number of points from P on the arc, which contains less points of P. We can without loss of generality assume that the point 0 is not on the shorter arc. Otherwise we can renumber the points P so that the point 0 is in other half-plane determined by s_j. Thus the length of s_j is $mt = b_j - a_j$ for some $m \leq \lfloor k/2 \rfloor$.



Now we are going to prove that there are not too many segments of Q which lie to the right of a_j. We can get the bound on the segments of Q lying to the left of b_j from the symmetry.

Define *u-length*[6] of s_i as $b_i - a_i$. There are two types of pairs (s_i, s_{i+1}), for $i \in \{j, \ldots, d-1\}$. Either s_i and s_{i+1} has the same u-length, or not.

There are at most t pairs of the same u-length: From the observation and because s_j is the shortest segment in Q (with length mt), each such a pair (s_i, s_{i+1}) has $x_i \geq m$. Since there are mt points under s_j, and every s_i for $i \in \{j+1, \ldots, d\}$ has a left endpoint a_i under s_j, there are at most $mt/m = t$ pairs of the same u-length.

Every s_i for $i \in \{j+1, \ldots, d\}$ has a left endpoint a_i under s_j. Thus u-length of s_d is at least $b_{d-1} - b_j$, but at most $(k-1)t$, what is the maximum u-length of a segment in $H(k,t)$. Therefore we get $X = \sum_{i=j}^{d} x_i \leq mt$ and $Y = \sum_{i=j}^{d-1} y_i \leq (k-1)t$. Each pair of different u-lengths contribute to $X + Y$ by at least t. Hence there are at most $(m+k-1)t/t \leq m+k-1$ pairs of different u-lengths.

Therefore there are at most $t+m+k-1$ pairs (s_i, s_{i+1}) for $i \in \{j, \ldots, d-1\}$. From the symmetry, the same bound holds for pairs (s_i, s_{i+1}), for $i \in \{1, \ldots, j-1\}$. Thus there are at most $2t + 2m + 2(k-1) + 1 \leq 2t + 3k - 1$ segments in clique Q.

Moreover we have $m < \lfloor k/2 \rfloor$ otherwise all segments has the same maximum length and there are no pairs different lengths. □

Proof of theorem 4.4.1. Set $\overline{G} := \overline{H}(k,k)$. Because $\overline{G}$ is a subgraph of $G(k,k)$, we have by Lemma 4.4.3 that $\alpha(\overline{G}) \leq k$. By Lemma 4.4.5 is $\omega(\overline{G}) \leq 5k$. Graph $\overline{G}$ has at least $\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} \lfloor (k^2+1)/i \rfloor + 1 \geq c \cdot k^2 \log k$ vertices for suitable constant c. □

[6] U-length of segment s is the number of points of P below s in unwrapped representation. It depends on the numbering of points P.

4.5 Size of circle graphs with $\omega \leq k$ and $\alpha \leq l$

In this section we will use the representation of circle graphs by chords of a circle (not the unwrapped representation as in the previous section). Chords are segments in the plane and thus we will often speak just about segments.

The goal is to determine or estimate the function

$$c_{k,l} = \max\{\, |V(G)| \,:\, G \in \textsc{cir} \;\&\; \omega(G) \leq k \;\&\; \alpha(G) \leq l \,\}.$$

Value $c_{k,l}$ is the maximum number of vertices of a circle graph G with $\omega(G) \leq k$ and $\alpha(G) \leq l$.

We show the upper bounds $c_{k,l} \leq \frac{3}{2}kl\log(l)$ and $c_{k,l} \leq 16(2^k - k - 1)l$. First bound is linear in k (for fixed l) and the second is linear in l (for fixed k). But it is interesting that there is no upper bound linear in both k and l. Moreover there is a lower bound by Kostochka [42], that shows $c_{m,m} = \Omega(m^2 \log m)$. For details see Theorem 4.4.1.

Later in this section we show almost tight constructions of lower bounds for small values of k, l.

4.5.1 General upper bounds

Permutation graph is an intersection graph of segments between two parallel lines. Hence it is a special case of circle graph.

Lemma 4.5.1. *Let G be a permutation graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. Then $|V(G)| \leq kl$.*

Proof. Assume that the supporting parallel lines are vertical. Two segments are independent if and only if one lies above the other. The binary relation "segment s lies above segment r" induce a partial order on the segments. Chains are sets of pairwise independent segments and antichains are sets of pairwise crossing segments. Hence the lemma follows from Dilworth's theorem. □

Let G be a circle graph. For a segment $s \in V(G)$ let $V_{cr}(s) = \{r \in V(G) \mid s \cap r \neq \emptyset\}$ be the set of segments crossing s.

Corollary 4.5.2. *Let G be a circle graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. Then $|V_{cr}(s)| \leq (k-1)l$ for every segment $s \in V(G)$.*

Proof. Segments $V_{cr}(s)$ determine a permutation graph. Moreover they contain at most $k-1$ pairwise crossing segments, because every clique in $V_{cr}(s)$ can be extended by segment s. Hence by Lemma 4.5.1 we get $|V_{cr}(s)| \leq (k-1)l$. □

Let G be the circle graph with $\omega(G) \leq k$ and $\alpha(G) \leq l$. We simply denote $\alpha = \alpha(G)$ and $\omega = \omega(G)$. The segment $s \in V(G)$ has *order* i if and only if there are exactly i independent segments of G in one half-plane determined by s and at least i independent segments of G in the other half-plane. In other words, s has order at least i, if it separates two independent sets of G, each of size at least i. Let $V_i \subseteq V$ be the set of segments of order i. Denote the graph $G|V_i$ induced by these segments by G_i and the size of independent set in G_i by α_i.

Let us define the *shell* $S(s)$ of a segment s. Segment s divides the supporting circle into two arcs. Shell of s is that of the two arcs, which induces smaller set of pairwise independent segments in G.

Observation 4.5.3. *There are no segments of order $\lceil \alpha/2 \rceil$ in a circle graph.*

Proof. Assume for a contradiction that there is a segment s of higher order. There will be at least $\lceil \alpha/2 \rceil$ independent segments in each half-plane determined by s so we would have $2\lceil \alpha/2 \rceil + 1 \geq \alpha + 1$ independent segments in G. □

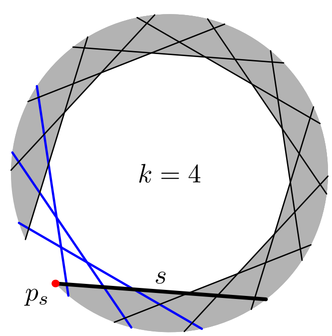
Observation 4.5.4. $\alpha(G_i) \leq \lfloor \alpha(G)/(i+1) \rfloor$.

Proof. Take independent segments $V_{\alpha_i} \subseteq V_i$. Shell of every $s \in V_{\alpha_i}$ contains no segment of V_i, but it contains i independent segments of G. All together we have $(i+1) \cdot |V_{\alpha_i}|$ independent segments of G. That gives $\alpha(G_i) \leq \alpha(G)/(i+1)$. Because $\alpha(G_i)$ is an integer, we take the lower integer part. □

Observation 4.5.5. *If $\alpha_i = 1$ then $|V_i| \leq k$ otherwise $|V_i| \leq k(\alpha_i + 1) - 1$.*

Proof. If $\alpha_i = 1$ then all segments V_i must cross each other. Thus $|V_i| \leq k$.

Assume that $\alpha_i > 1$. Let p_s be an endpoint of some segment $s \in V_i$. Cut the supporting circle at point p_s and unwrap it to a segment. (As we did with the unwrapped representation in section 4.2). Split V_i into two sets X, Y. Set $X \subseteq V_i$ contains the segments, whose shell strictly contains p_s. Set $Y \subseteq V_i$ contains the other segments. Segment s is in Y.



There are at most $k - 1$ segments in X, because all of them have to cross each other and segment s cross them too.

Two segments of Y can either lie next to each other (their shells are disjoint) or they cross. They cannot lie one under the other. We introduce a partial ordering on disjoint segments in Y. One segment is smaller than the other if and only if the segment lies to the left of the other.

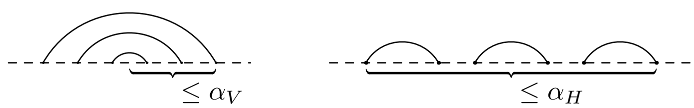
By Dilworth theorem there are at most $k\alpha_i$ segments in Y. Altogether there are at most $k\alpha_i + k - 1 = k(\alpha_i + 1) - 1$ segments in V_i. □

Remark: Observation 4.5.5 can be improved by one, if the independent set of G contains at least one segment of other order.

Remark: Assume that we have unwrapped representation. Observation 4.5.5 and corollary 4.5.2 bring two partial orderings on disjoint segments:

$u \prec_V v \iff u$ lies below v.

$u \prec_H v \iff u$ lies to the left of v.



Let α_V and α_V be the maximum length of a chain in orderings $\prec_V$, $\prec_H$ (for given circle graph G). We suggest to split the constrain $\alpha \leq l$ into two constrains $\alpha_V \leq l$ and $\alpha_H \leq l$. It might be interesting to investigate

$$c_{k,l_V,l_H} = \{\, |V(G)| \,:\, G \in \textsc{cir} \,\&\, \omega(G) \leq k \,\&\, \alpha_V(G) \leq l_V \,\&\, \alpha_H(G) \leq l_H \,\}.$$

Clearly $c_{k,l} \leq c_{k,l,l}$.

An easy upper bound from Dilworth theorem is $c_{k,l_V,l_H} \leq k \cdot l_V \cdot l_H$. The idea of proof is similar to the proof of Theorem 2.4.5. There are at most l_V classes of segments incomparable by $\prec_V$ and similarly at most l_H classes of segments incomparable by $\prec_H$. Thus there are at most $l_V \cdot l_H$ classes of segments incomparable by $\prec_V$ or $\prec_H$. Each such a class contains pairwise crossing segments and thus its size is at most k.

That immediately shows $c_{k,l} \leq kl^2$.

Theorem 4.5.6. *$c_{k,l} \leq \frac{3}{2}kl\log(l)$ for $l \geq 2$.*

Proof. Proof of this theorem is adapted from Kratochvíl and Kostochka [43].

Remind $\alpha \leq l$. By the definition of the order of a segment and by Observation 4.5.3 we have $V = V_0 \cup V_1 \cup \cdots \cup V_{\lceil \alpha/2 \rceil - 1}$. By Observations 4.5.4 and 4.5.5, the size of V_i is at most $k(\lfloor \alpha/(i+1) \rfloor + 1) - 1$ for $i \in \{0, \ldots, \lceil \alpha/2 \rceil - 1\}$. Moreover if α is odd then $\alpha_{\lceil \alpha/2 \rceil - 1} = 1$ and $|V_{\lceil \alpha/2 \rceil - 1}| \leq k$. Hence altogether we have

$$|V| \leq \sum_{i=0}^{\alpha/2-1} k(\left\lfloor \frac{\alpha}{i+1} \right\rfloor + 1) - 1$$

for α even. For α odd we have something slightly better:

$$|V| \leq k + \sum_{i=0}^{\lceil \alpha/2 \rceil - 2} k(\left\lfloor \frac{\alpha}{i+1} \right\rfloor + 1) - 1$$

After a little more calculations we can get $|V| \leq k \cdot (\alpha \log(\alpha) + \alpha/2) - \alpha/2 \leq \frac{3}{2}k\alpha\log(\alpha)$ for $\alpha \geq 2$. □

For small values of k the recurrence from the proof of theorem 4.5.6 gives

$c_{k,1} \le k$	$c_{k,5} \le 10k - 2$
$c_{k,2} \le 3k - 1$	$c_{k,6} \le 14k - 3$
$c_{k,3} \le 5k - 1$	$c_{k,7} \le 16k - 3$
$c_{k,4} \le 8k - 2$	$c_{k,8} \le 20k - 4$

Theorem 4.5.7. $c_{k,l} \le 16(2^k - k - 1)l$.

Proof. We proved in Theorem 4.3.1 that $\chi(G) \le 16 \cdot 2^{\omega(G)} - 16\omega(G) - 16$ in circle graphs. Let G be the circle graph with $\omega(G) \le k$ and $\alpha(G) \le l$. Color the segments, which correspond to the vertices of the graph G. Each color class has at most l segments and we have at most $16(2^k - k - 1)$ colors. □

4.5.2 Better upper bounds for small l

Lemma 4.5.8. $c_{k,1} \le k$ *and* $c_{k,2} \le 3k - 1$.

Proof. If $\alpha(G) = 1$ then G is a clique and has at most k vertices.

Now consider $\alpha(G) = 2$. Take some segment $s \in V(G)$ and split $V(G)$ into two sets: segments S_1, which cross s and segments S_2, which do not. Set S_2 contains no two disjoint segments. Otherwise there are three disjoint segments (the two disjoint and s). Hence $|S_2| \le k$. By Lemma 4.5.2 we have $|S_1| \le 2k - 2$. □

Lemma 4.5.9. $c_{k,3} \le 5k - 2$.

Proof. Either there is a segment s of order one or not.

In the first case if s is a segment of order one, then there are at most $c_{1,k} \le k$ segments in each half-plane. By lemma 4.5.2 the segment s is crossed by at most $3(k-1)$ other segments. Altogether there are at most $5k - 2$ segments.

In the second case all segments are of order zero. Thus by Observation 4.5.5 there are at most $4k - 1$ segments. □

Lemma 4.5.10. $c_{k,4} \le 8k - 4$.

Proof. If there exists a segment s of order one then it has at most $c_{1,k} \le k$ segments in one half-plane and at most $c_{2,k} \le 3k-1$ in the other half-plane. By lemma 4.5.2 segment s is crossed by at most $4(k-1)$ other segments. Altogether there are at most $8k - 4$ segments.

If there is no segment of order one then every segment has order zero. By Observation 4.5.5 there are at most $5k - 1$ segments. □

We cannot improve the additive constant in $c_{k,5}$ and we don't know any better upper bound than $10k - 2$, which follows from Theorem 4.5.6.

Lemma 4.5.11. $c_{k,6} \le 14k - 8$.

Proof. If there exists a segment s of order two then it has at most $c_{2,k} \le 3k-1$ segments in one half-plane and at most $c_{3,k} \le 5k-2$ segments in the other half-plane. By lemma 4.5.2 segment s is crossed by at most $6(k-1)$ other segments. Altogether there are at most $14k - 8$ segments.

If there is no segment of order two then every segment has order zero or one. By Observations 4.5.4 we have $\alpha_0 = 6$ and $\alpha_1 = 3$ and therefore by Observation 4.5.5 we get $|V_0| \leq 7k - 1$ and $|V_1| \leq 4k - 1$. So in this case there are at most $11k - 2$ segments. □

4.5.3 Lower bounds

In this section we construct lower bounds for $c_{k,l}$. We only describe the construction. The proof that the constructed set of segments contains no more than k pairwise crossing segments or no more then l pairwise disjoint segments is a little technical verification and we omit it.

Every construction starts with points on the circle. We split the points into several sets and for each set we call procedure `LAYER`, which constructs the segments between given points.

procedure `LAYER`(P, k)**:**

The procedure gets $2m$-point set P and k and produces a set of segments with endpoints in P and with at most k pairwise crossing segments.

Put $n = 2m$ and denote the points by numbers from zero to $n - 1$ in the order as they appear on the circle. For every even p connect point p with point $p + 2k - 1 \bmod n$ by an segment. The constructed set of segments contains at most k pairwise crossing segments and at most $\lfloor m/k \rfloor$ independent segments. For illustration see figures in 4.7.1. Let us note that each call of `LAYER` constructs the segments of the same order.

Lemma 4.5.12. $c_{k,2} \geq 3k - 1$.

Take $n = 2 \cdot (3k - 1)$ points on a circle and apply procedure `LAYER`. See 4.7.1 for examples.

Lemma 4.5.13. $c_{k,3} \geq 5k - 2$.

Take $n = 2 \cdot (4k - 2)$ points on a circle and apply `LAYER`. Then add k halving segment so that they do not create a clique of size $k + 1$. See 4.7.2 for examples.

Lemma 4.5.14. *$c_{k,4} \geq 7.5k - 3$ for k even and $c_{k,4} \geq \lfloor 7.5k \rfloor - 4$ for k odd.*

If k is even then take $n = 3 \cdot (5k - 2)$. If k is odd, take $n = 3 \cdot (5k - 3)$. Take set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ of n points on a circle. Let $P_0 = \{ p_i \in P \,|\, i \mod 3 = 0, 1 \}$ and $P_1 = P \setminus P_0$. Apply `LAYER` on P_i to get segments of order i for $i = 0, 1$. See 4.7.3 for examples.

Lemma 4.5.15. *$c_{k,5} \geq 10k - 6$ for $k > 0$.*

Take set $P = \{p_0, p_1, \ldots, p_{n-1}\}$ of $n = 3 \cdot (6k - 2)$ points on a circle. Let $P_0 = \{ p_i \in P \,|\, i \mod 3 = 0, 1 \}$ and $P_1 = P \setminus P_0$. Apply `LAYER` on P_i to get segments of order i for $i = 0, 1$. At the end we add k segments of order two. They are halving segments such that they do not create a clique larger than k. See 4.7.4 for examples.

Note that for k odd, there is an asymmetric construction with $\geq 10k - 5$ segments. See 4.7.4 for examples.

4.5.4 Overview of values $c_{k,l}$ for small k, l

Here we summarize the bounds for $l \le 6$.

$$\begin{array}{ll} c_{k,1} = k & 7.5k - 3 \le c_{k,4} \le 8k - 2 \\ c_{k,2} = 3k - 1 & 10k - 6 \le c_{k,5} \le 10k - 2 \\ c_{k,3} = 5k - 2 & c_{k,6} \le 14k - 8 \end{array}$$

What abount $c_{2,l}$? Currently, the best upper bounds are $5l$ (Theorem 4.5.7 gives only $2^4(4-2-1)l = 16l$, but we can modify its proof and use the result of Ageev that $f(2) = 5$) and $2l \log l + l/2$. The simple lower bound is $\frac{19}{6}l$ for $l \bmod 6 = 0$ (place copies of the construction for $c_{2,6}$ next to each other).

From the upper and lower bounds in this section we get the range of values for small k, l. They are in following table. The values in brackets mean the truth found by a computer program, which uses brute force.

$c_{row,col}$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	5	8	12	15–18(15)	?–20(19)	?–29($\ge$ 22)
3	3	8	13	18–20(19)	24–28	?–34	?–45
4	4	11	18	27–28	35–38	?–48	?–61
5	5	14	23	33–36	44–48	?–62	?–77
6	6	17	28	42–44	55–58	?–76	?–93
7	7	20	33	48–52	64–68	?–90	?–109

Note: To improve the lower bound on Ramsey-type question on segments (Theorem 1.2.3 in the first chapter), we would need $c_{5,5} \ge 46$, or $c_{6,6} \ge 71$. That is the hope for a tiny improvement.

4.6 Open Problems

Open Problem: In Section 4.3 we presented a problem of coloring circle graphs. Theorem 4.3.1 shows that every circle graph with $\omega = \omega(G)$ has

$$\chi(G) \le 16(2^{\omega} - \omega - 1).$$

On the other hand there are circle graphs, which require $c \cdot \omega \log \omega$ colors, for some constant c (Theorem 4.4.2). Still, there is an exponential gap. Can you improve these bounds?

Ageev [4] showed that $\chi(G) \le 5$ for circle graphs with $\omega(G) = 2$. What can you show for $\chi(G)$ when $\omega(G) = 3$? Theorem 4.3.1 gives only $\chi(G) \le 64$.

Open Problem: In Section 4.4 we presented the proof of the lower bound on $f(k)$. May be it is possible to modify the construction and get a better bound. Interesting questions are:

- How much is $\chi(\overline{G}(k,t))$? We showed just the simple lower bound $\chi(G) \geq |V(G)|/\alpha(G) \geq c \cdot kt$.
- The graph $G(k,t)$ has $\chi(G(k,t)) \geq |V(G)|/\alpha(G) \geq c \cdot k^2 t/k = c \cdot kt$. Graph $\overline{G}(k,t) \subseteq G(k,t))$ is a subgraph with $\omega(\overline{G}(k,t)) \leq O(k+t)$. Is it possible to choose a better subgraph? We would like to find a subgraph G' with $\omega(G') = O(k + \log t)$ or with $\omega(G') = O(k + \sqrt{t})$, but on the other hand is must have sufficiently many vertices.

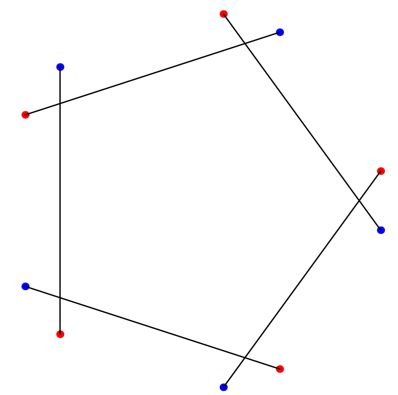
Open Problem: Section 4.5 brings some open problems too. We have $7.5k - const \leq c_{k,4} \leq 8k - const$. The gap brings us to the following question.

By Observation 4.5.5 the number of segments in first two layers is $|V_0 \cup V_1| \leq (\alpha+1)k - 1 + (\lfloor \alpha/2 + 1 \rfloor)k - 1 \leq (3\alpha/2 + 2)k - 2$. Can you show that $|V_0 \cup V_1| \leq (3\alpha/2 + 3/2)k - 2$?

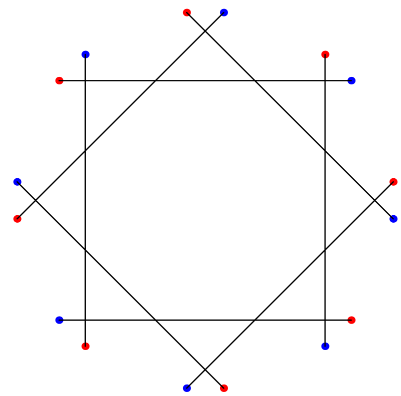
Open Problem: We had shown that $c_{2,l}$ is between $\frac{19}{6}l$ (for l mod $6 = 0$) and $min\{\, 5l,\, 2l \log l + l/2 \,\}$. What can you show for $c_{3,l}$? The upper bound from Theorem 4.5.7 gives only $2^4(8-3-1)l = 64l$.

4.7 Attachments

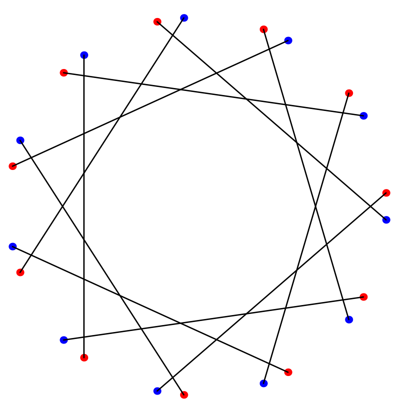
4.7.1 Lower bound for $c_{k,2}$



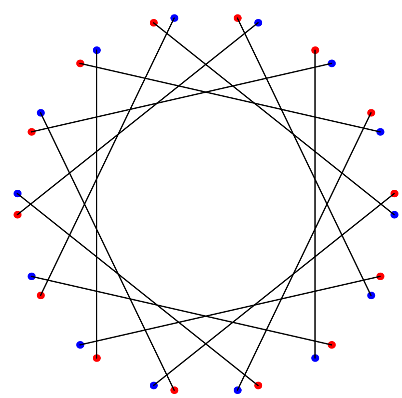
n=5 $k = 2,\ l = 2$



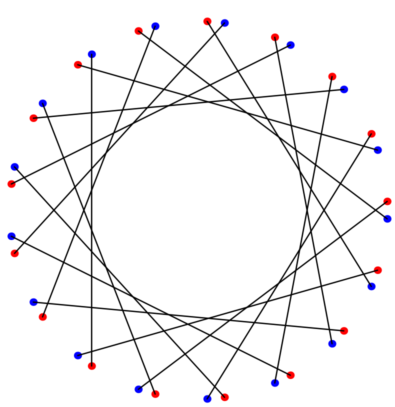
n=8 $k = 3,\ l = 2$



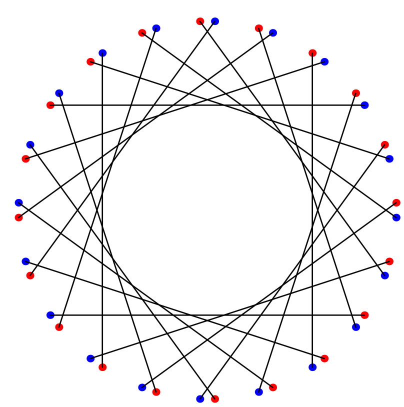
n=11 $k = 4,\ l = 2$



n=14 $k = 5,\ l = 2$

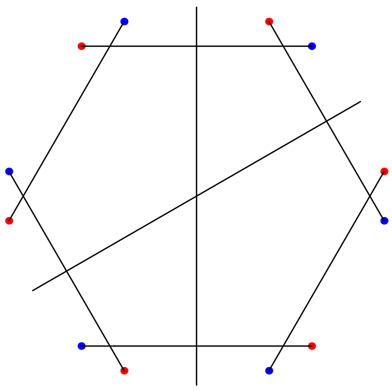


n=17 $k = 6,\ l = 2$

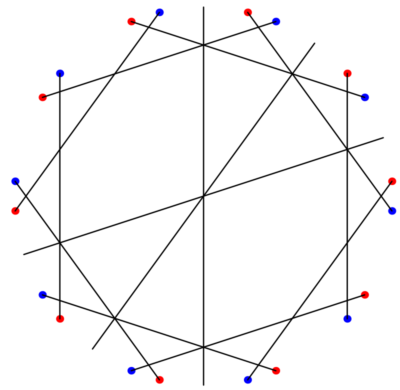


n=20 $k = 7,\ l = 2$

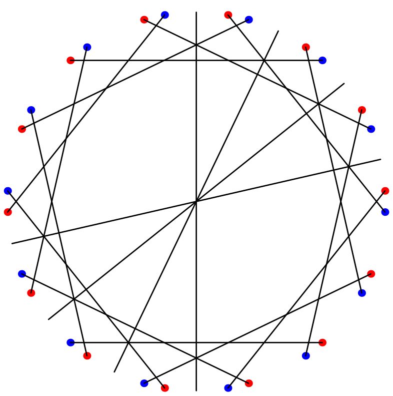
4.7.2 Lower bound for $c_{k,3}$



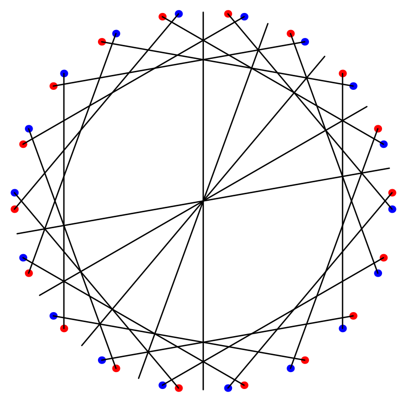
n=8 $k = 2$, $l = 3$



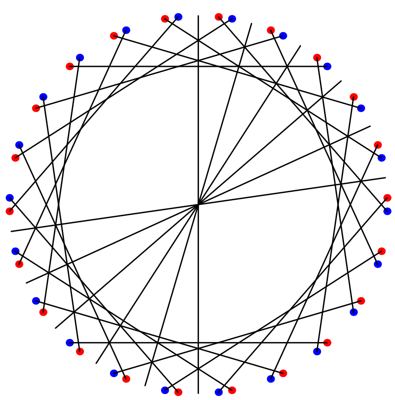
n=13 $k = 3$, $l = 3$



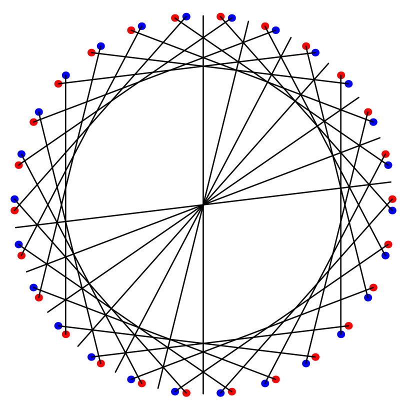
n=18 $k = 4$, $l = 3$



n=23 $k = 5$, $l = 3$

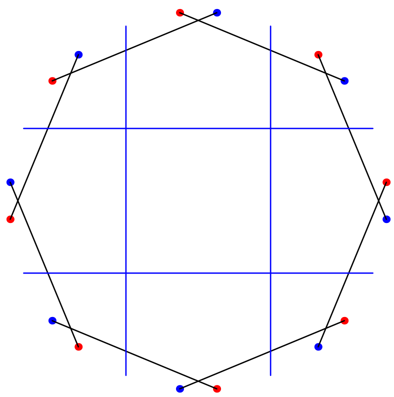


n=28 $k = 6$, $l = 3$

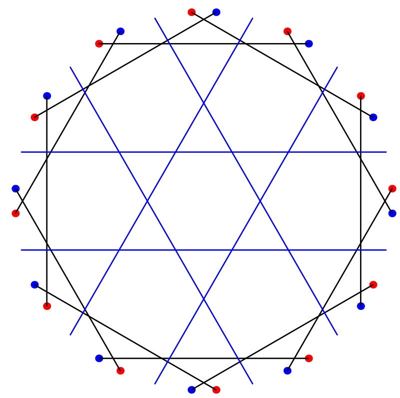


n=33 $k = 7$, $l = 3$

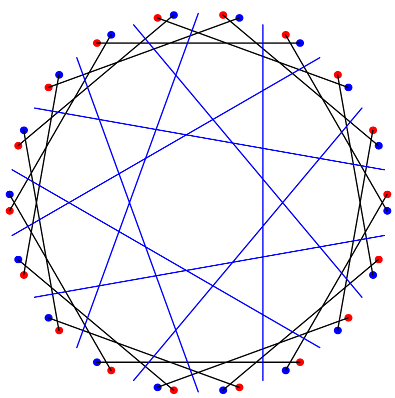
4.7.3 Lower bound for $c_{k,4}$



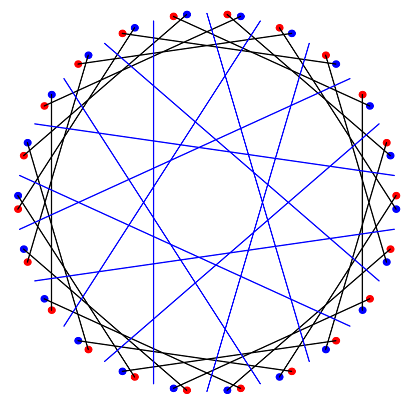
n=12 $k = 2$, $l = 4$



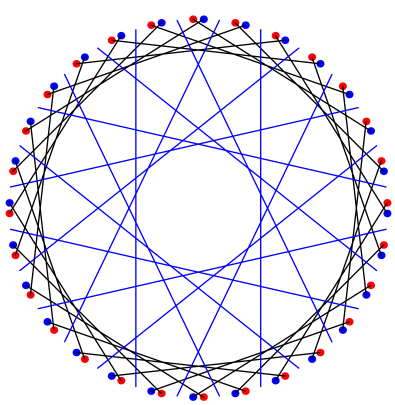
n=18 $k = 3$, $l = 4$



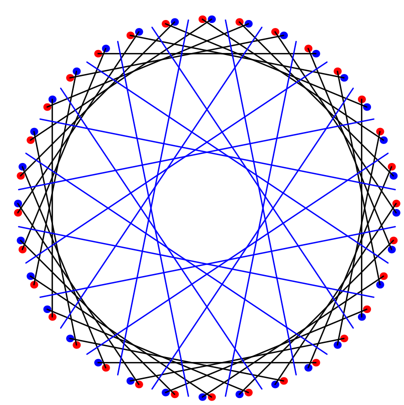
n=27 $k = 4$, $l = 4$



n=33 $k = 5$, $l = 4$



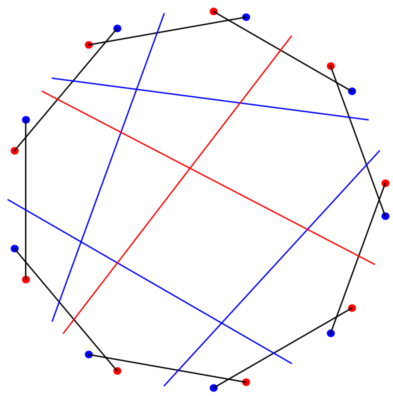
n=42 $k = 6$, $l = 4$



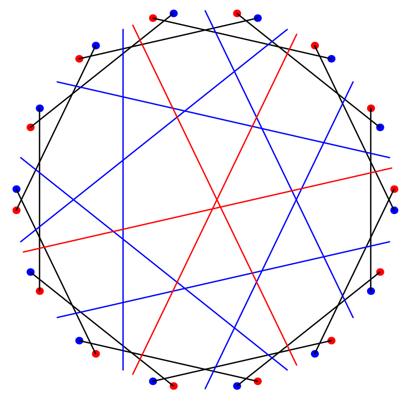
n=48 $k = 7$, $l = 4$

4.7.4 Lower bound for $c_{k,5}$

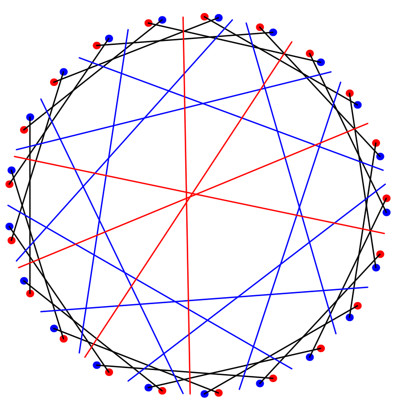
There is also symmetric construction for k even but it has one less segment.



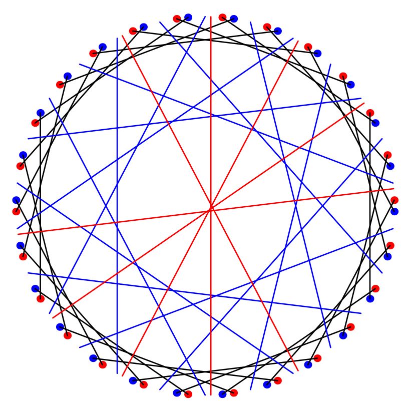
n=15 $k = 2$, $l = 5$



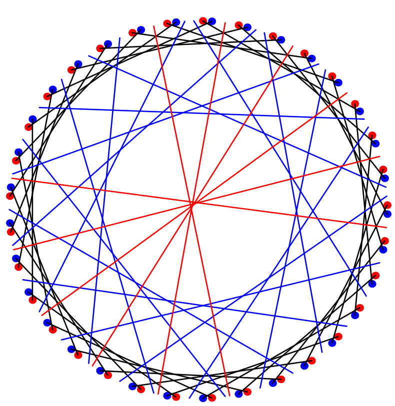
n=24 $k = 3$, $l = 5$



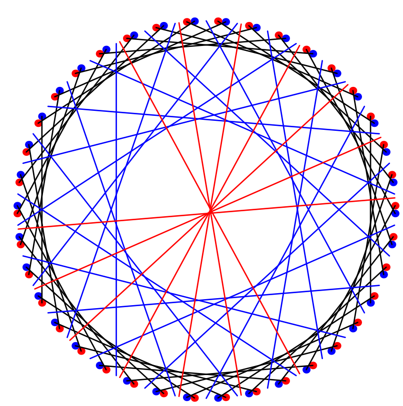
n=35 $k = 4$, $l = 5$



n=44 $k = 5$, $l = 5$

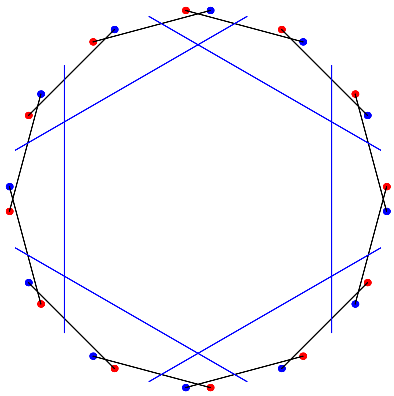


n=55 $k = 6$, $l = 5$

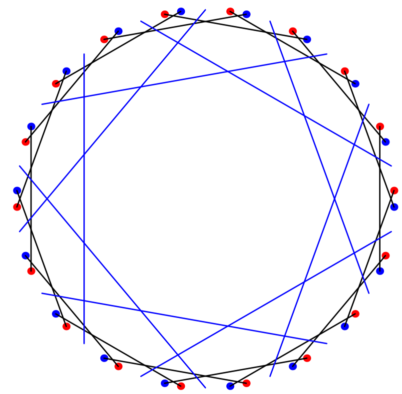


n=64 $k = 7$, $l = 5$

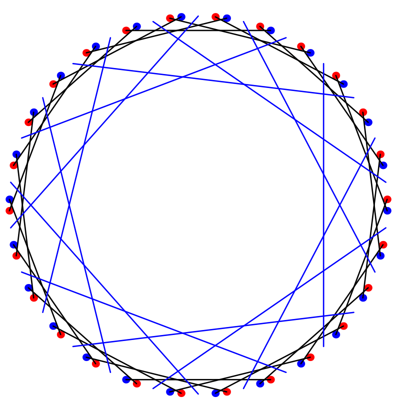
4.7.5 Lower bound for $c_{k,6}$



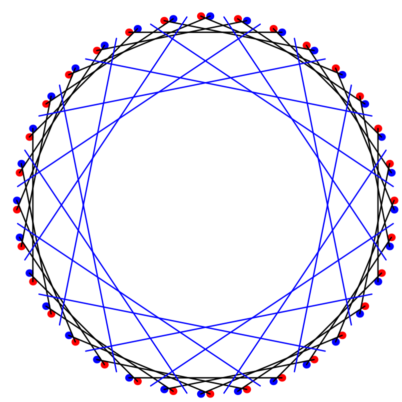
n=18 $k = 2$, $l = 6$



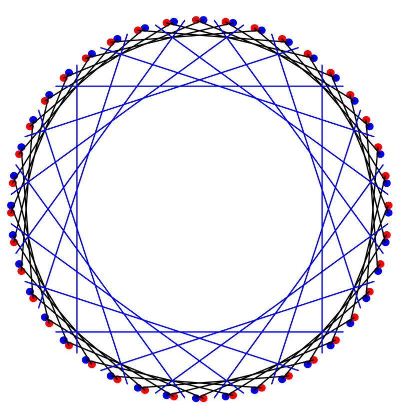
n=27 $k = 3$, $l = 6$



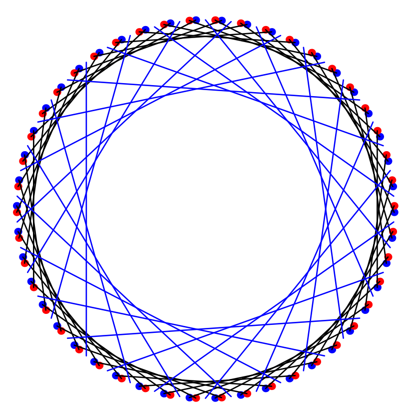
n=39 $k = 4$, $l = 6$



n=48 $k = 5$, $l = 6$



n=60 $k = 6$, $l = 6$



n=69 $k = 7$, $l = 6$

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