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Combinatorial identities in dual sequences

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Abstract

In this paper we derive a general combinatorial identity in terms of polynomials with dual sequences of coefficients. Moreover, combinatorial identities involving Bernoulli and Euler polynomials are deduced. Also, various other known identities are obtained as particular cases.

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1. Introduction

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers where $\mathbb{N} = \{0, 1, 2, \ldots\}$. We call the sequence $\{a_n^*\}_{n\in\mathbb{N}}$ given by

$$a_n^* = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i \tag{1.1}$$

the *dual sequence* of $\{a_n\}_{n\in\mathbb{N}}$. It is well known that $a_n^{**} = a_n$ for all $n \in \mathbb{N}$ (see, e.g. [2, pp. 192–193]). Those self-dual sequences are of particular interest and were investigated in [5]. The Bernoulli numbers $B_0, B_1, \ldots$ are given by $B_0 = 1$ and $\sum_{i=0}^{n}\binom{n+1}{i}B_i = 0\,(n = 1, 2, 3, \ldots)$; since $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k = 1, 2, \ldots$ the sequence $\{(-1)^n B_n\}_{n\in\mathbb{N}}$ is self-dual as observed in [5]. Like the definition of Bernoulli polynomials (see, e.g. [6]), we introduce

$$A_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i x^{n-i} \qquad \text{and} \qquad A_n^*(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i^* x^{n-i}. \tag{1.2}$$

Obviously $A_n(0) = (-1)^n a_n$, $A_n(1) = a_n^*$ and

$$A'_{n+1}(x) = \sum_{i=0}^{n} \binom{n+1}{i} (-1)^i a_i (n + 1 - i) x^{n-i} = (n + 1) A_n(x).$$

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In 1995 Kaneko [3] found the following new recursion formula for Bernoulli numbers:

$$\sum_{j=0}^{k}\binom{k+1}{j}(k+j+1)B_{k+j}=0 \qquad \text{for } k=1,2,\ldots.$$

By means of the Volkenborn integral, Momiyama [4] got the following symmetric extension: if $k,l\in\mathbb{N}$ and $k+l>0$, then

$$(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)B_{l+j}+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}(k+j+1)B_{k+j}=0.$$

Recently Wu et al. [7] proved further that for $k,l\in\mathbb{N}$ we have

$$\begin{aligned}(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)B_{l+j}(t)&+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}(k+j+1)B_{k+j}(-t)\\&=(-1)^k(k+l+2)(k+l+1)t^{k+l}.\end{aligned}\tag{1.3}$$

Motivated by the above work, we obtain the following general theorem.

Theorem 1.1. *Let $k,l\in\mathbb{N}$ and $x+y+z=1$. Then*

$$\begin{aligned}(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{A_{l+j+1}(y)}{l+j+1}&+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{A^*_{k+j+1}(z)}{k+j+1}\\&=\frac{a_0(-x)^{k+l+1}}{(k+l+1)\binom{k+l}{k}}.\end{aligned}\tag{1.4}$$

Also,

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}A_{l+j}(y)=(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}A^*_{k+j}(z)\tag{1.5}$$

and

$$\begin{aligned}(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}x^{k-j+1}(l+j+1)A_{l+j}(y)&\\+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}x^{l-j+1}(k+j+1)A^*_{k+j}(z)&\\=(k+l+2)((-1)^{k+1}A_{k+l+1}(y)+(-1)^{l+1}A^*_{k+l+1}(z)).&\end{aligned}\tag{1.6}$$

Remark 1.1. (1.5) in the case $l=0$ yields that

$$\begin{aligned}\sum_{j=0}^{k}\binom{k}{j}x^{k-j}A_j(y)=(-1)^kA^*_k(1-x-y)&=\sum_{j=0}^{k}\binom{k}{j}0^{k-j}A_j(x+y)\\&=A_k(x+y).\end{aligned}$$

Corollary 1.1. *Let k be a nonnegative integer. If $a_n^* = a_n$ for all $n \in \mathbb{N}$, then*

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j (2t-1)^{k-j} \frac{A_{k+j+1}(t)}{k+j+1} = \frac{a_0(2t-1)^{2k+1}}{2(2k+1)\binom{2k}{k}} \tag{1.7}$$

and

$$\sum_{j=0}^{k} \binom{k+1}{j} (k+j+1)(1-2t)^{k-j+1} A_{k+j}(t) = -2(k+1)A_{2k+1}(t). \tag{1.8}$$

If $a_n^* = -a_n$ for all $n \in \mathbb{N}$, then

$$\sum_{j=0}^{k} \binom{k}{j} (1-2t)^{k-j} A_{k+j}(t) = 0 \tag{1.9}$$

and in particular

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j a_{k+j} = 0.$$

Proof. To obtain (1.7)–(1.9), we simply take $l = k$, $x = 1 - 2t$ and $y = z = t$ in (1.4)–(1.6). (1.9) in the case $t = 0$ gives the last identity. □

Example 1.1. Let $u_0 = 0$, $u_1 = 1$, and $u_{s+1} = au_s + bu_{s-1}$ for $s = 1, 2, \ldots$, where a and b are complex numbers with $a^2 + 4b \neq 0$. It is well known that $u_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ for all $n \in \mathbb{N}$, where α and β are the two roots of the equation $x^2 - ax - b = 0$. Observe that

$$u_n^* = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \frac{\alpha^i - \beta^i}{\alpha - \beta} = \frac{(1-\alpha)^n - (1-\beta)^n}{\alpha - \beta}$$

(and hence $u_n^* = -u_n$ if $a = 1$). Also,

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i u_i^* (1-a)^{n-i} \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} \frac{(\alpha-1)^i - (\beta-1)^i}{\alpha - \beta} (1-a)^{n-i} \\
&\quad = \frac{(\alpha-a)^n - (\beta-a)^n}{\alpha-\beta} = (-1)^n \frac{\beta^n - \alpha^n}{\alpha - \beta} = (-1)^{n-1} u_n.
\end{aligned}$$

Let $k, l \in \mathbb{N}$. Applying (1.4)–(1.6) with $x = a$, $y = 0$ and $z = 1 - a$, we obtain the following identities:

$$\sum_{j=0}^{k} \binom{k}{j} (-1)^j a^{k-j} \frac{u_{l+j+1}}{l+j+1} = \sum_{j=0}^{l} \binom{l}{j} (-1)^j a^{l-j} \frac{u_{k+j+1}}{k+j+1}, \tag{1.10}$$

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j a^{k-j}u_{l+j}=-\sum_{j=0}^{l}\binom{l}{j}(-1)^j a^{l-j}u_{k+j}, \tag{1.11}$$

$$\sum_{j=0}^{k}\binom{k+1}{j}(-1)^j a^{k-j+1}(l+j+1)u_{l+j}-\sum_{j=0}^{l}\binom{l+1}{j}(-1)^j a^{l-j+1} \times (k+j+1)u_{k+j}=((-1)^k-(-1)^l)(k+l+2)u_{k+l+1}.$$

In the case $l=k$, (1.11) yields the following recursion formula:

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j a^{k-j}u_{k+j}=0.$$

Theorem 1.1 has the following important application.

Theorem 1.2. *Let k and l be nonnegative integers.*

(i) *If $x+y+z=0$ then*

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{y^{l+j+1}}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{z^{k+j+1}}{k+j+1} = \frac{(-x)^{k+l+1}}{(k+l+1)\binom{k+l}{k}}. \tag{1.12}$$

In particular,

$$\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{2^j(k+j+1)}=\frac{2^k}{(2k+1)\binom{2k}{k}}. \tag{1.13}$$

(ii) *For $n\in\mathbb{N}$ let $B_n(x)$ denote the Bernoulli polynomial of degree n. Suppose that $x+y+z=1$. Then*

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{B_{l+j+1}(y)}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{B_{k+j+1}(z)}{k+j+1} = \frac{(-x)^{k+l+1}}{(k+l+1)\binom{k+l}{k}}. \tag{1.14}$$

Also,

$$(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}B_{l+j}(y)=(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}B_{k+j}(z) \tag{1.15}$$

and

$$(-1)^k\sum_{j=0}^{k}\binom{k+1}{j}x^{k-j+1}(l+j+1)B_{l+j}(y)$$

$$+(-1)^l\sum_{j=0}^{l}\binom{l+1}{j}x^{l-j+1}(k+j+1)B_{k+j}(z)$$
$$=(-1)^k(k+l+2)(B_{k+l+1}(x+y)-B_{k+l+1}(y)). \tag{1.16}$$

(iii) *The second part remains valid if we replace all the Bernoulli polynomials in* (1.14)–(1.16) *by corresponding Euler polynomials defined by* $2e^{xz}/(e^z+1)=\sum_{n=0}^{\infty}E_n(x)z^n/n!$.

Proof. (i) Let $a_0=1$ and $a_i=0$ for $i=1,2,3,\ldots$. For any $n\in\mathbb{N}$ we clearly have $a_n^*=1$, $A_n(t)=t^n$ and $A_n^*(t)=(t-1)^n$. If $x+y+z=0$, then $x+y+(1+z)=1$ and $A_n^*(1+z)=z^n$, therefore (1.12) follows from (1.4). When $l=k$, $x=-1$ and $y=z=1/2$, (1.12) yields (1.13).

(ii) Let $a_n=(-1)^nB_n$ for $n\in\mathbb{N}$. Then $A_n^*(x)=A_n(x)=B_n(x)$. Applying Theorem 1.1 we obtain the identities (1.14)–(1.16). (Note that $B_n(1-t)=(-1)^nB_n(t)$.)

(iii) By the definition of the Euler polynomials, $E_n(1-x)=(-1)^nE_n(x)$ and $E_n(x+y)=\sum_{i=0}^{n}\binom{n}{i}E_i(x)y^{n-i}$ for any $n\in\mathbb{N}$. Let $a_n=(-1)^nE_n(0)$ for $n=0,1,2,\ldots$. Then $a_n^*=\sum_{i=0}^{n}\binom{n}{i}E_i(0)=E_n(1)=a_n$ and $A_n(x)=A_n^*(x)=E_n(x)$. So we have part (iii) by Theorem 1.1.

The proof of Theorem 1.2 is now complete. □

Remark 1.2. In the case $x=1$, $y=t$ and $z=-t$, (1.16) turns out to be (1.3) since $B_n(1+t)=B_n(t)+nt^{n-1}$. (1.15) in the case $x=1$, and its analogue with respect to Euler polynomials were recently discovered in [7]. When $l=k$, $x=1-2t$ and $y=z=t$, (1.14) and (1.16) yield the following interesting identities:

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j(2t-1)^{k-j}\frac{B_{k+j+1}(t)}{k+j+1}=\frac{(2t-1)^{2k+1}}{2(2k+1)\binom{2k}{k}}, \tag{1.17}$$

$$\sum_{j=0}^{k}\binom{k+1}{j}(k+j+1)(1-2t)^{k-j+1}B_{k+j}(t)=-2(k+1)B_{2k+1}(t). \tag{1.18}$$

They remain valid if the Bernoulli polynomials are replaced by corresponding Euler polynomials.

In the next section we will give more applications of Theorem 1.1. Section 3 will be devoted to a proof of Theorem 1.1.

2. More applications of Theorem 1.1

Theorem 2.1. *Let* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of complex numbers, and let* $k,l\in\mathbb{N}$. *Then we have*

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{a_{l+j+1}}{l+j+1}+\sum_{j=0}^{l}\binom{l}{j}(-1)^j\frac{a_{k+j+1}^*}{k+j+1}=\frac{a_0}{(k+l+1)\binom{k+l}{k}}. \tag{2.1}$$

Also,

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j a_{l+j} = \sum_{j=0}^{l}\binom{l}{j}(-1)^j a^*_{k+j} \tag{2.2}$$

and

$$\sum_{j=0}^{k}\binom{k+1}{j}(-1)^j(l+j+1)a_{l+j} + \sum_{j=0}^{l}\binom{l+1}{j}(-1)^j(k+j+1)a^*_{k+j}$$
$$= (k+l+2)((-1)^k a_{k+l+1} + (-1)^l a^*_{k+l+1}). \tag{2.3}$$

Proof. (2.1)–(2.3) follow from (1.4)–(1.6) in the case $x = 1$ and $y = z = 0$. □

Example 2.1. Let $k, l, m \in \mathbb{N}$. The Stirling numbers $S(m, n)(n \in \mathbb{N})$ of the second kind are given by $x^m = \sum_{n=0}^{\infty} S(m, n)(x)_n$, where $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$ for $n = 1, 2, \ldots$. Observe that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i((-1)^i i!S(m,i)) = \sum_{i=0}^{\infty} S(m,i)(n)_i = n^m \qquad \text{for } n \in \mathbb{N}.$$

Applying (2.2) we obtain the identity

$$\sum_{j=0}^{k}\binom{k}{j}(l+j)!S(m,l+j) = \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(k+j)^m. \tag{2.4}$$

Example 2.2. For $n = 1, 2, 3, \ldots$ the nth Bell number b_n expresses the number of partitions of a set of cardinality n, in addition $b_0 = 1$. It is well known that

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i((-1)^i b_i) = b_{n+1} \qquad \text{for } n \in \mathbb{N}$$

(see, e.g. [2, p. 359]). Applying Theorem 2.1 to the sequence $\{(-1)^n b_n\}_{n\in\mathbb{N}}$ we obtain the following three identities for $k, l \in \mathbb{N}$.

$$\sum_{j=0}^{l}\binom{l}{j}(-1)^j\frac{b_{k+j+2}}{k+j+1} - (-1)^l\sum_{j=0}^{k}\binom{k}{j}\frac{b_{l+j+1}}{l+j+1} = \frac{1}{(k+l+1)\binom{k+l}{k}}, \tag{2.5}$$

$$\sum_{j=0}^{k}\binom{k}{j}b_{l+j} = \sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}b_{k+j+1}, \tag{2.6}$$

$$\sum_{j=0}^{k}\binom{k+1}{j}(l+j+1)b_{l+j} + \sum_{j=0}^{l}\binom{l+1}{j}(-1)^{l-j}(k+j+1)b_{k+j+1}$$
$$= (k+l+2)(b_{k+l+2} - b_{k+l+1}). \tag{2.7}$$

Theorem 2.2. *Let $k, l, m \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^{k}\binom{k}{j}\frac{(-1)^j}{l+j+1}\binom{x+l+j+1}{m} = (-1)^k \sum_{k<j\le m}\frac{1}{j}\binom{l}{j-k-1}\binom{x}{m-j} + \frac{\binom{x}{m}}{(k+l+1)\binom{k+l}{k}}. \tag{2.8}$$

Also,

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\binom{y}{l+j+1}}{\binom{x-1}{l+j}} + \sum_{j=0}^{l}\binom{l}{j}(-1)^j\frac{\binom{x-y}{k+j+1}}{\binom{x-1}{k+j}} = \frac{x}{(k+l+1)\binom{k+l}{k}} \tag{2.9}$$

and

$$\sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\binom{y}{l+j}}{\binom{x}{l+j}} = \sum_{j=0}^{l}\binom{l}{j}(-1)^j\frac{\binom{x-y}{k+j}}{\binom{x}{k+j}}. \tag{2.10}$$

Proof. Let $a_n = \binom{x+n}{m}$ for $n \in \mathbb{N}$. By identity (3.47) of Gould [1] or (5.24) of [2], we have

$$a_n^* = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\binom{x+i}{m} = \begin{cases}(-1)^n\binom{x}{m-n} & \text{if } m \ge n,\\ 0 & \text{otherwise.}\end{cases}$$

Applying (2.1) we obtain (2.8).

Let $c_n = \binom{y}{n}\Big/\binom{x}{n}$ for $n \in \mathbb{N}$. Then $c_n^* = \binom{x-y}{n}\Big/\binom{x}{n}$ by (7.1) of [1], in fact

$$\binom{x}{n}c_n^* = \sum_{i=0}^{n}\binom{x-i}{n-i}(-1)^i\binom{y}{i} = (-1)^n\sum_{i=0}^{n}\binom{n-1-x}{n-i}\binom{y}{i} = (-1)^n\binom{n-1-x+y}{n} = \binom{x-y}{n}.$$

Note that $(n+1)\binom{x}{n+1} = x\binom{x-1}{n}$ for any $n \in \mathbb{N}$. So (2.9) and (2.10) follow from the first and the second identities in Theorem 2.1. □

3. Proof of Theorem 1.1

Lemma 3.1. *Let $i, k, l \in \mathbb{N}$. Then*

$$\sum_{j=0}^{l}\binom{l}{j}\binom{k+j}{i}(-1)^{l-j}(1+t)^{k+j-i} = \sum_{j=0}^{k}\binom{k}{j}\binom{l+j}{i}t^{l+j-i}. \tag{3.1}$$

Proof. Clearly

$$\begin{aligned}
&\sum_{j=0}^{k}\binom{k}{j}\binom{l+j}{i}t^{l+j-i}-\sum_{j=0}^{l}\binom{l}{j}\binom{k+j}{i}(-1)^{l-j}(1+t)^{k+j-i}\\
&\quad=\sum_{j=0}^{k}\binom{k}{j}\frac{(t^{l+j})^{(i)}}{i!}-\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}\frac{((1+t)^{k+j})^{(i)}}{i!}\\
&\quad=\frac{1}{i!}\cdot\frac{\mathrm{d}^i}{\mathrm{d}t^i}\left(\sum_{j=0}^{k}\binom{k}{j}t^{l+j}-\sum_{j=0}^{l}\binom{l}{j}(-1)^{l-j}(1+t)^{k+j}\right)\\
&\quad=\frac{1}{i!}\cdot\frac{\mathrm{d}^i}{\mathrm{d}t^i}(t^l(1+t)^k-(1+t)^k(1+t-1)^l)=0.
\end{aligned}$$

This proves (3.1). □

Proof of Theorem 1.1. Observe that

$$\begin{aligned}
&(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{A_{l+j+1}(y)}{l+j+1}+(-1)^l\sum_{j=0}^{l}\binom{l}{j}x^{l-j}\frac{A^*_{k+j+1}(z)}{k+j+1}\\
&\quad=(-1)^k\sum_{j=0}^{k}\binom{k}{j}\frac{x^{k-j}}{l+j+1}\left(a_0y^{l+j+1}+\sum_{i=0}^{l+j}\binom{l+j+1}{i+1}(-1)^{i+1}\right.\\
&\qquad\left.\times a_{i+1}y^{l+j+1-(i+1)}\right)+(-1)^l\sum_{j=0}^{l}\binom{l}{j}\frac{x^{l-j}}{k+j+1}\\
&\qquad\times\sum_{r=0}^{k+j+1}\binom{k+j+1}{r}(-1)^r a^*_r z^{k+j+1-r}\\
&\quad=ca_0+\sum_{i=0}^{k+l}c_i(-1)^{i+1}\frac{a_{i+1}}{i+1}
\end{aligned}$$

where

$$\begin{aligned}
c=&(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\frac{y^{l+j+1}}{l+j+1}\\
&+(-1)^l\sum_{j=0}^{l}\binom{l}{j}\frac{x^{l-j}}{k+j+1}\sum_{r=0}^{k+j+1}\binom{k+j+1}{r}(-1)^r z^{k+j+1-r}
\end{aligned}$$

and

$$\begin{aligned}
&c_i-(-1)^k\sum_{j=0}^{k}\binom{k}{j}x^{k-j}\binom{l+j}{i}y^{l+j-i}\\
&\quad=(-1)^l\sum_{j=0}^{l}\binom{l}{j}\frac{x^{l-j}}{k+j+1}\sum_{i<r\le k+j+1}\binom{k+j+1}{r}
\end{aligned}$$

$$
\begin{aligned}
&\times (-1)^r \binom{r}{i+1}(i+1)z^{k+j+1-r}\\
&= (-1)^l \sum_{j=0}^{l} \binom{l}{j} x^{l-j} \binom{k+j}{i} (-1)^{k+j+1}\\
&\quad \times \sum_{i<r\le k+j+1} \binom{k+j-i}{r-1-i} (-z)^{k+j-i-(r-1-i)}\\
&= (-1)^{k-1} \sum_{j=0}^{l} \binom{l}{j}\binom{k+j}{i} (-x)^{l-j}(1-z)^{k+j-i}.
\end{aligned}
$$

Obviously $c_i = 0$ when $x = 0$. If $x \neq 0$, then

$$
\begin{aligned}
\frac{(-1)^k c_i}{x^{k+l-i}} &= \sum_{j=0}^{k} \binom{k}{j}\binom{l+j}{i}\left(\frac{y}{x}\right)^{l+j-i}\\
&\quad - \sum_{j=0}^{l} \binom{l}{j}\binom{k+j}{i}(-1)^{l-j}\left(1+\frac{y}{x}\right)^{k+j-i}\\
&= 0
\end{aligned}
$$

by Lemma 3.1.

Let us now calculate the value of c. Clearly

$$
\begin{aligned}
(-1)^k c &= \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \frac{y^{l+j+1}}{l+j+1}\\
&\quad + (-1)^{k+l} \sum_{j=0}^{l} \binom{l}{j} x^{l-j} \frac{(-1)^{k+j+1}}{k+j+1}(1-z)^{k+j+1}\\
&= \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \int_0^y t^{l+j}\,\mathrm{d}t - \sum_{j=0}^{l} \binom{l}{j}(-x)^{l-j} \int_0^{x+y} t^{k+j}\,\mathrm{d}t\\
&= \int_0^y \sum_{j=0}^{k} \binom{k}{j} x^{k-j} t^{l+j}\,\mathrm{d}t - \int_0^{x+y} \sum_{j=0}^{l} \binom{l}{j}(-x)^{l-j} t^{k+j}\,\mathrm{d}t\\
&= \int_0^y t^l (x+t)^k\,\mathrm{d}t - \int_0^{x+y} t^k (t-x)^l\,\mathrm{d}t\\
&= \int_x^{x+y} s^k (s-x)^l\,\mathrm{d}s - \int_0^{x+y} t^k (t-x)^l\,\mathrm{d}t = -\int_0^x s^k (s-x)^l\,\mathrm{d}s\\
&= -\int_0^1 (tx)^k (tx-x)^l x\,\mathrm{d}t = (-1)^{l+1} x^{k+l+1} B(k+1, l+1),
\end{aligned}
$$

where $B(k+1, l+1) := \int_0^1 t^k(1-t)^l\,\mathrm{d}t$ is known to be

$$\frac{\Gamma(k+1)\Gamma(l+1)}{\Gamma(k+l+2)} = \frac{k!\,l!}{(k+l+1)!} = \frac{1}{(k+l+1)\binom{k+l}{k}}.$$

So we have $c = (-x)^{k+l+1}/\big((k+l+1)\binom{k+l}{k}\big)$.

In view of the above, (1.4) holds.

If we replace z in (1.4) by $1-x-y$ and take partial derivation with respect to y, then we obtain (1.5) from (1.4). Substituting $k+1$ and $l+1$ for k and l in (1.5) and taking derivation with respect to y, we then get (1.6).

The proof of Theorem 1.1 is now complete. □

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