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Partitions of n into $t\sqrt{n}$ parts

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Abstract

Szekeres proved, using complex analysis, an asymptotic formula for the number of partitions of n into at most k parts. Canfield discovered a simplification of the formula, and proved it without complex analysis. We re-prove the formula, in the asymptotic regime when k is at least a constant times $\sqrt{n}$, by showing that it is equivalent to a local central limit theorem in Fristedt's model for random partitions. We then apply the formula to derive asymptotics for the number of minimal difference d partitions with a given number of parts. As a corollary, we find (explicitly computable) constants $c_d, \beta_d, \gamma_d, \sigma_d$ such that the number of minimal difference d partitions of n is $(1 + o(1))c_d n^{-3/4} \exp(\beta_d \sqrt{n})$ (a result of Meinardus), almost all of them (fraction $a(1 + o(1))$) have approximately $\gamma_d \sqrt{n}$ parts, and the distribution of the number of parts in a random such partition is asymptotically normal with standard deviation $(1 + o(1))\sigma_d n^{1/4}$. In particular, $\gamma_2 = \sqrt{15} \log[(1 + \sqrt{5})/2]/\pi$.

1. Introduction

Let $p(n, k)$ be the number of unordered partitions of n into exactly k parts. Let $P(n, k) = \sum_{j=1}^{k} p(n, j)$ be the number of partitions of n into at most k parts, or equivalently the number of partitions of n into parts all of which do not exceed k. Hardy and Ramanujan [7] proved the famous asymptotic formula

$$P(n, \infty) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}} \tag{1}$$

for the total number of all partitions, where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. Some 35 years later, Szekeres [11] derived an asymptotic formula for $P(n, k)$. A few years ago, Canfield [2] discovered a simpler way to write the formula. This is easiest to understand

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when k is approximately a constant t times $\sqrt{n}$. The formula then takes the form

$$P(n, t\sqrt{n}) \sim \frac{G(t)}{n} \mathrm{e}^{H(t)\sqrt{n}} \tag{2}$$

where $G(t)$ and $H(t)$ are functions defined as follows: for $0 \le x \le 1$, let

$$\mathrm{Li}_2(x) = \int_0^x -\log(1-t)\frac{\mathrm{d}t}{t} = \sum_{m=1}^{\infty} \frac{x^m}{m^2}$$

be the dilogarithm function. Define a function α: $[0,\infty] \to [0, \pi/\sqrt{6}]$ by the implicit equation

$$\alpha(t)^2 = \mathrm{Li}_2(1 - \mathrm{e}^{-\alpha(t)t}).$$

It is easy to check that $\alpha(t)$ is an increasing function that satisfies $\alpha(0) = 0, \alpha(\infty) = \pi/\sqrt{6}$. Then $G(t)$, $g(t)$ and $H(t)$ are given by

$$G(t) = \frac{\alpha(t)}{2\pi[2 - (t^2+2)\mathrm{e}^{-\alpha(t)t}]^{1/2}}$$
$$H(t) = 2\alpha(t) - t\log(1 - \mathrm{e}^{-\alpha(t)t}).$$

Define also

$$g(t) = \mathrm{e}^{-t\alpha(t)} G(t).$$

One can obtain from (2) also an asymptotic formula for $p(n,k)$, namely

$$p(n, \lfloor t\sqrt{n} \rfloor) \sim \frac{g(t)}{n} \mathrm{e}^{H(t)\sqrt{n}}. \tag{3}$$

Theorem 1. *As $n \to \infty$, (2) holds uniformly for $t \in [T,\infty]$ for every $T > 0$. (3) holds uniformly as t ranges over compact subsets of $(0,\infty)$.*

Szekeres's proof of Theorem 1 used complex analysis and the saddle point method, and required considerable analytic insight, especially given his more complicated formulation of (2). As well as simplifying it, Canfield re-proved (2) without recourse to complex analysis, by using only the recurrence equation satisfied by $P(n,k)$ and elementary real analysis. Our first main goal in this paper is to give a new probabilistic proof of Theorem 1. Our proof uses Fristedt's conditioning device for random partitions [6]. We show that the proof of (2) reduces to proving a local limit theorem in Fristedt's model. We then apply the standard methodology of probability theory, namely representing the probabilities as inverse Fourier integrals. This is formally equivalent to the use of contour integration and the saddle point method in Szekeres's paper, but in our opinion the probabilistic outlook gives important insight into the technique. A similar use of local limit theorems can be found, e.g., in [3, 6, 9].

The form of the functions $G(t)$, $g(t)$ and $H(t)$ may seem unwieldy. Our second main goal in this paper is to show that it is nevertheless possible to extract useful information from them. We describe an application to the asymptotics of minimal difference partitions: for $d \in \mathbb{N}$, *a minimal difference d* partition is a partition $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k$ such that $\forall i\ \lambda_i - \lambda_{i+1} \ge d$. Note that for $d = 1$ these are just partitions into distinct parts. Let $q_d(n)$

be the total number of minimal difference d partitions of n, and let $q_d(n,k)$ be the number of minimal difference d partitions of n into exactly k parts. Then we have the formula

$$q_d(n,k) = p\left(n - \sum_{j=1}^{k-1} jd, k\right) = p\left(n - d\binom{k}{2}, k\right), \tag{4}$$

since the mapping $(\lambda_i)_{i=1}^k \to (\lambda_i - d(k-i))_{i=1}^k$ gives a bijection between the set of minimal difference d partitions of n into k parts and the set of partitions of $(n - dk(k-1)/2)$ into k parts. (3) may now be used to prove:

Theorem 2. *For each $d \in \mathbb{N}$ we have as $n \to \infty$, uniformly, as t ranges over compact subsets of $(0, \sqrt{2/d})$:*

$$q_d(n, \lfloor t\sqrt{n} \rfloor) \sim \frac{k_d(t)}{n} \mathrm{e}^{K_d(t)\sqrt{n}} \tag{5}$$

where

$$k_d(t) = \frac{1}{1 - dt^2/2} g\left(\frac{t}{\sqrt{1 - dt^2/2}}\right) \exp\left(\frac{dt}{2\sqrt{1 - dt^2/2}} \alpha\left(\frac{t}{\sqrt{1 - dt^2/2}}\right)\right)$$

$$K_d(t) = \sqrt{1 - dt^2/2}\, H\left(\frac{t}{\sqrt{1 - dt^2/2}}\right).$$

Theorem 3. *For each $d \in \mathbb{N}$, define y_d as the unique solution in the interval $(0, 1)$ of the equation*

$$(1-y)^d = y.$$

Define

$$\beta_d = \frac{2\mathrm{Li}_2(y_d) + \log y_d \cdot \log(1 - y_d)}{(\mathrm{Li}_2(y_d) + \frac{d}{2} \log^2(1 - y_d))^{1/2}}$$

$$\gamma_d = \frac{-\log(1 - y_d)}{(\mathrm{Li}_2(y_d) + \frac{d}{2} \log^2(1 - y_d))^{1/2}}$$

$$\sigma_d = \frac{1}{(-K_d''(\gamma_d))^{1/2}}$$

$$c_d = k_d(\gamma_d)\sqrt{2\pi}\sigma_d.$$

Then

(a) *(Meinardus [8]; see also [1, Example 8, p. 99])*

$$q_d(n) \sim \frac{c_d}{n^{3/4}} \mathrm{e}^{\beta_d \sqrt{n}}.$$

(b) *A "typical" minimal difference d partition of n has approximately $\gamma_d\sqrt{n}$ parts. That is, for any $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{q_d(n)} \sum_{|k - \gamma_d\sqrt{n}| < \epsilon\sqrt{n}} q_d(n,k) = 1.$$

Table 1
Values of c_d, β_d, γ_d and σ_d

d	c_d	β_d	γ_d	σ_d
1	$\frac{1}{4\times 3^{1/4}}$	$\frac{\pi}{\sqrt{3}}$	$\frac{\sqrt{12}\log 2}{\pi}$	0.478815...
2	0.216122...	$\frac{2\pi}{\sqrt{15}}$	$\frac{\sqrt{15}\log[(1+\sqrt{5})/2]}{\pi}$	0.351859...
3	0.231676...	1.505235...	0.507887...	0.294510...
4	0.242867...	1.42124...	0.453526...	0.259883...
5	0.251663...	1.35607...	0.414727...	0.236017...

(c) *The number of parts in a random minimal difference d partition of n has asymptotically the normal distribution with expectation $\gamma_d\sqrt{n}$ and standard deviation $\sigma_d n^{1/4}$. That is, for any $u \in \mathbb{R}$,*

$$\frac{1}{q_d(n)} \sum_{k<\gamma_d\sqrt{n}+u\sigma_d n^{1/4}} q_d(n,k) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} \mathrm{e}^{-x^2/2}\mathrm{d}x.$$

The first few values of $c_d, \beta_d, \gamma_d, \sigma_d$ are shown in Table 1 above. The explicit values for $d = 1, 2$ are derived using elementary properties of the dilogarithm function. The case $d = 1$ of Theorem 3(a) is the well-known fact $q_1(n) \sim (4 \times 3^{1/4})^{-1} \exp(\pi\sqrt{n/3})$ proved by Hardy and Ramanujan [7]. The result that almost all partitions of n into distinct parts have about $(\sqrt{12}\log(2)/\pi)\sqrt{n}$ parts was first proved by Erdös and Lehner [5]. The case $d = 2$ of Theorem 3(a) is in accordance with (and can be deduced from) the first Rogers–Ramanujan identity, which states that $q_2(n)$ is equal to the number of partitions of n into parts which are congruent to 1 or 4 modulo 5. The result that a typical minimal difference 2 partition has approximately $(\sqrt{15}\log[(1+\sqrt{5})/2]/\pi)\sqrt{n}$ parts is apparently new. In a forthcoming paper [10] we show a new method of deriving this result, based on the computation of stationary probabilities for a certain Markov chain. The method gives more general results on the "limit shape" of this class of partitions, i.e. the function of s which gives the "typical" number of parts which are greater than $s\sqrt{n}$ in a random minimal difference 2 partition. Also, see [4] for a recent work on classes of partitions defined by inequalities (of which minimal difference partitions are an example).

In the next section, we outline the steps required for the proof of Theorem 1. In Section 3 we complete the proof, and in Section 4 we show how Theorems 2 and 3 follow as easy corollaries to Theorem 1.

2. Theorem 1—preparation for the proof

In the next two sections, we use the following notation:

$$F_k(z) = \sum_{n=0}^{\infty} P(n,k)z^n = \prod_{j=1}^{k} \frac{1}{1-z^j} \qquad (|z| < 1)$$

is the generating function for $P(n, k)$, k fixed. Let $t \in (0, \infty]$ be fixed, and write

$$k_n = t\sqrt{n}, \qquad s_n = \frac{\alpha(t)}{\sqrt{n}}, \qquad x_n = \mathrm{e}^{-s_n} = 1 - \frac{\alpha(t)}{\sqrt{n}} + O\left(\frac{1}{\sqrt{n}}\right).$$

All our estimates will be uniform in t for $t > T > 0$. Therefore we may assume for simplicity of notation that t varies slightly with n in such a manner that $t\sqrt{n}$ is always an integer.

We now describe a version of Fristedt's probabilistic model for partitions [6]. Fix $0 < x < 1$ and $k \in \mathbb{N}$. Define independent random variables $R_1, R_2, R_3, \ldots, R_k$ such that $R_j + 1$ has a geometric distribution with parameter $1 - x^j$. More precisely

$$P_{x,k}(R_j = l) = (1 - x^j)x^{jl} \qquad l = 0, 1, 2, \ldots$$

where $P_{x,k}$ denotes probability, with parameters x and k. Let $N = \sum_{j=1}^{k} jR_j$. Then $(R_1, R_2, \ldots, R_k)$ can be thought of as the frequential coding of a random partition of the (random) integer N into parts not exceeding k, i.e. the partition in which 1 appears R_1 times, 2 appears R_2 times, etc. For any (nonrandom) partition

$$n = 1 \cdot r_1 + 2 \cdot r_2 + \cdots + k \cdot r_k$$

of n into parts not exceeding k, given in frequential coding, the probability of it appearing in the random model is

$$\begin{aligned} P_{x,k}(R_1 = r_1, R_2 = r_2, \ldots, R_k = r_k) &= \prod_{j=1}^{k} P_{x,k}(R_j = r_j) \\ &= \prod_{j=1}^{k} ((1 - x^j)x^{jr_j}) = \frac{x^n}{F_k(x)}. \end{aligned}$$

Therefore the probability that $N = n$ is a sum over all $P(n, k)$ different partitions of n into parts not exceeding k, of this quantity, namely

$$P_{x,k}(N = n) = \frac{P(n, k)x^n}{F_k(x)}.$$

This is the key observation that we will require for our proof; we have constructed a random variable whose value probabilities are related to $P(n, k)$ in a relatively simple way. Furthermore, this random variable is a sum of lattice random variables, and thus we can expect it to be an approximately normal lattice random variable and satisfy a local limit theorem.

The proof of (2) will now follow from the following propositions:

Proposition 1. *As $n \to \infty$,*

$$\begin{aligned} \log F_{k_n}(x_n) = (\alpha(t) - t \log(1 - \mathrm{e}^{-\alpha(t)t}))\sqrt{n} - \frac{1}{4} \log n \\ + \frac{1}{2} \log\left(\frac{\alpha(t)}{2\pi(1 - \mathrm{e}^{-\alpha(t)t})}\right) + o(1). \end{aligned}$$

Proposition 2. *For choice of parameters x_n, k_n, N is a random variable with expectation*

$$E_{x_n,k_n}(N) = n\left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right)$$

and variance

$$\sigma^2_{x_n,k_n}(N) \sim \left(\frac{2}{\alpha(t)} - \frac{t^2}{\alpha(t)}\frac{\mathrm{e}^{-\alpha(t)t}}{1-\mathrm{e}^{-\alpha(t)t}}\right) n^{3/2}.$$

Proposition 3. *For choice of parameters x_n, k_n, the random variable N "satisfies a local limit theorem at 0", that is*

$$P_{x_n,k_n}(N = n) \sim \frac{1}{\sqrt{2\pi}\sigma_{x_n,k_n}(N)}$$

as $n \to \infty$.

In the next section we prove these claims. To see that (2) follows from them, write

$$\begin{aligned}
P(n, k_n) &= x_n^{-n} \cdot F_{k_n}(x_n) \cdot P_{x_n,k_n}(N = n) \sim \mathrm{e}^{\alpha(t)\sqrt{n}} \\
&\quad \times \left(\left(\frac{\alpha(t)}{2\pi(1-\mathrm{e}^{-\alpha(t)t})}\right)^{1/2} \frac{1}{n^{1/4}} \mathrm{e}^{(\alpha(t)-t\log(1-\mathrm{e}^{-\alpha(t)t}))\sqrt{n}}\right) \\
&\quad \times \left(\frac{1}{\sqrt{2\pi}}\left(\frac{2}{\alpha(t)} - \frac{t^2}{\alpha(t)}\frac{\mathrm{e}^{-\alpha(t)t}}{1-\mathrm{e}^{-\alpha(t)t}}\right)^{-1/2} n^{-3/4}\right) \\
&= \frac{\alpha(t)}{2\pi[2-(t^2+2)\mathrm{e}^{-\alpha(t)t}]^{1/2} n} \mathrm{e}^{(2\alpha(t)-t\log(1-\mathrm{e}^{-\alpha(t)t}))\sqrt{n}} = \frac{G(t)}{n}\mathrm{e}^{H(t)\sqrt{n}}.
\end{aligned}$$

(3) follows easily from (2) using the relation $p(n,k) = P(n-k,k)$ together with the equation (which is easy to verify)

$$-\tfrac{1}{2}tH(t) + \tfrac{1}{2}t^2H'(t) = -t\alpha(t). \tag{6}$$

3. Proof of the propositions

3.1. Proof of Proposition 1

We use Euler–Maclaurin summation: write as usual $\{x\} = x - \lfloor x \rfloor$; then

$$\begin{aligned}
\log F_{k_n}(x_n) &= \log F_{k_n}(\mathrm{e}^{-\alpha(t)/\sqrt{n}}) = -\sum_{j=1}^{k_n} \log(1-\mathrm{e}^{-\alpha(t)j/\sqrt{n}}) \\
&= \int_1^{t\sqrt{n}} -\log(1-\mathrm{e}^{-\alpha(t)u/\sqrt{n}})\mathrm{d}u + \frac{1}{2}(-\log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) \\
&\quad - \log(1-\mathrm{e}^{-\alpha(t)t})) + \int_1^{t\sqrt{n}} \frac{-\mathrm{e}^{-\alpha(t)u/\sqrt{n}}\alpha(t)/\sqrt{n}}{1-\mathrm{e}^{-\alpha(t)u/\sqrt{n}}}\left(\{u\} - \frac{1}{2}\right)\mathrm{d}u
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{n}}{\alpha(t)} \int_{\alpha(t)/\sqrt{n}}^{\alpha(t)t} -\log(1-\mathrm{e}^{-v})\mathrm{d}v - \frac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) \\
&\quad - \frac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)t}) - \int_{\alpha(t)/\sqrt{n}}^{\alpha(t)t} \frac{\mathrm{e}^{-v}}{1-\mathrm{e}^{-v}} \left(\{\sqrt{n}v/\alpha(t)\} - \frac{1}{2}\right) \mathrm{d}v \\
&= \frac{\sqrt{n}}{\alpha(t)} (\mathrm{Li}_2(\mathrm{e}^{-\alpha(t)/\sqrt{n}}) - \mathrm{Li}_2(\mathrm{e}^{-\alpha(t)t})) - \frac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) \\
&\quad - \frac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)t}) - \int_{\alpha(t)/\sqrt{n}}^{\alpha(t)t} \frac{\mathrm{e}^{-v}}{1-\mathrm{e}^{-v}} \left(\{\sqrt{n}v/\alpha(t)\} - \frac{1}{2}\right) \mathrm{d}v.
\end{aligned}
$$

Recall that the dilogarithm function satisfies the identity

$$
\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log x \cdot \log(1-x),
$$

which is easily verified by differentiating both sides. Write

$$
I_x(s) = \int_s^x \frac{\mathrm{e}^{-v}}{1-\mathrm{e}^{-v}} \left(\{v/s\} - \frac{1}{2}\right) \mathrm{d}v.
$$

We will prove shortly that

$$
I_x(s) \xrightarrow[s \searrow 0]{} \tfrac{1}{2}\log(2\pi) - 1 \tag{7}
$$

uniformly in x for $x > X > 0$. Assuming this, for the moment, we have

$$
\begin{aligned}
\log F_{k_n}(x_n) &= \frac{\sqrt{n}}{\alpha(t)} \left[-\mathrm{Li}_2(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) + \frac{\alpha(t)}{\sqrt{n}} \log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) \right. \\
&\quad \left. +\ \mathrm{Li}_2(1-\mathrm{e}^{-\alpha(t)t}) - \alpha(t)t\log(1-\mathrm{e}^{-\alpha(t)t}) \right] \\
&\quad - \tfrac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) - \tfrac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)t}) - I_{\alpha(t)t}(\alpha(t)/\sqrt{n}) \\
&= \sqrt{n} \left(\frac{\mathrm{Li}_2(1-\mathrm{e}^{-\alpha(t)t})}{\alpha(t)} - t\log(1-\mathrm{e}^{-\alpha(t)t}) \right) \\
&\quad + \tfrac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)/\sqrt{n}}) - \tfrac{1}{2}\log(1-\mathrm{e}^{-\alpha(t)t}) - \tfrac{1}{2}\log(2\pi) + o(1) \\
&= \sqrt{n}(\alpha(t) - t\log(1-\mathrm{e}^{-\alpha(t)t})) + \tfrac{1}{2}(\log(\alpha(t)/\sqrt{n}) \\
&\quad - \log(1-\mathrm{e}^{-\alpha(t)t}) - \log(2\pi)) + o(1) \\
&= \sqrt{n}(\alpha(t) - t\log(1-\mathrm{e}^{-\alpha(t)t})) - \tfrac{1}{4}\log n \\
&\quad + \frac{1}{2}\log\left(\frac{\alpha(t)}{2\pi(1-\mathrm{e}^{-\alpha(t)t})}\right) + o(1)
\end{aligned}
$$

which was the expression that we wanted.

Proof of (7). To prove (7) and thus finish the proof of Proposition 1, write $I_x(s)$, in the range $x > X$, as

$$I_x(s) = \int_s^X \frac{1}{v}\left(\{v/s\} - \frac{1}{2}\right)\mathrm{d}v$$
$$+ \int_s^\infty \left[\left(\frac{\mathrm{e}^{-v}}{1-\mathrm{e}^{-v}} - \frac{1}{v}\right)1_{[s,X]}(v) + \frac{\mathrm{e}^{-v}}{1-\mathrm{e}^{-v}}1_{[X,x]}(v)\right]\left(\{v/s\} - \frac{1}{2}\right)\mathrm{d}v.$$

The second integral is a scalar product in $L_2([0,\infty))$ of the sawtooth function $\{v/s\} - 1/2$ with a bounded, square-integrable function, and so can easily be seen to converge to 0, with the required uniformity in x, as $s \searrow 0$ (this is a version of the Riemann–Lebesgue lemma). For the first integral, we compute

$$\begin{aligned}
\int_s^X \frac{1}{v}\left(\{v/s\} - \frac{1}{2}\right)\mathrm{d}v &= \sum_{k=1}^{\lfloor X/s\rfloor}\int_{ks}^{(k+1)s}\frac{1}{v}\left(\frac{v}{s} - k - \frac{1}{2}\right)\mathrm{d}v + O(s) \\
&= \sum_{k=1}^{\lfloor X/s\rfloor}\left(1 - \left(k+\frac{1}{2}\right)\log((k+1)/k)\right) + O(s) \\
&= \lfloor X/s\rfloor - \tfrac{1}{2}\log(\lfloor X/s\rfloor + 1) \\
&\quad + \sum_{k=1}^{\lfloor X/s\rfloor} k\log k - \sum_{k=2}^{\lfloor X/s\rfloor+1}(k-1)\log k + O(s) \\
&= \lfloor X/s\rfloor - \tfrac{1}{2}\log(\lfloor X/s\rfloor + 1) + \log(\lfloor X/s\rfloor!) \\
&\quad - \lfloor X/s\rfloor\log(\lfloor X/s\rfloor + 1) + O(s) \xrightarrow[s\searrow 0]{} \tfrac{1}{2}\log(2\pi) - 1
\end{aligned}$$

by Stirling's formula. □

3.2. *Proof of Proposition 2*

We use the simple probabilistic fact that if X is a random variable such that $X+1$ has geometric distribution with parameter $0 < p < 1$, that is

$$P(X = l) = p(1-p)^l \qquad l = 0, 1, 2, 3, \ldots,$$

then

$$E(X) = \sum_{l=0}^{\infty} l p(1-p)^l = \frac{1-p}{p}$$
$$\sigma^2(X) = \sum_{l=0}^{\infty} l^2 p(1-p)^l - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}.$$

Now with choice of parameters $x_n, k_n, N = \sum_{j=1}^{k_n} jR_j$, so

$$E_{x_n,k_n}(N) = \sum_{j=1}^{k_n} j\frac{x_n^j}{1-x_n^j} = n\sum_{j=1}^{k_n}\frac{1}{\sqrt{n}}\frac{j}{\sqrt{n}}\frac{\mathrm{e}^{-\alpha(t)j/\sqrt{n}}}{1-\mathrm{e}^{-\alpha(t)j/\sqrt{n}}}.$$

The sum is a Riemann sum, with $\Delta u = 1/\sqrt{n}$, for the integral

$$\int_0^t \frac{u\mathrm{e}^{-\alpha(t)u}}{1-\mathrm{e}^{-\alpha(t)u}}\mathrm{d}u = \frac{1}{\alpha(t)^2}\int_0^{1-\mathrm{e}^{-\alpha(t)t}} \frac{-\log(1-v)}{v}\mathrm{d}v = \frac{\mathrm{Li}_2(1-\mathrm{e}^{-\alpha(t)t})}{\alpha(t)^2} = 1.$$

The difference between the Riemann sum and the integral is easily seen to be $O(1/\sqrt{n})$, so

$$E_{x_n,k_n}(N) = n(1+O(1/\sqrt{n})).$$

Similarly, the variance

$$\begin{aligned}\sigma^2_{x_n,k_n}(N) &= \sum_{j=1}^{k_n} j^2 \frac{x_n^j}{(1-x_n^j)^2} = n^{3/2}\sum_{j=1}^{k_n}\frac{1}{\sqrt{n}}\left(\frac{j}{\sqrt{n}}\right)^2 \frac{\mathrm{e}^{-\alpha(t)j/\sqrt{n}}}{(1-\mathrm{e}^{-\alpha(t)j/\sqrt{n}})^2} \\ &\sim n^{3/2}\int_0^t \frac{u^2\mathrm{e}^{-\alpha(t)u}}{(1-\mathrm{e}^{-\alpha(t)u})^2}\mathrm{d}u.\end{aligned}$$

The integral can be evaluated to be

$$\begin{aligned}\int_0^t \frac{u^2\mathrm{e}^{-\alpha(t)u}}{(1-\mathrm{e}^{-\alpha(t)u})^2}\mathrm{d}u &= \frac{1}{\alpha(t)^3}\int_0^{1-\mathrm{e}^{-\alpha(t)t}} \frac{\log^2(1-v)}{x^2}\mathrm{d}x \\ &= \frac{1}{\alpha(t)^3}\left[2\mathrm{Li}_2(v) - \frac{1-v}{v}\log^2(1-v)\right]_{v=0}^{v=1-\mathrm{e}^{-\alpha(t)t}} \\ &= \frac{1}{\alpha(t)^3}\left[2\mathrm{Li}_2(1-\mathrm{e}^{-\alpha(t)t}) - \alpha(t)^2t^2\frac{\mathrm{e}^{-\alpha(t)t}}{1-\mathrm{e}^{-\alpha(t)t}}\right] \\ &= \frac{2}{\alpha(t)} - \frac{t^2}{\alpha(t)}\frac{\mathrm{e}^{-\alpha(t)t}}{1-\mathrm{e}^{-\alpha(t)t}}.\end{aligned}$$

3.3. *Proof of Proposition 3*

We now reach the most delicate part of the analysis, namely the proof of the claim that N satisfies a local limit theorem at 0. The idea is to use Fourier inversion. Denote by $\phi_{x,k}(s) = E_{x,k}(\mathrm{e}^{isN})$ the characteristic function of N for parameter choice x, k. Then

$$\phi_{x,k}(s) = \sum_{n=0}^{\infty} P_{x,k}(N=n)\mathrm{e}^{ins} = \sum_{n=0}^{\infty}\frac{P(n,k)x^n}{F_k(x)}\mathrm{e}^{ins} = \frac{F_k(x\mathrm{e}^{is})}{F_k(x)},$$

and using Fourier inversion we get what is really a disguised contour integral:

$$\begin{aligned}P_{x_n,k_n}(N=n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\phi_{x_n,k_n}(s)\mathrm{e}^{-ins}\mathrm{d}s \\ &= \frac{1}{2\pi\sigma_{x_n,k_n}(N)}\int_{-\pi\sigma_{x_n,k_n}(N)}^{\pi\sigma_{x_n,k_n}(N)}\phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}\mathrm{d}u.\end{aligned}$$

So it is enough to prove that

$$\int_{-\pi\sigma_{x_n,k_n}(N)}^{\pi\sigma_{x_n,k_n}(N)} \phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}\mathrm{d}u \xrightarrow[n\to\infty]{} \sqrt{2\pi}. \tag{8}$$

Indeed, probabilistic thinking leads us to expect that for any $u \in \mathbb{R}$,

$$\phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)} \xrightarrow[n\to\infty]{} \mathrm{e}^{-u^2/2}, \tag{9}$$

which will give us (8) if we can prove some additional boundedness estimates. Note that (9) is equivalent to the claim that N satisfies a (non-local) central limit theorem, i.e. that $(N-n)/\sigma_{x_n,k_n}(N) \to N(0,1)$ in the distribution as $n \to \infty$. This can be deduced e.g. by using the Lindeberg–Feller central limit theorem for triangular arrays. Instead, we give a direct proof. First, we need a technical lemma:

Lemma 1. *For* $0 < x < 1$, $s \in \mathbb{R}$, *let*

$$f_x(s) = \log\left(\frac{1-x}{1-\mathrm{e}^{is}x}\right) - i\frac{x}{1-x}s + \frac{1}{2}\frac{x}{(1-x)^2}s^2.$$

Then there exists a constant $C > 0$ *such that*

$$|f_x(s)| \le C\frac{x|s|^3}{(1-x)^3} \qquad (0 < x < 1, s \in \mathbb{R}).$$

Proof. First, consider the case $|s| \le (1-x)/2$:

$$\begin{aligned}
\log\left(\frac{1-x}{1-\mathrm{e}^{is}x}\right) &= \sum_{j=1}^{\infty}\frac{x^j}{j}(\mathrm{e}^{ijs}-1) \\
&= \sum_{j=1}^{\infty}\frac{x^j}{j}\sum_{k=1}^{\infty}\frac{i^k j^k s^k}{k!} \\
&= \sum_{k=1}^{\infty}\frac{i^k}{k!}\left(\sum_{j=1}^{\infty} j^{k-1}x^j\right)s^k = i\frac{x}{1-x}s - \frac{1}{2}\frac{x}{(1-x)^2}s^2 \\
&\quad + \sum_{k=3}^{\infty} i^k\left(\frac{1}{k!}\sum_{j=1}^{\infty} j^{k-1}x^j\right)s^k
\end{aligned}$$

so that

$$\begin{aligned}
|f_x(s)| &\le \sum_{k=3}^{\infty}\left(\frac{1}{k!}\sum_{j=1}^{\infty} j^{k-1}x^j\right)|s|^k \\
&\le \sum_{k=3}^{\infty}\left(\frac{1}{k!}\sum_{j=1}^{\infty} j(j+1)\cdots(j+k-2)x^j\right)|s|^k \\
&= \sum_{k=3}^{\infty}\frac{x}{k}\frac{|s|^k}{(1-x)^k} \le \sum_{k=3}^{\infty}\frac{x}{3}\left(\frac{|s|}{1-x}\right)^k = \frac{x}{3}\frac{|s|^3/(1-x)^3}{1-|s|/(1-x)}.
\end{aligned}$$

When $|s| \le (1-x)/2$ this gives us $|f_x(s)| \le 2x|s|^3/3(1-x)^3$. Next, for $|s| > (1-x)/2$ we have

$$\left| -i\frac{x}{1-x}s + \frac{1}{2}\frac{x}{(1-x)^2}s^2 \right| \le \frac{x|s|^3}{1-x}\frac{1}{s^2} + \frac{x|s|^3}{(1-x)^2}\frac{1}{|s|} \le (4+2)\frac{x|s|^3}{(1-x)^3},$$

so it remains to prove

$$\left| \log\left(\frac{1-x}{1-x\mathrm{e}^{is}}\right) \right| \le C\frac{x|s|^3}{(1-x)^3} \qquad (|s| > (1-x)/2).$$

For $|s| \ge 1/4$, clearly

$$\begin{aligned}\left| \log\left(\frac{1-x}{1-x\mathrm{e}^{is}}\right) \right| &\le \sum_{j=1}^{\infty} \frac{x^j}{j}|\mathrm{e}^{ijs}-1| \le -2\log(1-x) \\ &\le C'\frac{x}{(1-x)^3} \le 64C'\frac{x|s|^3}{(1-x)^3}.\end{aligned}$$

Finally, for $0 \le (1-x)/2 \le |s| \le 1/4$ (which implies in particular $1/2 \le x \le 1$),

$$\begin{aligned}\log\left(\frac{1-x}{1-x\mathrm{e}^{is}}\right) &= -\log\left(1 - \frac{x}{1-x}(\mathrm{e}^{is}-1)\right) \\ &= -\log\left(1 - \frac{x}{1-x}2i\mathrm{e}^{is/2}\sin(s/2)\right) = -\log(1-i\mathrm{e}^{is/2}S),\end{aligned}$$

where we write $S = 2\sin(s/2)x/(1-x)$. We have

$$\frac{1}{40} \le \frac{x}{1-x}\frac{|s|}{10} \le |S| \le \frac{x}{1-x}|s|$$

and therefore, since $\pi/2 - 1/8 \le \arg(iS\mathrm{e}^{is/2}) \le \pi/2 + 1/8$,

$$\begin{aligned}\left| \log\left(\frac{1-x}{1-x\mathrm{e}^{is}}\right) \right| &= |\log(1-iS\mathrm{e}^{is/2})| \le C''|S|^3 \\ &\le C''\left(\frac{|s|}{1-x}\right)^3 \le 2C''\frac{x|s|^3}{(1-x)^3}. \quad \square\end{aligned}$$

Proof of (9).

$$\begin{aligned}&\log(\phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}) \\ &\quad = \log F_{k_n}(x_n\mathrm{e}^{iu/\sigma_{x_n,k_n}(N)}) - \log F_{k_n}(x_n) - \frac{inu}{\sigma_{x_n,k_n}(N)} \\ &\quad = \sum_{j=1}^{k_n}\log\left(\frac{1-x_n^j}{1-x_n^j\mathrm{e}^{iju/\sigma_{x_n,k_n}(N)}}\right) - \frac{inu}{\sigma_{x_n,k_n}(N)} \\ &\quad = \sum_{j=1}^{k_n} f_{x_n^j}(ju/\sigma_{x_n,k_n}(N)) + i\left(\sum_{j=1}^{k_n}\frac{jx_n^j}{1-x_n^j} - n\right)\frac{u}{\sigma_{x_n,k_n}(N)}\end{aligned}$$

$$- \frac{1}{2} \left(\sum_{j=1}^{k_n} \frac{j^2 x_n^j}{(1 - x_n^j)^2} \right) \frac{u^2}{\sigma^2_{x_n,k_n}(N)}$$
$$= i(E_{x_n,k_n}(N) - n) \frac{u}{\sigma_{x_n,k_n}(N)} - \frac{u^2}{2} + R_n(u)$$
$$= O(n^{-1/4})u - \frac{u^2}{2} + R_n(u),$$

where

$$|R_n(u)| = \left| \sum_{j=1}^{k_n} f_{x_n^j}(ju/\sigma_{x_n,k_n}(N)) \right| \le C \frac{|u|^3}{\sigma^3_{x_n,k_n}(N)} \sum_{j=1}^{k_n} \frac{j^3 x_n^j}{(1 - x_n^j)^3} = |u|^3 O(n^{-1/4}),$$

since

$$\sum_{j=1}^{k_n} \frac{j^3 x_n^j}{(1 - x_n^j)^3} = n^2 \sum_{j=1}^{k_n} \frac{1}{\sqrt{n}} \left(\frac{j}{\sqrt{n}} \right)^3 \frac{\mathrm{e}^{-\alpha(t)j/\sqrt{n}}}{(1 - \mathrm{e}^{-\alpha(t)j/\sqrt{n}})^3}$$
$$\sim n^2 \int_0^t \frac{v^3 \mathrm{e}^{-\alpha(t)v}}{(1 - \mathrm{e}^{-\alpha(t)v})^3} \mathrm{d}v,$$

so altogether we have shown that for all $u \in \mathbb{R}$,

$$\log(\phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N)) \mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}) \xrightarrow[n \to \infty]{} -\frac{u^2}{2}. \quad \square$$

Proof of (8). To prove that (8) follows from (9), note first that for $|z| < 1$,

$$F_k(z) = \exp\left(-\sum_{j=1}^{k} \log(1 - z^j) \right) = \exp\left(\sum_{j=1}^{k} \sum_{l=1}^{\infty} \frac{z^{jl}}{l} \right),$$

so for $z = x_n \mathrm{e}^{is/\sqrt{n}}$,

$$|\phi_{x_n,k_n}(s/\sqrt{n})| = \left| \frac{F(z)}{F(x_n)} \right|$$
$$= \exp\left(\sum_{j=1}^{k_n} (\mathrm{Re}(z^j) - x_n^j) + \mathrm{Re}\left[\sum_{l=2}^{\infty} \frac{1}{l} \sum_{j=1}^{k_n} (z^{jl} - x_n^{jl}) \right] \right)$$
$$\le \exp\left(\sum_{j=1}^{k_n} (\mathrm{Re}(z^j) - x_n^j) \right) = \exp\left(\sum_{j=1}^{k_n} x_n^j (\cos(js/\sqrt{n}) - 1) \right)$$
$$= \exp\left(-\sqrt{n} \sum_{j=1}^{k_n} \mathrm{e}^{-\alpha(t)j/\sqrt{n}} (1 - \cos(js/\sqrt{n})) \frac{1}{\sqrt{n}} \right).$$

Around $s = 0$ the Taylor expansion

$$\sum_{j=1}^{k_n} \mathrm{e}^{-\alpha(t)j/\sqrt{n}}(1-\cos(js/\sqrt{n}))\frac{1}{\sqrt{n}} = \frac{1}{2}\left(\sum_{j=1}^{k_n} \mathrm{e}^{-\alpha(t)j/\sqrt{n}}\frac{j^2}{n}\frac{1}{\sqrt{n}}\right)s^2 + O(s^4)$$

holds uniformly in n, since the coefficient of s^{2k} is asymptotically

$$\frac{(-1)^{k-1}}{(2k)!}\int_0^t \mathrm{e}^{-\alpha(t)u}u^{2k}\mathrm{d}u.$$

Therefore for s in some neighborhood $[-S_0, S_0]$ of 0, we have for some $A > 0$,

$$|\phi_{x_n,k_n}(s/\sqrt{n})| \le \exp(-A\sqrt{n}s^2). \tag{10}$$

For $|s| > S_0$, it is easy to check that for some $B > 0$ not depending on n and not depending on t for $t > T$,

$$\sum_{j=1}^{k_n} \mathrm{e}^{-\alpha(t)j/\sqrt{n}}(1-\cos(js/\sqrt{n}))\frac{1}{\sqrt{n}} > B$$

(approximate the sum by an integral, and take $B = \frac{1}{2}\inf_{|s|>S_0}\int_0^T \mathrm{e}^{-\pi u/\sqrt{6}}(1 - \cos(su))\mathrm{d}u > 0$). This leads to the estimate

$$|\phi_{x_n,k_n}(s/\sqrt{n})| \le \exp(-B\sqrt{n}), \qquad (|s| > S_0). \tag{11}$$

Now (8) follows readily from (10) and (11), because

$$\begin{aligned}\int_{-\pi\sigma_{x_n,k_n}(N)}^{\pi\sigma_{x_n,k_n}(N)} &\phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}\mathrm{d}u \\ &= \int_{|\sqrt{n}u/\sigma_{x_n,k_n}|\le S_0} \phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}\mathrm{d}u \\ &\quad + \int_{S_0<|\sqrt{n}u/\sigma_{x_n,k_n}|<\pi\sigma_{x_n,k_n}(N)} \phi_{x_n,k_n}(u/\sigma_{x_n,k_n}(N))\mathrm{e}^{-inu/\sigma_{x_n,k_n}(N)}\mathrm{d}u.\end{aligned}$$

In the first term, the integrand is dominated by $\exp(-A(n^{3/2}/\sigma^2_{x_n,k_n}(N))u^2) = \exp(-A'u^2)$; therefore this term converges to $\sqrt{2\pi}$ by the dominated convergence theorem. The second term is bounded in absolute value by

$$2\pi\sigma_{x_n,k_n}(N)\exp(-B\sqrt{n}) \xrightarrow[n\to\infty]{} 0. \quad \square$$

4. Proofs of Theorems 2 and 3

4.1. *Proof of Theorem 2*

Use (4):

$$q_d(n, t\sqrt{n}) = p\left(n - \frac{dt\sqrt{n}(t\sqrt{n}-1)}{2}, t\sqrt{n}\right) = p(n', t'\sqrt{n'}) \sim \frac{g(t')}{n'}\mathrm{e}^{H(t')\sqrt{n'}},$$

where

$$n' = n - \frac{dt^2}{2}n + \frac{dt}{2}\sqrt{n} = n(1 - dt^2/2)\left(1 + \frac{dt}{2(1 - dt^2/2)\sqrt{n}}\right)$$
$$t' = t\sqrt{n/n'} = \frac{t}{\sqrt{1 - dt^2/2}} \cdot \left(1 + \frac{dt}{2(1 - dt^2/2)\sqrt{n}}\right)^{-1/2}.$$

(Again, it may be checked that assuming that the relevant quantities are integers does no harm.) Now we have

$$g(t') \sim g\left(\frac{t}{1 - dt^2/2}\right)$$
$$\left(1 + \frac{dt}{2(1 - dt^2/2)\sqrt{n}}\right)^{\pm 1/2} = 1 \pm \frac{dt}{4(1 - dt^2/2)\sqrt{n}} + O(1/n).$$

Therefore,

$$\begin{aligned}
H(t')\sqrt{n'} &= \sqrt{n}\sqrt{1 - dt^2/2}\left(1 + \frac{dt}{4(1 - dt^2/2)\sqrt{n}} + O(1/n)\right) \\
&\quad \times \left(H\left(\frac{t}{\sqrt{1 - dt^2/2}}\right) - H'\left(\frac{t}{\sqrt{1 - dt^2/2}}\right)\right. \\
&\quad \left. \times \frac{dt^2}{4(1 - dt^2/2)^{3/2}\sqrt{n}} + O(1/n)\right) \\
&= \left(\sqrt{1 - dt^2/2}H\left(\frac{t}{\sqrt{1 - dt^2/2}}\right)\right)\sqrt{n} \\
&\quad + \left(\frac{dt}{4\sqrt{1 - dt^2/2}}H\left(\frac{t}{\sqrt{1 - dt^2/2}}\right)\right. \\
&\quad \left. - \frac{dt^2}{4(1 - dt^2/2)}H'\left(\frac{t}{\sqrt{1 - dt^2/2}}\right)\right) + O(1/\sqrt{n}) \\
&= K_d(t)\sqrt{n} + \frac{dt}{2\sqrt{1 - dt^2/2}}\alpha\left(\frac{t}{\sqrt{1 - dt^2/2}}\right) + O(1/\sqrt{n})
\end{aligned}$$

(using (6)). This implies (5). □

4.2. *Proof of Theorem 3*

We begin by showing that γ_d, β_d are the coordinates of the global maximum of the function $K_d(t)$. Introduce auxiliary variables

$$x = \frac{t}{\sqrt{1 - dt^2/2}}, \qquad y = 1 - \mathrm{e}^{-\alpha(x)x}.$$

Then we have

$$t = \frac{x}{\sqrt{1 + dx^2/2}}, \qquad x = \frac{-\log(1-y)}{\sqrt{\mathrm{Li}_2(y)}}$$
$$H(x) = 2\sqrt{\mathrm{Li}_2(y)} + \frac{\log y \cdot \log(1-y)}{\sqrt{\mathrm{Li}_2(y)}}$$
$$K_d(t) = \frac{2\mathrm{Li}_2(y) + \log y \cdot \log(1-y)}{(\mathrm{Li}_2(y) + (d/2)\log^2(1-y))^{1/2}}$$
$$t = \frac{-\log(1-y)}{(\mathrm{Li}_2(y) + (d/2)\log^2(1-y))^{1/2}}.$$

Differentiating $K_d(t)$ as a function of the variable y now gives, after a lengthy computation (which is best done by computer),

$$\frac{\mathrm{d}}{\mathrm{d}y} K_d(t) = \frac{\sqrt{2} \cdot (d\log(1-y) - \log y) \cdot (2y\mathrm{Li}_2(y) - (1-y)\log^2(1-y))}{(1-y) \cdot y \cdot [d\log^2(1-y) + 2\mathrm{Li}_2(y)]^{3/2}}.$$

Note that y is in the range $0 < y < 1$. The function $2y\mathrm{Li}_2(y) - (1-y)\log^2(1-y)$ is positive in $(0, 1)$ (its derivative is $2\mathrm{Li}_2(y) + \log^2(1-y)$). It follows that the critical point y_d is the solution of the equation $(1-y)^d = y$, and by substituting t and $K_d(t)$ above one obtains the expressions for γ_d, β_d given in Theorem 3.

Theorem 3 will now follow by summing $q_d(n, k)$ over the appropriate range of values of k, and expanding $K_d(t)$ into a Taylor series around its critical point $t = \gamma_d$. The only potential obstacle is the lack of complete uniformity in (5), that prevents ruling out a significant contribution for $q_d(n, k)$ coming from very small or very large values of k. Only an upper bound on $q(n, k)$ is necessary, since in the vicinity of the maximum point, where a lower bound might be necessary, (5) holds uniformly. We will make use of the following lemma.

Lemma 2. *For any $n \in \mathbb{N}$ and $t > 0$,*

$$P(n, t\sqrt{n}) \leq \mathrm{e}^{H(t)\sqrt{n}}.$$

Proof. Note that for any $k \in \mathbb{N}$ and $0 < x < 1$, since $F_k(x) = \sum_{n=0}^{\infty} P(n, k)x^n$,

$$P(n, k)x^n \leq F_k(x),$$

or

$$\log P(n, k) \leq -n\log x + \log F_k(x).$$

Set $k = t\sqrt{n}$ and $x = \mathrm{e}^{-u/\sqrt{n}}$. Then

$$\log F_k(x) = -\sum_{j=1}^{k} \log(1 - x^j) \leq \sqrt{n} \int_0^t -\log(1 - \mathrm{e}^{-us})\mathrm{d}s$$
$$= \sqrt{n}\frac{1}{u}(\mathrm{Li}_2(1) - \mathrm{Li}_2(\mathrm{e}^{-ut})) = \sqrt{n}\left(\frac{\mathrm{Li}_2(1 - \mathrm{e}^{-ut})}{u} - t\log(1 - \mathrm{e}^{-ut})\right).$$

So

$$\log P(n,k) \le \sqrt{n}\left(u + \frac{\mathrm{Li}_2(1-\mathrm{e}^{-ut})}{u} - t\log(1-\mathrm{e}^{-ut})\right).$$

Setting $u = \alpha(t)$ gives the desired bound. $\square$

We are now ready to conclude the proof. Write

$$K_d(t) = \beta_d - \frac{1}{2\sigma_d^2}(t-\gamma_d)^2 + O((t-\gamma_d)^3),$$
$$k_d(t) = k_d(\gamma_d) + O(t-\gamma_d),$$

both big O's being uniform in some neighborhood $[\gamma_d - \epsilon, \gamma_d + \epsilon]$ of γ_d. Let

$$m_d = \max\{K_d(t) : t \in [0, \sqrt{2/d}] \setminus [\gamma_d - \epsilon, \gamma_d + \epsilon]\} < \beta_d.$$

For an integer $k = \gamma_d\sqrt{n} + u\sigma_d n^{1/4}$, $u \in \mathbb{R}$, (5) gives

$$q_d(n,k) = \frac{(1+O(u/n^{1/4}))k_d(\gamma_d)}{n}\exp\left(\beta_d\sqrt{n} - \frac{u^2}{2} + O(u^3/n^{1/4})\right)$$
$$= (1+O(u^3/n^{1/4}))\frac{k_d(\gamma_d)}{n}\mathrm{e}^{\beta_d\sqrt{n}}\cdot \mathrm{e}^{-u^2/2}.$$

Summing over k, and using Lemma 2 outside $[\gamma_d - \epsilon, \gamma_d + \epsilon]$ and the uniformity in (5) inside, gives

$$\sum_{k \le \gamma_d\sqrt{n} + u\sigma_d n^{1/4}} q_d(n,k) \sim \frac{k_d(\gamma_d)\sigma_d}{n^{3/4}}\mathrm{e}^{\beta_d\sqrt{n}}\int_{-\infty}^{u}\mathrm{e}^{-x^2/2}\mathrm{d}x.$$

Setting $u = \infty$ (this is permitted, again because of Lemma 2) gives

$$q_d(n) = \sum_k q_d(n,k) \sim \frac{k_d(\gamma_d)\sqrt{2\pi}\sigma_d}{n^{3/4}}\mathrm{e}^{\beta_d\sqrt{n}}.$$

This is Theorem 3(a). The ratio of the last two equations gives Theorem 3(c), which implies Theorem 3(b). $\square$

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