Fano colourings of cubic graphs and the Fulkerson Conjecture

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Abstract

A Fano colouring is a colouring of the edges of a cubic graph by points of the Fano plane such that the colours of any three mutually adjacent edges form a line of the Fano plane. It has recently been shown by Holroyd and Škoviera (J. Combin. Theory Ser. B, to appear) that a cubic graph has a Fano colouring if and only if it is bridgeless. In this paper we prove that six, and conjecture that four, lines of the Fano plane are sufficient to colour any bridgeless cubic graph. We establish connections of our conjecture to other conjectures concerning bridgeless cubic graphs, in particular to the well-known conjecture of Fulkerson about the existence of a double covering by 1-factors in every bridgeless cubic graph.

Keywords: cubic graph, edge-colouring, Fano plane, snark, Fulkerson Conjecture

1 Introduction

A *Fano colouring* is an edge-colouring of a cubic graph which uses points of the Fano plane as colours subject to the condition that any three colours meeting at a vertex form a line. With the classical concept of a Tait colouring Fano colourings share the property that the colours of any two adjacent edges determine the colour of the third edge adjacent to them. Moreover, a colouring which uses the same line at all vertices is nothing but the usual 3-edge-colouring. This makes Fano colourings a natural generalization of Tait colourings and a suitable tool for investigating cubic graphs that are not 3-edge-colourable – that is, *snarks*.

It is convenient to consider Fano colourings within a broader context of edge-colourings by Steiner triple systems. Recall that a *Steiner triple system* $\mathcal{S} = (X, B)$ of order n is a collection B of three-element subsets (called *triples* or *blocks*) of a set X of n *points* such

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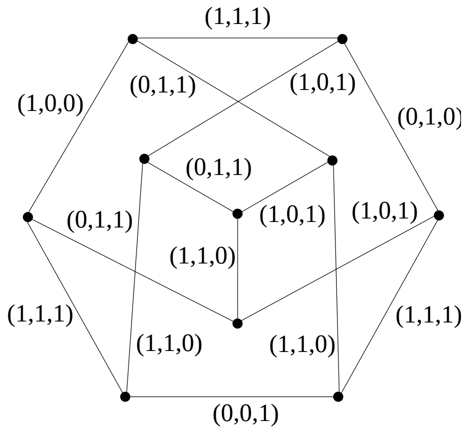

Figure 1: A Fano colouring of the Petersen graph

that each pair of points is together present in exactly one triple. Given a Steiner triple system $\mathcal{S}$, an $\mathcal{S}$-*colouring* of a cubic graph G is a colouring of the edges of G by points of $\mathcal{S}$ such that the three colours occurring at any vertex form a block of $\mathcal{S}$. (We allow our graphs to have multiple edges and sometimes even loops. However, edge-colourings make sense only for loopless graphs.)

The study of edge-colourings by Steiner triple systems was initiated by Archdeacon [1], and the first result in this direction is due to Fu [7] who described two classes of bridgeless cubic graphs that admit a Fano colouring. Recently, Holroyd and Škoviera [10] substantially improved Fu's results by showing that every bridgeless cubic graph has an $\mathcal{S}$-colouring for every Steiner triple system $\mathcal{S}$ of order greater than 3.

This paper is devoted to an investigation of colourings which employ the smallest non-trivial Steiner triple system, the Fano plane. As follows from [10], every bridgeless cubic graph has a Fano colouring. Here we will deal with further properties of Fano colourings. Our main concern is the following problem: *How many lines of the Fano plane are needed to colour a given cubic graph?*

It transpires that the answer to this question does not actually involve the structure of lines employed by a Fano colouring – it only depends on their number. In particular, any Fano colouring of a non-3-edge-colourable graph requires at least four lines, and there is only one admissible configuration of four lines which can occur. On the other hand, all seven lines are never needed.

Theorem 1.1 *Every bridgeless cubic graph has a Fano colouring which uses at most six different lines.*

These facts suggest that all snarks fall into one of three classes according to the number of lines required by a Fano colouring. While the class of graphs that require four lines is infinite (see Theorem 4.1 and Example 4.2), we have not been able to find any representatives of the remaining two classes. Moreover, with the help of a computer we have verified that up to 30 vertices no snarks of this sort exist. This justifies the following conjecture.

Conjecture 1.2 *Every bridgeless cubic graph has a Fano colouring which uses at most four lines.*

It was conjectured by Fulkerson [8] that in every bridgeless cubic graph there is a collection of six perfect matchings such that each edge belongs to exactly two of them. We show that Fulkerson's conjecture implies Conjecture 1.2, and that the latter conjecture is equivalent to the statement that every bridgeless cubic graph contains three perfect matchings with empty intersection, conjectured by Fan and Raspaud in [6]. Finally we propose two weaker versions of Conjecture 1.2 involving configurations of five lines.

2 Colourings and configurations

We start with a definition of the Fano plane. Here is one of the possibilities: The *Fano plane* is an incidence structure $\mathcal{F} = (P, L)$ consisting of a set P of seven *points*, say $P = \{1, 2, \ldots, 7\}$, and a collection of seven *lines* $L = \{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}\}$. A point p and a line l such that $p \in l$ are said to be *incident*.

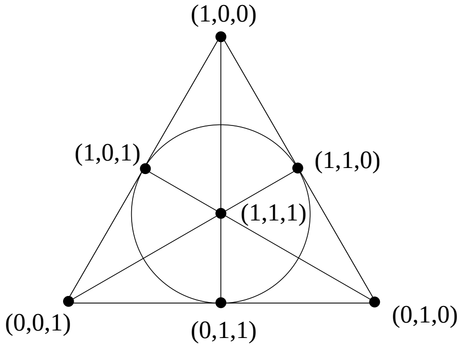

Figure 2: The Fano plane

If we label each point $i \in P$ by its binary code $(i_1, i_2, i_3) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, we obtain the usual representation of $\mathcal{F}$ as the projective plane $PG(2, 2)$ of order 7. Hence, the following three axioms are satisfied in $\mathcal{F}$:

(P1) There is exactly one line through any pair of distinct points.

(P2) Any two lines intersect in exactly one point.

(P3) There are at least four points in general position.

In this model of the Fano plane, three points i, j, and k form a line precisely when $i + j + k = 0$ in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The representation of $\mathcal{F}$ as $PG(2, 2)$ also shows that the Fano plane is highly symmetrical. Automorphisms of $\mathcal{F}$ are permutations of P which take lines to lines; they are often called *collineations*. Since the cyclic permutation (1372456) is a collineation, the Fano plane is point-transitive. In fact, the collineation group of $\mathcal{F}$ is 2-transitive on P (see [2], Chapter 2) which means that any two ordered pairs of points can be mapped onto each other by a collineation. In particular, the Fano plane is line-transitive. In addition to this, the Fano plane is *self-dual* (cf. [2], Chapter 3). Roughly speaking, this means that the role of points and lines in $\mathcal{F}$ can be interchanged and the incidence relation between points and lines can be reversed to obtain an isomorphic projective plane. More precisely, the *dual* structure $\mathcal{F}^* = (P^*, L^*)$, where $P^* = L$ and L^* consists of all bundles of lines through a point, is isomorphic to $\mathcal{F}$. Due to this isomorphism, any two ordered pairs of lines can be mapped to each other by a collineation of $\mathcal{F}$. So the collineation group of $\mathcal{F}$ is 2-transitive on L, too.

Symmetries of the Fano plane have a strong impact on the properties of Fano colourings ($\mathcal{F}$-colourings, for short). In this context, again, the representation of $\mathcal{F}$ as $PG(2, 2)$ is very useful, because it offers an important alternative approach to Fano colourings based on the concept of a nowhere-zero flow. For any graph G let $D(G)$ denote the set which is obtained by replacing each edge of G with a pair of oppositely directed *darts*. Each dart z, including those on loops, has its *inverse* dart $z^{-1} \neq z$ which is incident with the same vertices but has opposite direction. For an arbitrary vertex v, we let $D(v)$ be the set of all darts emanating from v. Clearly, these sets partition the whole dart-set.

Let A by an Abelian group with additive notation. Define an A-*flow* on G to be a function $\xi : D(G) \to A$ satisfying the following two conditions:

(F1) $\xi(z^{-1}) = -\xi(z)$, for each dart $z \in D(G)$,

(F2) $\sum_{z \in D(v)} \xi(z) = 0$, for each vertex $v \in V(G)$.

A flow ξ is said to be *nowhere-zero* if $\xi(z) \neq 0$ for each dart $z \in D(G)$.

Observe that if every element of A is self-inverse, then $\xi(z) = \xi(z^{-1})$ for each dart z, and we may simply view an A-flow on G as a function defined on the edges of G rather than on darts. Note that in this case the group A will be isomorphic to a direct product of copies of $\mathbb{Z}_2$.

Since the lines of $\mathcal{F}$ correspond to triples of points whose sum is 0, it follows immediately from the definition that an $\mathcal{F}$-colouring of a graph G is just a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow on G. An important consequence of this approach is that a cubic graph which has a bridge cannot be $\mathcal{F}$-coloured because an arbitrary flow takes value 0 on any bridge. Conversely, every bridgeless cubic graph G admits a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow (see [5], Chapter 6, or [12]), and hence G can be $\mathcal{F}$-coloured. Thus a cubic graph is Fano-colourable if and only if it is bridgeless (see Theorem 1.1 of [10]).

On the other hand, the original geometric approach to Fano colourings suggests the question about the structure of configurations of points and lines in the Fano plane that can occur in a colouring of a cubic graph. Here a *configuration* is simply a set of lines (that is, three-element blocks) together with all incident points. If $\mathcal{C}$ is an arbitrary configuration, not necessarily contained in the Fano plane, then a $\mathcal{C}$-*colouring* is a colouring by points of $\mathcal{C}$ such that the colours at any vertex form a block of $\mathcal{C}$.

Sometimes it is useful to transform a $\mathcal{C}$-colouring into a $\mathcal{D}$-colouring for a suitable configuration $\mathcal{D}$. This is done simply by mapping the points of $\mathcal{C}$ to points of $\mathcal{D}$ in such a way that each triple of $\mathcal{C}$ becomes a triple of $\mathcal{D}$. Examples of such transformations can be found further in this section and in Section 5.

The following proposition describes the structure of all configurations in the Fano plane according to the number of lines. Two of them have to be specifically mentioned: $C_{15} = \{\{a,b,c\},\{a,d,e\},\{a,f,g\},\{c,e,g\}\}$, a configuration of four lines covering seven points, and $C_{16} = \{\{a,b,c\},\{a,d,e\},\{b,d,f\},\{c,e,f\}\}$, a configuration of four lines covering six points which is also known as the *Pasch configuration*.

Proposition 2.1 *For $m = 1,2,5,6$ there is only one m-line configuration in the Fano plane, up to collineation. For $m = 3,4$ there are exactly two non-isomorphic m-line configurations in the Fano plane: C_{15} and C_{16}, if $m = 4$, and their complements, if $m = 3$.*

Proof. For $m = 1$ and $m = 2$ the result follows from the above-mentioned fact that the collineation group acts 2-transitively on the set of lines. Since the configurations of five or six lines are complements of these, the result follows for $m = 5$ and $m = 6$ as well.

As regards the second statement, Grannell et al. [9] show that there exist only three non-isomorphic configurations of four lines covering at most seven points, but only two of them, C_{15} and C_{16}, occur in the Fano plane. This proves the result for $m = 4$. Again, for $m = 3$ the result follows by passing to the complements. □

Let us consider a Fano colouring of a bridgeless cubic graph with minimum number of lines. If the graph is 3-edge-colourable, then the colouring uses only one line. We therefore concentrate on bridgeless cubic graphs that have no 3-edge-colouring. For simplicity, and in accordance with [4], [13] and [15], we call these graphs *snarks.*

For snarks we have the following result.

Proposition 2.2 *Let G be a snark which is $\mathcal{C}$-colourable for some configuration $\mathcal{C}$ in the Fano plane. Then $\mathcal{C}$ covers all seven points and has at least four lines.*

Proof. Suppose that $\mathcal{C}$ misses a point of $\mathcal{F}$. Since $\mathcal{F}$ is point-transitive, we may assume that the point $(1,1,1)$ is not contained in $\mathcal{C}$. By mapping $(0,0,1)$ and $(1,1,0)$ to 1, $(0,1,0)$ and $(1,0,1)$ to 2, and $(1,0,0)$ and $(0,1,1)$ to 3, we obtain a 3-edge-colouring of G, contradicting the assumption that G is a snark.

The smallest number of lines that cover all seven points of the Fano plane is clearly 3. By Proposition 2.1, there is only one such configuration, up to collineation, namely the bundle of three lines through a point. We show that this configuration cannot occur in a colouring of a snark. Without loss of generality, consider the set of lines containing the point $(1,1,1)$, that is $\mathcal{C} = \{\{(0,0,1),(1,1,0),(1,1,1)\},\{(0,1,0),(1,0,1),(1,1,1)\}, \{(0,1,1),(1,0,0),(1,1,1)\}\}$. Suppose that G admits a $\mathcal{C}$-colouring. Then, by mapping $(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ to 1, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$ to 2, and $(1,1,1)$ to 3, we obtain a 3-edge-colouring of G, which is a contradiction again. Thus the least number of lines of the Fano plane which can colour a snark is 4, as claimed. □

The previous two propositions have the following important consequence.

Corollary 2.3 *For each $m \leq 7$ there exists, up to collineation, at most one configuration of m lines in the Fano plane which can occur in a colouring of a snark.*

Thus the problem of colouring a cubic graph by the minimum number of lines of the Fano plane does not, in fact, involve the structure formed by these lines.

Remark 2.4 The third configuration of three-element blocks on at most seven points mentioned in the proof of Proposition 2.1 is $C_{14} = \{\{a,b,f\},\{a,c,e\},\{b,c,d\},\{e,f,g\}\}$. This configuration plays an important role in general Steiner colourings, as shown in [10]. However, C_{14} itself does not colour any snark. Indeed, any C_{14}-colouring of a cubic graph can easily be transformed into a 3-edge-colouring via the mapping $a,d,g \mapsto 1$, $b,e \mapsto 2$, and $c,f \mapsto 3$. Thus a cubic graph is C_{14}-colourable if and only if it is 3-edge-colourable, and this occurs precisely when it is C_{16}-colourable. To see the latter equivalence, one simply embeds C_{16}, the Pasch configuration, into the Fano plane and uses the argument of the proof of Proposition 2.2. The problem of which graphs are C_{15}-colourable will be discussed separately in Section 4.

3 Six-line colourings

The aim of this section is to show that any bridgeless cubic graph can be coloured by at most six lines of the Fano plane. Up to collineation, the Fano plane contains only one configuration of six lines (Proposition 2.1), so we can restrict ourselves to colourings by the configuration $\mathcal{K}$ consisting of all lines different from $\{(0,1,0),(1,0,0),(1,1,0)\}$.

Our proof uses the following improvement of the famous Petersen's one-factor theorem.

Theorem 3.1 (Plesník [14]) *Let G be an $(r-1)$-edge-connected r-valent graph ($r>0$) of even order, and let A be an arbitrary set of $r-1$ edges. Then $G-A$ has a 1-factor.*

The crucial step in the proof of Theorem 1.1 is the following lemma which solves the problem for cyclically 4-edge-connected graphs. Recall that a graph is said to be *cyclically k-edge-connected* if no edge-cut involving fewer than k edges leaves two or more components containing circuits. It is easy to see that for $k \le 3$ a cubic graph is cyclicaly k-edge-connected precisely when it is k-(edge)-connected.

Lemma 3.2 *Let G be a cyclically 4-edge-connected cubic graph, and let v be a vertex of G. Then G admits a $\mathcal{K}$-colouring such that the edges incident with v receive the colours $(0,0,1)$, $(0,1,0)$, and $(0,1,1)$ in any chosen order.*

Proof. It is sufficient to prove that G has a $\mathcal{K}$-colouring under which the edges e_1, e_2 and e_3 incident with v will receive the colours $(0,0,1)$, $(0,1,0)$, and $(0,1,1)$ in the given order. By Theorem 3.1, there is a 1-factor F_1 in G which contains the edge e_2. Let us contract each circuit of the complementary 2-factor F_2 to a vertex. Note that the resulting graph G' may contain multiple edges and loops even when G was simple. Since G is cyclically 4-edge-connected, G' is 4-edge-connected and so, by a theorem of Jaeger (see [12], Theorem 4.7 or [5] Proposition 6.4.4), G' has a nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2$-flow ϕ. Clearly, ϕ can be chosen in such a way that $\phi(e_2')=(0,1)$ where e_2' is the edge of G' corresponding to e_2 in G.

Let λ be the mapping which embeds $\mathbb{Z}_2\times\mathbb{Z}_2$ into $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ by sending an arbitrary element (g,h) of $\mathbb{Z}_2\times\mathbb{Z}_2$ to $(g,h,0)$. Then $\lambda\phi$ is a nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$-flow on G'. Expand the vertices of G' back to the original circuits of G, so that the edges of G' become edges of G, with the flow values determined by $\lambda\phi$. Now let $C=x_0x_1\dots x_{q-1}$ be any circuit of F_2, and let y_i be the edge of F_1 adjacent to x_i and x_{i+1}, for $i\in\mathbb{Z}_q$. If C has chords, each of them will have two labels. Moreover, in the circuit of F_2 which contains the vertex v we choose the labelling so that $e_1=x_0$ and $e_3=x_{q-1}$.

For $r=0,1,\dots,q-1$ we now define

$$\lambda\phi(x_r)=(0,0,1)+\sum_{i=0}^{r-1}\lambda\phi(y_i).$$

Since $\sum\lambda\phi(y_i)=0$, this definition is unambiguous, and the Kirchhoff law (condition (F2)) is satisfied at each vertex of G. It follows that $\lambda\phi$ is a Fano colouring of G with $\phi(e_1)=(0,0,1)$, $\phi(e_2)=(0,1,0)$, and $\phi(e_3)=(0,1,1)$. Observe that any line of $\mathcal{F}$ which occurs in $\lambda\phi$ contains two points whose third coordinate is 1. Hence the line $\{(0,1,0),(1,0,0),(1,1,0)\}$ is never used, and the colouring employs at most six lines. □

Proof of Theorem 1.1 We show that every bridgeless cubic graph G admits a $\mathcal{K}$-colouring. We proceed by induction on the number of vertices of G. For the basis of induction we observe that by Lemma 3.2 the result holds for all cyclically 4-edge-connected graphs.

Now let G contain a cycle-separating edge-cut of size 2 or 3. Among such cuts there must be one with the property that at least one of the resulting components, say Q, has no cycle-separating edge-cut of size smaller than 4. Take both the cut and the component Q to be of minimum size. Denote the cut by S, and let R be the other component of $G-S$. Let $G_1 = G/R$ and $G_2 = G/Q$ be the graphs formed from G by contracting R or Q, respectively, to a single vertex. Let r and q be the respective vertices resulting from the contraction.

Assume that $|S| = 3$. Since the order of G_2 is smaller than that of G, the induction hypothesis implies that G_2 has a $\mathcal{K}$-colouring. Observe that the configuration $\mathcal{K}$ is line-transitive because $\mathcal{F}$ is 2-transitive. Thus our $\mathcal{K}$-colouring can be chosen in such a way that the colours at q form the line $\{(0,0,1),(0,1,0),(0,1,1)\}$. By Lemma 3.2, G_1 can be $\mathcal{K}$-coloured in such a way that at the vertex r the colours $(0,0,1)$, $(0,1,0)$, and $(0,1,1)$ appear in any prescribed order. We choose the ordering so that the colours at r exactly match the colours at q. The resulting colourings of G_1 and G_2 can now be combined into a $\mathcal{K}$-colouring of the whole G.

If $|S| = 2$, we suppress the 2-valent vertices q and r to obtain cubic graphs G_1 and G_2, and use similar arguments as above to establish a $\mathcal{K}$-colouring of G. The details are left to the reader. □

4 Four-line colourings

Proposition 2.2 shows that a Fano colouring of a snark gives rise to a configuration of at least four lines covering all seven points of the Fano plane. It is therefore natural to examine Fano colourings which use exactly four lines. By Corollary 2.3, there is only one such configuration in the Fano plane, up to collineation, so we can restrict ourselves to colourings by the configuration $\mathcal{L} = \{\{(0,0,1),(1,1,0),(1,1,1)\},\{(0,1,0),(1,0,1),(1,1,1)\}, \{(1,0,0),(0,1,1),(1,1,1)\},\{(0,1,1),(1,0,1),(1,1,0)\}\}$. Clearly, $\mathcal{L}$ is isomorphic to C_{15}.

Theorem 1.1 and Corollary 2.3 suggest that all snarks can be divided into three classes according to whether they require four, five or six lines in a Fano colouring, respectively. Our next result implies that that the first of these classes is infinite.

Theorem 4.1 *Let G be a bridgeless cubic graph which has a 2-factor F whose odd circuits can be arranged into pairs $\{C_1, D_1\}, \{C_2, D_2\}, \ldots, \{C_n, D_n\}$ such that for each pair $\{C_i, D_i\}$ there exist two distinct edges e_i and f_i in $G-F$ incident with both C_i and D_i. Then G has a Fano colouring by at most four lines.*

Proof. We shall construct a Fano colouring ϕ of G such that the colours meeting at each vertex form a line of $\mathcal{L}$. Let $F_1 = G-F$ be the 1-factor of G complementary to F. First, we set $\phi(e_i) = (0,0,1)$ and $\phi(f_i) = (0,1,0)$ for each $i = 1,2,\ldots,n$, and assign $(0,1,1)$ to the remaining edges of F_1. Now we colour the 2-factor. On each even circuit we will alternate the colours $(1,0,1)$ and $(1,1,0)$; there are two possibilities for each even circuit and both of them are feasible. It remains to colour the edges of odd circuits of F. Let T be an odd circuit and assume that it is incident with the edges e_r and f_r. These two edges divide T into two paths, one even and the other odd. On the odd path we will alternate the colours $(1,1,1)$ and $(1,0,0)$ beginning with $(1,1,1)$. On the even path we colour the edge adjacent to e_r by $(1,1,0)$, and then alternate $(1,0,1)$ and $(1,1,0)$. It is a routine matter to check that the resulting colouring ϕ of G is indeed an $\mathcal{L}$-colouring. □

Example 4.2 Let us look at Fano colourings of the Isaacs snarks I_k, also known as the flower snarks [11]. For $k \geq 3$ odd, let I_k be the graph which has $\{v_i, w_i, u_i, z_i; i \in \mathbb{Z}_k\}$ as its vertex-set, and $\{v_iw_i, v_iu_i, v_iz_i, w_iu_{i+1}, u_iw_{i+1}, z_iz_{i+1}; i \in \mathbb{Z}_k\}$ as its edge-set. It is well known that these graphs are not 3-edge-colourable. The whole vertex set of an Isaacs snark I_k can be covered by two disjoint odd circuits $C = w_0v_0u_0w_1v_1u_1 \dots w_{k-1}v_{k-1}u_{k-1}w_0$ and $D = z_0z_1 \dots z_{k_1}z_0$. Since C and D are connected by the edges v_0z_0 and v_1z_1 (for example), Theorem 4.1 yields that I_k has a Fano colouring which uses only four lines. □

Snarks which need more than four lines in every Fano colouring appear to be very difficult to find. In fact, we do not know any single example of a cubic graph which would require exactly five or exactly six lines. With the help of a computer we have checked all 31646 cyclically 4-edge-connected snarks of girth at least 5 on at most 30 vertices compiled by Brinkmann [3]. We have found that for each snark G from this catalogue, and for each vertex v of G, there exists an $\mathcal{L}$-colouring of G such that the edges incident with v can be coloured by the points of any line of $\mathcal{L}$ in any order. This fact, together with a suitable treatment of short circuits (similar to one used in the proof of Theorem 4.8) enables us to extend $\mathcal{L}$-colourings of these snarks to all (bridgeless) snarks of order 30 or less, and to conclude that *all bridgeless cubic graphs of order not exceeding* 30 *are $\mathcal{L}$-colourable.* This leads us to propose the following conjecture:

Conjecture 4.3 (Four-Line Conjecture) *Every bridgeless cubic graph has a Fano colouring which uses at most four lines.*

Figure 1 shows an $\mathcal{L}$-colouring of the Petersen graph. If G is a cubic graph which has a homomorphism into the Petersen graph such that any three mutually adjacent edges of G are mapped to three mutually adjacent edges of the Petersen graph, then G is clearly $\mathcal{L}$-colourable, too. However, the *Petersen Colouring Conjecture* of Jaeger [12] claims that such a homomorphism exists for any bridgeless cubic graph. Thus the Four-Line Conjecture is implied by the Petersen Colouring Conjecture.

In the next two theorems we explore the relationship of Conjecture 4.3 to other conjectures concerning bridgeless cubic graphs. We start with the well-known conjecture of Fulkerson [8], sometimes attributed also to Berge.

Conjecture 4.4 (Berge-Fulkerson) *In every bridgeless cubic graph, there exists a family of six* 1*-factors such that each edge appears in exactly two of them.*

Theorem 4.5 *The Berge-Fulkerson conjecture implies the Four-Line Conjecture.*

Proof. Let G be a bridgeless cubic graph that fulfils the Berge-Fulkerson conjecture. Then there are six 1-factors $M_1, M_2, \dots, M_6$ in G which together cover each edge twice. We will show that G has an $\mathcal{L}$-colouring ϕ which derives from the chosen family of 1-factors. In order to define ϕ, we assign a value $\phi(e)$ in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ to each edge of G depending on which of the fifteen possible combinations of 1-factors it belongs to. The assignment is given in the following table.

$e \in$	$\phi(e)$
$M_1 \cup M_2$	$(0,0,1)$
$M_1 \cup M_3$	$(0,1,0)$
$M_1 \cup M_4$	$(0,1,1)$
$M_1 \cup M_5$	$(0,1,1)$
$M_1 \cup M_6$	$(0,1,1)$

$e \in$	$\phi(e)$
$M_2 \cup M_3$	$(1,0,0)$
$M_2 \cup M_4$	$(1,0,1)$
$M_2 \cup M_5$	$(1,0,1)$
$M_2 \cup M_6$	$(1,0,1)$
$M_3 \cup M_4$	$(1,1,0)$

$e \in$	$\phi(e)$
$M_3 \cup M_5$	$(1,1,0)$
$M_3 \cup M_6$	$(1,1,0)$
$M_4 \cup M_5$	$(1,1,1)$
$M_4 \cup M_6$	$(1,1,1)$
$M_5 \cup M_6$	$(1,1,1)$

It is straightforward to verify that for any possible distribution of 1-factors at a vertex, the three assigned values of ϕ form a line of $\mathcal{L}$. $\square$

In 1994, Fan and Raspaud stated the following interesting conjecture ([6], Conjecture 4.2.).

Conjecture 4.6 (Fan-Raspaud) *Every bridgeless cubic graph contains three perfect matchings with empty intersection.*

Theorem 4.7 *The Fan-Raspaud Conjecture and the Four-Line Conjecture are equivalent.*

Proof. ($\Rightarrow$) Assume that a bridgeless cubic graph G contains three perfect matchings M_1, M_2, and M_3 with $M_1 \cap M_2 \cap M_3 = \emptyset$. We show that G can be $\mathcal{L}$-coloured. Define a mapping $\phi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by setting $\phi(e) = (\phi_1(e), \phi_2(e), \phi_3(e))$ where $\phi_i(e) = 1$ if and only if $e \notin M_i$, $i = 1, 2, 3$. Since each coordinate mapping ϕ_i is the characteristic function of a 2-factor, ϕ_i is a $\mathbb{Z}_2$-flow. In turn, ϕ is a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Define the *weight* of an edge of G to be the number coordinates in $\phi(e)$ equal to 1. Then for each vertex v, the sum of weights of the edges incident with v is 6. There are three possible distributions of weights at a vertex, namely $2 + 2 + 2$, $3 + 2 + 1$, and $3 + 3 + 0$. The last possibility is excluded because the edge with weight 0 would belong to $M_1 \cap M_2 \cap M_3$. Thus ϕ is a nowhere-zero flow, that is, a Fano colouring. Moreover, both remaining distributions of weights represent a line of $\mathcal{F}$ which belongs to $\mathcal{L}$, so ϕ is indeed an $\mathcal{L}$-colouring.

($\Leftarrow$) Now assume that G has an $\mathcal{L}$-colouring ϕ. Let $\phi = (\phi_1, \phi_2, \phi_3)$, and for $i = 1, 2, 3$ define C_i to be the set of all edges e for which $\phi_i(e) = 1$. Each ϕ_i is a $\mathbb{Z}_2$-flow because ϕ is an nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-flow. Thus each C_i is a cycle, that is, a set of disjoint circuits. We show that C_i is, in fact, a 2-factor. Since ϕ only uses lines of $\mathcal{L}$, at each vertex of G the distribution of weights of edges (defined as above) is either $3 + 2 + 1$ or $2+2+2$. Note that the weight of an edge equals the number of C_i's containing that edge. In both types of distributions it is then easy to see that any vertex in question is covered by each of C_1, C_2 and C_3, implying that C_i is a 2-factor. Define M_i to be the 1-factor complementary to C_i, and suppose that there is an edge $e = uv$ in $M_1 \cap M_2 \cap M_3$. The sum of weights at u (and also at v) is 6 but the distribution is $3+3+0$ because the weight of e is 0. However, this contradicts the fact that ϕ is a nowhere-zero flow. $\square$

The following theorem examines the properties of the smallest potential counterexample to our Four-Line Conjecture.

Theorem 4.8 *The smallest counterexample to the Four-Line Conjecture, if it exists, must be snark of girth at least* 5*.*

Proof. Let G be a bridgeless cubic graph of minimum order which is not $\mathcal{L}$-colourable. Clearly, G must be a snark. We show that G has no short cycles.

Suppose that G has a 2-circuit D formed by a pair of parallel edges f and g. Contract D to a vertex v incident with edges e_1 and e_2 which were originally adjacent to both f and g, and then suppress v thereby creating a new edge e. The resulting cubic graph G' has an $\mathcal{L}$-colouring ϕ'. This colouring extends to an $\mathcal{L}$-colouring ϕ of G by setting $\phi(e_1) = \phi(e_2) = \phi'(e)$, $\phi(f) = a$, and $\phi(g) = b$ where $\{\phi(e), a, b\}$ is any line of $\mathcal{L}$ containing $\phi(e)$. Thus G has girth at least 3.

If G contains a triangle, say T, we proceed in the standard way. Form G' by contracting T into a vertex v. Again, G' has an $\mathcal{L}$-colouring ϕ'. We extend ϕ' to an $\mathcal{L}$-colouring ϕ

of G. If g is and edge of T adjacent to edges e and f which in G' are incident with the vertex v, then we set $\phi(g) = \phi(e) + \phi(f)$. By doing this for each edge of T we obtain an $\mathcal{L}$-colouring of G. Thus the girth of G is at least 4.

Finally assume that G contains a circuit $Q = defg$ of length 4. Let us remove from G two opposite edges of Q, say e and g. Suppress the 2-valent vertices incident with d and f to create new edges d' and f', respectively, thereby obtaining a cubic graph G'. This smaller graph has an $\mathcal{L}$-colouring ϕ'. We now extend ϕ' to an $\mathcal{L}$-colouring ϕ of G. Let d_1dd_2 and f_1ff_2 be the paths in G obtained by reinserting the original end-vertices of d and f into d' and f', respectively. Define $\phi(d) = \phi'(d') + (1,1,1)$, $\phi(d_1) = \phi(d_2) = \phi'(d')$, $\phi(f) = \phi'(f') + (1,1,1)$, $\phi(f_1) = \phi(f_2) = \phi'(f')$, and $\phi(e) = \phi(g) = (1,1,1)$. Since the point $(1,1,1)$ lies on each line of $\mathcal{L}$, it is easy to see that the resulting mapping ϕ is indeed an $\mathcal{L}$-colouring. Thus the girth of G is at least 5. □

It is generally accepted that a "non-trivial" snark is cyclically 4-edge-connected and has girth at least 5. It would therefore be desirable to establish a lower bound on the cyclic connectivity of a smallest counterexample to the Four-Line Conjecture. A natural approach to this problem is through finding a suitable sufficient condition. For example, to prove that the smallest counterexample to the Four-Line Conjecture is (cyclically) 3-edge-connected it is enough to show that the following statement:

If a graph G is $\mathcal{L}$-colourable, then for each edge e it admits $\mathcal{L}$-colourings ϕ and ψ such that $\phi(e)$ and $\psi(e)$ have different weights.

However, we have been unable to prove it.

5 Concluding remarks

The comparison of Conjecture 1.2 to Theorem 1.1 suggests an obvious problem: to improve the bound upper bound of six lines in Theorem 1.1 by (at least) one, that is, to show – if true – that every bridgeless cubic graph can be coloured by at most five lines of the Fano plane. As we know from Section 2, there is, up to collineation, only one configuration to consider. Nevertheless, one can think of another possible relaxation of the Four-Line Conjecture. In Remark 2.4 combined with Corollary 2.3 we have shown that the configuration C_{15} is the only possible configuration of four lines that can occur in a Steiner colouring of a snark. However, C_{15} is contained in every non-trivial Steiner triple system (see [9]). Thus there is no reason to exclude from consideration colourings by other possible five-block extensions of C_{15} that can occur in a Steiner triple system.

A straightforward but slightly tedious analysis shows that, up to isomorphism, there are only two such configurations. For all other configurations $\mathcal{C}$ there exists a block-preserving mapping of $\mathcal{C}$ into one of these two configurations which transforms a $\mathcal{C}$-colouring into a colouring by one of them. The first configuration is $\mathcal{M}_1 = \{\{a,b,c\}, \{a,d,e\}, \{a,f,g\}, \{c,e,g\}, \{c,d,f\}\}$ which extends the representation of C_{15} given in Section 2 with the block $\{b,c,f\}$. It occurs in the Fano plane as $\mathcal{L} \cup \{(0,0,1),(0,1,0), (0,1,1)\}$, for example. The second configuration, $\mathcal{M}_2 = \{\{a,b,c\},\{a,d,e\},\{a,f,g\}, \{c,e,g\},\{b,c,f\}\}$, extends C_{15} with the block $\{b,c,f\}$. Since $\mathcal{M}_2$ contains a pair of parallel lines, it cannot be found in the Fano plane. It is embedded in the affine plane $AG(2,3)$ of order 9 as $\{\{(0,0),(0,1),(0,2)\},\{(2,0),(2,1),(2,2)\}, \{(0,0),(1,1),(2,2)\},\{(0,1),(1,1), (2,1)\},\{(0,2),(1,1),(2,0)\}\}$, for example. Recall that $AG(2,3)$ is a Steiner triple system whose points are the elements of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and a triple $\{x,y,z\}$ of points forms a block if and only if $x + y + z = 0$.

As a result of these considerations, there are two weaker versions of the Four-Line Conjecture involving a configuration of five lines:

Conjecture 5.1 *Every bridgeless cubic graph has an* $\mathcal{M}_1$*-colouring.*

Conjecture 5.2 *Every bridgeless cubic graph has an* $\mathcal{M}_2$*-colouring.*

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