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European Journal of Combinatorics 24 (2003) 649–663

European Journal of Combinatorics

www.elsevier.com/locate/ejc

On the order of countable graphs

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Received 5 March 2003; accepted 28 April 2003

Abstract

A set of graphs is said to be *independent* if there is no homomorphism between distinct graphs from the set. We consider the existence problems related to the independent sets of countable graphs. While the maximal size of an independent set of countable graphs is 2^{ω} the *On Line* problem of extending an independent set to a larger independent set is much harder. We prove here that singletons can be extended ("partnership theorem"). While this is the best possible in general, we give structural conditions which guarantee independent extensions of larger independent sets.

This is related to universal graphs, rigid graphs (where we solve a problem posed in J. Combin. Theory B 46 (1989) 133) and to the density problem for countable graphs.

MSC: 03C; 05C; 05E; 06A

Keywords: Density; Partially ordered sets; Rigid graphs; Universal graphs

1. Introduction and statement of results

Given graphs $G = (V, E)$, $G' = (V', E')$ a homomorphism is any mapping $V \to V'$ which preserves all the edges of G:

$$\{x, y\} \in E \Longrightarrow \{f(x), f(y)\} \in E'$$

This is briefly denoted by $f : G \to G'$. We indicate the existence of a homomorphism by $G \to G'$ and in the context of partially ordered sets this will be also denoted by $G \leq G'$. $\leq$ is obviously a quasiorder.

$\leq$ is a very rich quasiorder which has been studied in several contexts, see [13] for a survey of this area. For example it has been shown (and this also not difficult to see)

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doi:10.1016/S0195-6698(03)00064-7

that any poset may be represented by $\leq$; see [11, 18] for less easy results in this area. A particular case is an *independent set* of graphs which can be defined as an independent set (or antichain) in this quasiorder. Here we are interested in a seemingly easy question:

Independence problem (shortly IP)

Given a set $\{G_\iota; \iota \in I\}$ of graphs does there exist a graph G such that $\{G_\iota; \iota \in I\}$ together with G form an independent set of graphs?

This problem has been solved for finite sets of (finite or infinite) graphs in [10, 11]. The general case is much harder and it is relatively consistent to assume the negative solution (this is related to the Vopěnka Axiom, see [7, 11]).

In this paper we discuss IP for countable graphs.

We prove the following:

Theorem 1.1. *For every countable graph G the following two statements are equivalent:*

(i) *there exists a countable graph G' such that G and G' are independent.*
(ii) *G is not bipartite and it does not contain an infinite complete subgraph.*

Both conditions given in (ii) are clearly necessary. The non-bipartite comes from the general (cardinality unrestricted) independence problem as the only finite exception and the absence of an infinite clique is due to the cardinality restriction.

This modest looking result (which we could call *Partnership theorem*: non-bipartite countable graphs without K_ω have independent partners) has a number of consequences and leads to several interesting problems. First, we want to mention that the above result (and the IP) is related to universal graphs.

Let $\mathcal{K}$ be a class of graphs. We say that a graph $U \in \mathcal{K}$ is *hom-universal* (with respect to $\mathcal{K}$) if $G \leq U$ for every $G \in \mathcal{K}$, [15].

Note that a graph U may be hom-universal with respect to a class $\mathcal{K}$ without being universal (in the usual sense: any graph from $\mathcal{K}$ is a subgraph of U; see [5, 6, 8, 9, 19] for an extensive literature about universal graphs). For example the triangle K_3 is hom-universal for the class $\mathcal{K}$ of all 3-colorable graphs and obviously this class does not have a finite universal graph. On the other hand clearly any universal graph is also hom-universal.

Let GRA_ω denote the class of all countable non-bipartite graphs without an infinite complete subgraph (which is denoted by K_ω). It is well known that the class GRA_ω does not have a universal graph. The same proof actually gives that GRA_ω has no hom-universal graph. (Here is a simple proof which we sketch for the completeness: suppose that U is hom-universal for GRA_ω. Denote by $U \oplus x$ the graph which we obtain from U by addition of a new vertex x joined to all the vertices of U. Then there exists $f : U \oplus x \to U$. Define the vertices $x_0, x_1, \ldots$ by induction: $x_0 = x, x_{i+1} = f(x_i)$. It is easy to see that all these vertices form a complete graph in U.)

Theorem 1.1 is a strengthening of the non-hom-universality of GRA_ω. In fact Theorem 1.1 is best possible in the following sense:

Corollary 1.1. *For a positive integer t the following two statements are equivalent:*

(i) *For every finite set $\{G_1, \ldots, G_t\}$ of graphs from GRA_ω there exists a graph $G \in GRA_\omega$ such that G and G_i are independent for all $i = 1, \ldots, t$.*
(ii) *$t = 1$.*

Proof. There are many examples proving (i) $\Rightarrow$ (ii). For example consider the complete graph K_n and let U_n be any universal (and thus also hom-universal) countable K_n-free graph (U_n exists by [4]). Then the set $\{K_n, U_n\}$ cannot be extended to a larger independent set as every graph G either contains K_n or is homomorphic to U_n.

An example for $t > 2$ consists of an independent set of finite graphs $G_0, \dots, G_{t-1}$ and a countable graph U, $U \ngeq G_i, i = 0, \dots, t-1$ which is universal for all graphs G satisfying $G \ngeq G_i, i = 0, \dots, t-1$. Such a graph exists by [1, 12]. (Note also that an analogous result does not hold for infinite sets. To see this let $G_i = C_{2i+3}$ be the set of all cycles of odd length. Then there is no G which is independent of all graphs G_i.) $\square$

Theorem 1.1 is in the finite (or cardinality unrestricted) case also known as the (Sparse) Incomparability Lemma [13, 15]. We can formulate this as follows:

Theorem 1.2. *For any choice of graphs G, H, G non-bipartite, satisfying $G < H, H \nleq G$ there exists a graph G' such that $G' < H$, $H \nleq G'$ and such that G and G' are independent.*

If G has a finite chromatic number then G' may be chosen finite.

(The last part of Theorem 1.2 may be seen as follows (sketch): if $\chi(G) = k$ then take G'' with $\chi(G'') > k$ and without cycles $\leq l$ such that G contains an odd cycle of length $\leq l$. Then G and G'' are independent. If $\chi(G')$ is large then also graphs G and $G'' \times H$ are independent, see [10].)

We do not know whether Theorem 1.2 holds if all the graphs are supposed to be countable. Partial results are included in Section 5.

Theorem 1.1 is also related to the notion of a rigid graph: A graph G is said to be *rigid* if its only homomorphism $G \to G$ is the identical mapping. We shall prove the following:

Theorem 1.3. *For any countable graph G not containing K_ω there exists a countable rigid graph G' containing G as an induced subgraph.*

The history of this result goes to [2] (the finite case), to [3] (the unrestricted cardinality case), and to [15] (the optimal chromatic number for the finite case). Theorem 1.3 solves an open problem proposed in [15].

Finally, let us mention that Theorem 1.1 is related to the concept of density.

Given a class $\mathcal{K}$ of graphs and two graphs $G_1, G_2 \in \mathcal{K}$, $G_1 < G_2$, we say that the pair (G_1, G_2) is a *gap* in $\mathcal{K}$ if there is no $G \in \mathcal{K}$ satisfying $G_1 < G < G_2$. The *density problem* for class $\mathcal{K}$ is the problem to characterize all gaps in $\mathcal{K}$. (If there are a "few" gaps then we have a tendency to say that class $\mathcal{K}$ is *dense*; see [10, 11, 16].)

Our theorem has the following corollary:

Corollary 1.2. *Any pair (G, K_ω) fails to be a gap in the class of all countable graphs.*

Proof. Let $G < K_\omega$, $G \in GRA_\omega$, be given. According to Theorem 1.1 there exists $G' \in GRA_\omega$ such that $G' \nrightarrow G$. Then we have $G < G + G' < K_\omega$. $\square$

Note that we used the easier part of Theorem 1.1. This is being discussed below and some particular positive examples of the density of the class GRA_ω are stated. However the characterization of all gaps for the class GRA_ω remains an open problem. In the class

GRA_ω there are infinitely many gaps. This is in a sharp contrast with finite graphs where the trivial gap (K_1, K_2) is the only gap, see [16].

The paper is organized as follows: In Section 2 we give some no-homomorphism conditions which will aid us in Section 3 in the proof of Theorem 1.1. In Section 4 we define high and low graphs and show their relationship to the independence problem. In Section 5 we prove Theorem 1.3. In Section 6 we give structural conditions which allow us to prove that certain graphs are high and thus generalize Theorems 1.1 and 1.3 to other graphs H than K_ω. We also modify the proof of Theorem 1.1 to this setting. This yields a more direct proof and allows us (at least in principle) to hunt for partners. We find classes of graphs where the independent extension property holds. Section 7 contains some remarks and problems.

2. Necessary conditions for the existence of a homomorphism

Given two graphs G_1, G_2 it is usually not easy to prove that $G_1 \nrightarrow G_2$. We shall use the following two basic facts:

Suppose $G_1 \to G_2$. Then

(i) If G_1 contains an odd cycle of length $<l$ then also G_2 contains an odd cycle of length $<l$.

(ii) $\chi(G_1) \leq \chi(G_2)$ (where χ denotes the chromatic number).

To this well known list (which cannot be expanded much more even in the finite case) we add the rank function which we are going to introduce as follows:

Let $G = (V, E)$ be a graph in GRA_ω. By K_n we denote the complete graph on $n = \{0, 1, \dots, n-1\}$. Consider the set $h(K_n, G)$ of all homomorphisms $K_n \to G$ and denote by T^G the union of all the sets $h(K_n, G), n = 1, 2, \dots$. We think of T^G as a (relational) tree ordered by the relation $f \subseteq g$.

It is clear and well known that

(i) T^G is a relational tree;

(ii) T^G has no infinite branches;

(iii) We can define ordinal $\mathrm{rk}(T^G) < \omega_1$ the *ordinal rank function* of T^G.

(For completeness recall the definition of the ordinal rank function: for a tree T without infinite branches $\mathrm{rk}(T)$ is defined as $\sup\{\mathrm{rk}(T_\iota) + 1\}$ over all branches of T at the root.) Put $\mathrm{rk}(G) = \mathrm{rk}(T^G)$. We have the following (perhaps folkloristic):

Lemma 2.1. *If* $G_1 \leq G_2$ *then* $\mathrm{rk}(G_1) \leq \mathrm{rk}(G_2)$.

Proof. Let $f : G_1 \to G_2$ be a homomorphism. Then for every n we have a natural mapping $h(f) : h(K_n, G_1) \to h(K_n, G_2)$ defined by $h(f)(g) = f \circ g$. $h(f)$ is a level preserving mapping $T^{G_1} \to T^{G_2}$ and thus $\mathrm{rk}(G_1) \leq \mathrm{rk}(G_2)$. ☐

For every ordinal $\alpha < \omega_1$ and graph G on ω consider the following undirected graph $K_\omega^{\langle\alpha\rangle}$:

the vertices of $K_\omega^{\langle\alpha\rangle}$: all decreasing sequences of ordinal numbers $<\alpha$;

the edges of $K_\omega^{\langle\alpha\rangle}$: all pairs $\{\nu, \mu\}$ satisfying $\nu \subseteq \mu$, by this symbol we mean the containment of sequences ν and μ (as initial segments).

One can say that $K_\omega^{\langle\alpha\rangle}$ is a tree of cliques with the total height α.

The following holds for any $\alpha < \omega_1$:

(i) $K_\omega^{\langle\alpha\rangle} \in GRA_\omega$;

(ii) $\mathrm{rk}(K_\omega^{\langle\alpha\rangle}) = \alpha$;

(iii) $K_\omega^{\langle\mathrm{rk}(G)+1\rangle} \nrightarrow G$.

(This gives yet another proof that there is no countable hom-universal K_ω-free graph.)

Put $G^0 = K_\omega^{\langle\mathrm{rk}(G)+1\rangle}$ and let us look at the statement of Theorem 1.1. We have $G^0 \not\leq G$ (by (iii)) and thus if also $G \not\leq G^0$ then we are done. So we can assume the following situation: $G \leq G^0$ and $G^0 \not\leq G$. Now if G^1 is any graph satisfying $G^1 \not\leq G^0$ then necessarily $G^1 \not\leq G$ (as otherwise $G^1 \leq G \leq G^0$) and thus by the same token we can assume $G \leq G^1$. This strategy of the proof will be followed in the next section.

3. Proof of Theorem 1.1

We proceed by contradiction: let $G \in GRA_\omega$ be a graph which is comparable to every other graph in GRA_ω. By Theorem 1.2 the chromatic number of G is infinite.

We shall construct graphs G^0, G^1, G^2 such that $G^0 \not\leq G$ (and thus $G < G^0$), $G^1 \not\leq G^0$ (thus $G < G^1$) and $G^2 \not\leq G^1$ (and thus $G < G^2$). Using a construction similar to the one of G^2, we define a family $\{G_\eta\}$ of graphs which satisfy $G_\eta \not\leq G^1$ and thus $G < G_\eta$. Then the existence of some η, such that $G_\eta < G$ will give rise to a contradiction.

The graph $G^0 = K_\omega^{\langle\mathrm{rk}(G)+1\rangle}$ was constructed in the previous section.

Definition of G^1. The vertices of G^1: $\omega \times 2$. The edges of G^1: all pairs of the form $\{(n, i), (m, i)\}$ where $\lfloor\sqrt{n}\rfloor = \lfloor\sqrt{m}\rfloor$, $i = 0, 1$ and of the form $\{(n, 0), (m, 1)\}$ where $n < m$.

Thus G^1 is a "half graph" where the vertices are "blown up" by complete graphs of increasing sizes.

Claim 1. $G^1 \nrightarrow G^0$.

Proof (of Claim 1). Assume to the contrary: Let $f : G^1 \to G^0$ be a homomorphism. As f is restricted to each of the complete graphs in each of the sets $\omega\times\{0\}$, $\omega\times\{1\}$ is monotone we can find an infinite set $X \subset \omega$ such that the mapping f restricted to the set $X \times \{0\}$ is injective. The set $Y = \{f(x); x \in X \times \{0\}\}$ is an infinite set in $V(G^0) = V(K_\omega^{\langle\mathrm{rk}(G)+1\rangle})$. The graph $K_\omega^{\langle\mathrm{rk}(G)+1\rangle}$ is defined by the tree $T, \mathrm{rk}(T) = \mathrm{rk}(G) + 1$ and thus by either the König lemma (or Ramsey theorem) the set Y either contains an infinite chain (i.e. a complete graph in G^0) which is impossible, or Y contains an infinite independent set in T and thus also in G^0.

So Y are the vertices of a star in T with center y. y is a function $y : K_n \to G$. Choose $n_\star \in \omega$ such that the set $X \cap (n_\star \times \{0\})$ has at least $n + 1$ elements.

Now the function f restricted to the set $\{n_\star^2, n_\star^2+1, \ldots, n_\star^2+n_\star+1\} \times \{1\}$ is injective and if $(i,1)$ is any vertex of this set then $f(i,1)$ is connected to all vertices $f(m,0)$ for $m \in X \cap [0, n_\star]$. This implies that $f(m,0) \subset y$ for every $m \in X \cap [0, n_\star]$. But this is a contradiction. $\square$

Construction of G^2. The vertices of $V(G^2) = A_0 \cup A_1 \cup A_2$ where $A_0 = \{r\}$, and A_1 and A_2 are infinite sets (all three mutually disjoint). The set A_1 is disjoint union of finite complete graphs denoted by K_i^1 (isomorphic to K_i), $i \in \omega$. The set A_2 is disjoint union of finite complete graphs denoted by $K_{x,j}^2$ (isomorphic to K_j), $j \in \omega$. The edges of G^2 are the edges of all indicated complete graphs together with all edges of the form $\{r, x\}$, $x \in A_1$ and all pairs of the form $\{x, y\}$, $x \in A_1$, $y \in \cup_{j\in\omega} V(K_{x,j}^2)$.

So the graph G^2 is a tree of depth 2 with infinite branching with all its vertices "blown up" by complete graphs of increasing sizes.

Claim 2. $G^2 \nrightarrow G^1$.

Proof. The proof is easy using the main property of the half graph: all the vertices of one of its "parts" (i.e. of the set $\omega \times \{1\}$) have finite degree.

Assume to the contrary that $f : G^2 \to G^1$ is a homomorphism (for G^1 we preserve all the above notation). We shall consider two cases according to the value of $f(r)$.

Case 1. $f(r) = (n, 1)$ for some $n \in \omega$.

But then the subgraph of G^1 induced by the neighborhood $N(n,1)$ of the vertex $(n,1)$ has a finite chromatic number (as $(n,1)$ has finite degree in G^1) whereas the neighborhood of r in the graph G^2 has the infinite chromatic number (as this neighborhood is the disjoint union of complete graphs K_i^1, $i \in \omega$).

Case 2. $f(r) = (n, 0)$ for some $n \in \omega$.

By a similar argument as in Case 1 we see that not all vertices $f(x)$, $x \in A_1$ can be mapped to the vertices of the set $\omega \times \{0\}$ (as by the connectivity of the subgraph of G^2 formed by $A_0 \cup A_1$ this graph would be mapped to a finite complete graph). Thus let $f(x_1) = (m, 1)$ for an $x_1 \in A_1$. But then the neighborhood $N(m,1)$ of $(m,1)$ in the graph G^2 has a finite chromatic number whereas x_1 has infinite chromatic number (in G^1). $\square$

Thus we see that $G^2 \nrightarrow G^1$ and consequently $G \to G^2$. The last example which we shall construct will be a family of graphs $\{G_\eta\}$. This has to be treated in a more general framework and we do it in a separate subsection.

3.1. *Tree like graphs*

We consider the following generalization of the above construction of G^2:

Let $\mathcal{G}$ be an infinite set of finite graphs of the form $G_{j,i}$, $i, j \in \omega$ which satisfies:

(i) $\chi(G_{j,i}) \geq i$;

(ii) $G_{j,i}$ does not contain odd cycles of length $\leq j$;

(iii) All the graphs are vertex disjoint.

Let $T = (V, E)$ be a graph tree (i.e. we consider just the successor relation) defined as follows: $V = A_0 \cup A_1 \cup A_2$ where $A_0 = \{r\}$, $A_1 = \omega$ and $A_2 = \omega \times \omega$. The edges of T are all edges of the form $\{(r, i)\}$, $i \in \omega$ and all pairs of the form $\{i, (i, j)\}$, $i, j \in \omega$.

Let $\eta : V \to \omega \times \omega$ be any function.

Define the graph G_η as follows: the set of vertices of G_η is the union of all graphs $G_{\eta(x)}, x \in V$. The edges of G_η are edges of all graphs $G_{\eta(x)}, x \in \omega$ together with all edges of the form $\{a, b\}$ where $a \in G_{\eta(x)}$, $b \in G_{\eta(y)}$ and $\{x, y\} \in E$.

Then we have analogously as in Claim 2:

Claim 3. *Let $\eta : V \to \omega$ be any function and let $\eta_1, \eta_2 : V \to \omega$ be defined by $\eta(x) = (\eta_1(x), \eta_2(x))$. If η_2 is unbounded on A_1 and on the subsets of A_2 of the form $\{i\} \times \omega, i \in \omega$, then $G_\eta \not\leq G^1$.*

Now, consider the graph G again. As $\chi(G)$ is infinite denote by K the minimal number of vertices of a subgraph G' of G with chromatic number 5 (by compactness it is K that is finite). Let $\eta : V \to \omega$ be any function which is unbounded on ω and each of the sets $\{i\} \times \omega, i \in \omega$ and moreover which satisfies $\eta_1(i) \geq K$ for every $i \in \omega$.

It is $G_\eta \not\leq G^1$ by Claim 3. Thus $G \leq G_\eta$. In this situation we prove the following (and this will conclude the proof of Theorem 1.1).

Claim 4. $G \not\leq G_\eta$.

Proof. Assume to the contrary: let $f : G \to G_\eta$. Then the vertices of the subgraph G' are mapped into a set $\cup_{i \in I} G_{\eta(i)}$ where I is a finite subset of V. Denote by G'' the image of G' in G_η. Due to the tree structure of G_η we have that $\chi(G'') \leq 2 \max_{i \in I} \chi(G'' \cap G_{\eta(i)})$.

As $\eta(i) \geq K$ and thus all graphs $G'' \cap G_{\eta(i)}$ are bipartite. This implies $\chi(G'' \cap G_{\eta(i)}) \leq 2$ and finally we get $\chi(G') \leq \chi(G'') \leq 4$, a contradiction. □

4. Independent families

In a certain sense Theorem 1.1 captures the difficulty of independent extension property. The pair K_3, U_3 (see proof following Theorem 1.1 in the Section 1) cannot be extended to a large independent set *because* U_3 is a rich graph. This can be made precise. Towards this end we first modify the ordinal rank function for graphs below a given graph H. We return to these results in Section 6.

Let G, H be infinite graphs. Assume that the vertices of H are ordered in a sequence of type ω. We can thus assume that H is a graph on ω. Denote by H_n the subgraph of H induced on the set $\{0, 1, \ldots, n-1\}$.

Consider the set $h(H_n, G)$ of all homomorphisms $H_n \to G$ and denote by T_H^G the union of all the sets $h(H_n, G)$, $n = 1, 2, \ldots$. We think of T_H^G as a (relational) tree ordered by the relation $f \subseteq g$. T_H^G is called the *H-valued tree of G* (with respect to a given ω-ordering of H).

It is clear that

(i) T_H^G is a (relational) tree;

(ii) T_H^G has no infinite branches.

Thus we can define ordinal $\mathrm{rk}(T_H^G) < \omega_1$ the *ordinal rank function* of T_H^G.

Put $\mathrm{rk}_H(G) = \mathrm{rk}(T_H^G)$ (*the ordinal H-rank* of G). We have then the following:

Lemma 4.1. *Let G_1, G_2 be graphs with $H \not\leq G_1$ and $H \not\leq G_1$. Then $G_1 \leq G_2$ implies* $\mathrm{rk}_H(G_1) \leq \mathrm{rk}_H(G_2)$.

Proof. Let $f : G_1 \to G_2$ be a homomorphism. Then for every n we have a natural mapping $h(f) : h(H_n, G_1) \to h(H_n, G_2)$ defined by $h(f)(g) = f \circ g$. The mapping $h(f)$ is level preserving mapping $T_H^{G_1} \to T_H^{G_2}$ and thus $\mathrm{rk}_H(G_1) \leq \mathrm{rk}_H(G_2)$. $\square$

For a countable graph G on ω and every ordinal $\alpha < \omega_1$ define the following graph $G^{\langle\alpha\rangle}$:

The vertices of $G^{\langle\alpha\rangle}$ are all decreasing sequences of ordinal numbers $<\alpha$; the edges of $G^{\langle\alpha\rangle}$ are all pairs $\{\nu, \mu\}$ satisfying $\nu \subseteq \mu$ and $\{\ell(\nu), \ell(\mu)\} \in E(G)$. (Recall that $\ell(\nu)$ is the length of the sequence ν.)

One can say that $G^{\langle\alpha\rangle}$ is a tree of copies of G_n (G_n is the graph induced by G on the set $\{0, 1, \ldots, n-1\}$ with the total height α. (This notation also explains the rather cumbersome notation $K_\omega^{\langle\alpha\rangle}$.)

We have the following:

Lemma 4.2. (i) *$G^{\langle\alpha\rangle} \leq G$;*

(ii) *If $\alpha \leq \beta$ then also $G^{\langle\alpha\rangle} \leq G^{\langle\beta\rangle}$;*

(iii) *$G \leq H$ if and only if $G^{\langle\alpha\rangle} \leq H$ for every $\alpha < \omega_1$.*

Proof. This is an easy statement. The existing homomorphisms are canonical level-preserving homomorphisms. Let us mention just (iii):

If $f : G \to H$ then $G^{\langle\alpha\rangle} \to H$ by composition of f with the map guaranteed by (i). Thus assume $G \not\leq H$ and $G^{\langle\alpha\rangle} \leq H$ for any $\alpha < \omega_1$. In this case the ordinal G-rank of H is defined and $\mathrm{rk}_G(H) = \alpha < \omega_1$. As $\mathrm{rk}_G(G^{\langle\alpha+1\rangle}) = \alpha + 1 > \mathrm{rk}_G(H)$ we get a contradiction. $\square$

We say that G is *α-low* if $G \leq G^{\langle\alpha\rangle}$. A *low graph* is a graph which is low for some $\alpha < \omega_1$, a graph is *high* if it is not low.

We have the following

Theorem 4.1. *Let $G_1, \ldots, G_t$ be an independent set of countable connected graphs including at least one high graph. Then there exists a countable graph G such that $G, G_1, \ldots, G_t$ is an independent set.*

Corollary 4.1. *Any finite set of high graphs can be extended to a larger independent set.*

Proof. Choose the notation such that the graphs $G_1, \ldots, G_{s-1}$ are low while graphs $G_s, \ldots, G_t$ are high (the case $s = 0$ corresponds to the set of all high graphs).

Choose $\alpha < \omega_1$ such that for any $m \in \{s, \ldots, t\}$ and $n \in \{1, \ldots, s-1\}$ the graph $G_n^{\langle\alpha\rangle}$ has no homomorphism to G_m. This is possible as by the high–low assumption for every m, n as above there is no homomorphism $G_m \to G_n$ and thus for some $\alpha(m,n) < \omega_1$ we have $G_n^{\langle\alpha(m,n)\rangle} \nrightarrow G_m$. (This also covers the case $s = 0$.) Put $\alpha' = \max \alpha(m,n)$ and $\alpha = \max \mathrm{rk}_{G_n^{\langle\alpha(m,n)\rangle}} G_m$.

We define

$$G = \sum_{i=0}^{t-s} G_{s+i}^{\langle\alpha\rangle}$$

and prove that G is the desired graph. Fix $n \in \{0, \dots, s-1\}$ and choose $m \in \{s, \dots, t\}$ arbitrarily. Then $G_m^{\langle\alpha\rangle} \nrightarrow G_n$ and thus $G \nrightarrow G_n$ as claimed.

In the opposite direction for every $m, n \in \{s, \dots, t\}$ we have $G_m \nrightarrow G_m^{\langle\alpha\rangle}$ by G_m high and $G_m \nrightarrow G_n^{\langle\alpha\rangle}$ by the choice of α (i.e. as α is large enough). As G_m is a connected graph G_m maps to G if and only if it maps to one of the components. Thus $G_m \nrightarrow G$ and we are done. □

Remark. Corollary 4.1 shows that we have an extension property providing we "play" with high graphs. This is in agreement with the "random building blocks" used in the proofs of universality, see [11].

5. Rigid graphs

We prove Theorem 1.3.

Let G be a countable graph not containing K_ω, we can assume that G is infinite. In fact we can assume without loss of generality that every edge of G belongs to a triangle and that G is connected (we simply consider a graph which contains G as an induced subgraph).

Let $G_1 \in GRA_\omega$ form an independent pair with G (G_1 exists by Theorem 1.1). We can assume without loss of generality that also every edge of G_1 belongs to a triangle. For that it is enough to attach to every edge of G_1 a pendant triangle; (as every edge of G belongs to a triangle) these triangles do not influence the non-existence of homomorphisms between G and G_1. G_1 can also be assumed to be connected.

Let G_0 be a countable rigid graph without triangles. The existence of G_0 follows from the existence of a countable infinite rigid relation (take a one way infinite path on ω together with arc $(0, 3)$) by replacing every edge by a finite triangle free rigid graph; see e.g. [13, 15, 18].

Let $\mu : V(G) \to V(G_0)$ and $\nu : V(G_0) \to V(G_1)$ be bijections. Define the graph G' as the disjoint union of graphs G, G_0, G_1 together with the matchings $\{\{x, \mu(x)\}; x \in V(G)\}$ and $\{\{x, \nu(x)\}; x \in V(G_0)\}$.

We prove that G' is rigid (G' obviously contains G as an induced subgraph).

Let $f : G' \to G'$ be a homomorphism. As the matching edges and the edges of G_0 do not lie in a triangle we have either $f(V(G)) \subseteq V(G)$ or $f(V(G)) \subseteq V(G_1)$. However the last possibility fails as G and G_1 are independent. Similarly, we have either $f(V(G_1)) \subseteq V(G_1)$ or $f(V(G)) \subseteq V(G)$ and the last possibility again fails.

Thus we have $f(V(G)) \subseteq V(G)$ and $f(V(G_1)) \subseteq V(G_1)$. As the vertices of G_0 are the only vertices joined both to $V(G)$ and $V(G')$ we also have $f(V(G_0)) \subseteq V(G_0)$. However G_0 is rigid and thus $f(x) = x$ for every $x \in V(G_0)$. Finally as G and G_0, G_0 and G_1 are joined by a matching we have that $f(x) = x$ for all $x \in V(G')$.

Remark. This "sandwich construction" may be the easiest proof of a statement of this type (cf. [2, 3, 15, 18]). This proves also the analogous statement for every infinite κ (also for the finite case) providing that we use the fact that on every set there exists a rigid relation. This has been proved in [21], and e.g. [14] for a recent easy proof.

6. Gaps below *H*

We say that a gap $G < H$ is a *gap below* H. In the introduction we derived from Theorem 1.1 that there are no gaps below K_ω. It is well known that finite undirected graphs have no non-trivial gap (except $K_1 < K_2$), see [16, 20]. Also infinite graphs (with unrestricted cardinalities) have no non-trivial gaps [10]. However note that classes of graphs with bounded cardinality (such as GRA_ω) may have many non-trivial gaps. For example if $H = K_n$ then let U_n be the hom-universal K_n-free universal graph. Consider the graph $G_n = U_n \times K_n$ (the product here is the categorical product defined by projection-homomorphisms). Then $G_n < K_n$ and it is easy to see that G_n is also a K_n-free hom-universal graph (universal for graphs below K_n). Now if $G < K_n$ then also $G \leq G_n$ and thus (G_n, K_n) is a gap (below K_n). In fact this holds for other finite graphs, see [12]. It seems to be difficult to find gaps formed by infinite graphs only. Here we give some explanation of this difficulty. We use the ordinal H-rank function for graphs below H which was introduced in Section 4.

It is not necessarily true that $H^{\langle\alpha\rangle} \in GRA_H$. We defined above H to be an *α-low graph* if $H^{\langle\alpha\rangle} \in GRA_H$. Here are sufficient conditions for low and high:

For a graph F we say that an infinite subset X of $V(F)$ is *separated by a subset* C if for any two distinct vertices x, y of X there is no path $x = x_0, x_1, \dots, x_t = y$ in F such that none of the vertices $x_1, \dots, x_{t-1}$ belong to C (thus possibly $x, y \in C$).

Recall, that graphs G and G' are said to be *hom-equivalent* if $G \leq G' \leq G$. This is denoted by $G \simeq G'$.

We say that graph F is *H-connected* if no infinite subset X of $V(F)$ is separated by a subset C such that $C \simeq H'$ for a finite subgraph H' of H. H is said to have *finite core* if H is equivalent to its finite subgraph. Any graph with infinite chromatic number has no finite core (and this is far from being a necessary condition). The following then holds:

(iv) If H is H-connected without a finite core then $H^{\langle\alpha\rangle} \in GRA_H$.

Proof. H is infinite. Let $f : H \to H^{\langle\alpha\rangle}$ be a homomorphism. As H is not equivalent to any of its finite subgraph there exists an infinite set $X \subset V(H)$ such that f restricted to the set X is injective. Then the set $f(X)$ is an infinite subset of $H^{\langle\alpha\rangle}$ and applying the König's lemma to the tree structure of $H^{\langle\alpha\rangle}$ we get that either there is an infinite chain (which is impossible as $H^{\langle\alpha\rangle}$ is H-free) or there is an infinite star. Its vertices form an independent set which is separated by the finite graph corresponding to the stem of the star. □

We have the following:

Theorem 6.1. *Let H be an H-connected graph without a finite core. Then the following holds:*

(i) *There is no gap below H;*

(ii) *GRA_H has no hom-universal graph;*

(iii) *For every $G < H$ there exists $G' < H$ such that G and G' are independent ("partners under H").*

Proof. (i) is easier. Let $G < H$. Then $\mathrm{rk}_H(G) = \alpha < \omega_1$. It is $H^{\langle\alpha+1\rangle} < H$ and thus $H^{\langle\alpha+1\rangle} \nrightarrow G$. Put $G^0 = H^{\langle\alpha+1\rangle}$ and thus we have $G < G + G^0 < H$ as needed. The same proof gives (ii).

However by Lemma 4.2 we also know that there exists $\beta > \alpha$ such that $G \not\leq H^{\langle\beta\rangle}$. This proves (iii). □

We give another proof of Theorem 6.1 (iii) which is an extension of the proof given in Sections 3 and 4. This proof is more direct and gives us more tools for hunting partners.

Proof (of Theorem 6.1 (iii)). Let $G < H$ be fixed. We proceed in a complete analogy to the above proof of Theorem 1.1 and we outline the main steps and stress only the differences. Thus let G be a counterexample. Consider $G^0 = H^{\langle\alpha+1\rangle}$. We have $G^0 \not\leq G$ and $G^0 < H$ and thus we have $G < G^0$. As G^0 has the tree structure we can find G^1 in a similar way such that $G^1 \not\leq G^0$ and $G^1 < H$. Given G^1 we then define graphs G^2 and G_μ with $G \leq G^2$ and $G \leq G_\mu$. However we have to continue (as possibly $\chi(H) \leq 4$) and also define graph G^4 with $G \leq G^4$. This will finally lead to a contradiction.

The details of this process are involved and we need several technical definitions.

An *H-partite graph* (G, c) is a graph together with a fixed homomorphism $c : G \to H$. The sets $c^{-1}(x)$ are color classes of (G, c). Given two H-partite graphs (G, c) and (G', c') the *H-join* $(G, c) \bowtie (G', c')$ is the disjoint union of (G, c) and (G', c') together with all edges $\{x, x'\}$ where $x \in V(G)$, $x' \in V(G')$ and $\{c(x), c(x')\} \in E(H)$. The graph $(G, c) \bowtie (G', c')$ is again H-partite (with the coloring denoted again by c).

Recall, that H_n is the graph H restricted to the set $\{0, \dots, n-1\}$. Let H_n^0 and H_n^1 be copies of H_n so that all the graphs H_n^0 and H_n^1, $n \in \omega$ are mutually disjoint. Without loss of generality the vertices of $V(H_n^i)$ belong to $\omega \times \{i\}$, $i = 0, 1$. The graphs H_n, H_n^0, H_n^1 are considered as H-partite graphs with the inclusion H-coloring.

Definition of G^1. The vertices of G^1 form the set $\omega \times 2$. The edges of G^1 are all pairs of the form $\{(x, i), (y, i)\}$ where $\{x, y\} \in H_n^i$ for some $n \in \omega$ together with all the edges of the graphs $H_m^0 \bowtie H_n^1$ where $m \leq n$.

G^1 is an H-partite graph with $c : G^1 \to H$ defined as the limit of all the inclusions $H_n \subset H$. We can still think of G^1 as a suitable blowing of a half graph. What is important is that the key property of half graphs holds for G^1: all the vertices in the class $\omega \times \{1\}$ have finite degree.

Claim 1. $G^1 \nrightarrow G^0$.

(Recall that $G^0 = H^{\langle\alpha+1\rangle}$.) Assume to the contrary, let $f : G^1 \to G^0$.

As H does not have a finite retract we get (by compactness) that for every m there exists n such that $H_m \not\leq H_n$. It follows that there exists an infinite set $X \subset \omega$ such that the mapping f restricted to the set $X \times \{0\}$ is injective. The set $Y = \{f(x); x \in X\}$ is then an infinite subset of the tree $H^{\langle\alpha\rangle}$, $\alpha = \mathrm{rk}_H(G)$ which defines the graph G^0 and thus by either the König lemma (or Ramsey theorem) the set Y either includes an infinite chain (i.e. a complete graph in G^0) which is impossible, or Y includes an infinite independent set in $H^{\langle\alpha\rangle}$ and thus also in G^0. So Y are the vertices of an infinite star in $T_{\alpha,H}$ with center y. y is in fact an injective homomorphism $y : H_n \to H$. Define the set C by $C = f^{-1}(\{0, \dots, n-1\})$. Then C separates X while $C \le H_n$. But this is a contradiction.

Construction of G^2. The vertices of $V(G^2) = A_0 \cup A_1 \cup A_2$ where $A_0 = \{r\}$, and A_1 and A_2 are infinite sets (all three mutually disjoint). The set A_1 is a disjoint union of graphs H_i denoted by H_i^1 (isomorphic to H_i), $i \in \omega$. The set A_2 is a disjoint union of graphs denoted by $H_{x,j}^2$ (isomorphic to H_j), $x \in A_1$, $j \in \omega$. The edges of G^2 are the edges of all indicated graphs H_i^1 and $H_{x,j}^2$ together with all edges of the form $\{r, x\}$, $x \in A_1$ and all pairs of the form $\{x, y\}$, $x \in A_1$, $y \in \cup_{j\in\omega} V(H_{x,j}^1)$, $\{c(x), c(y)\} \in E(H))$.

So the graph G^2 is a tree of depth 2 with infinite branching with all its vertices “blown up” by graphs H_n of increasing sizes, the graph induced by vertices $V(H_i^1) \cup V(H_{x,j}^2)$ is isomorphic to $H_i^1 \bowtie H_{x,j}^2$.

Claim 2. $G^2 \nrightarrow G^1$.

Proof. Assume to the contrary that $f : G^2 \to G^1$ is a homomorphism (for G^1 we preserve all the above notation). We shall consider two cases according to the value of $f(r)$.

Case 1. $f(r) = (n, 1)$ for some $n \in \omega$.

(We proceed similarly as in Case 1 of the proof of Theorem 1.1.) But then the subgraph of G^1 induced by the neighborhood $N(n, 1)$ of the vertex $(n, 1)$ can be mapped to a finite subgraph of H (as $(n, 1)$ has finite degree in G^1) whereas the neighborhood of r in the graph G^2 cannot be mapped to the finite subset of H (as this neighborhood is the disjoint union of graphs H_i^1, $i \in \omega$).

Case 2. $f(r) = (n, 0)$ for some $n \in \omega$.

This is a similar adaptation of Case 2 of the proof of Theorem 1.1. □

Next we shall define graphs G_η. We consider the following generalization of the above construction of G^2:

Let $\mathcal{G}$ be an infinite set of finite graphs of the form $G_{j,i}$ which satisfies:

(i) $G_{j,i} \nrightarrow H_i$;
(ii) $G_{j,i}$ do not contain odd cycles of length $\le j$;
(iii) $G_{j,i} \to H$ (this homomorphism will be denoted again by c);
(iv) All the graphs are vertex disjoint.

By now it is easy to get such examples, see e.g. [13, 15].

Let $T = (V, E)$ be a graph tree (i.e. we consider just the successor relation) defined as follows: $V = A_0 \cup A_1 \cup A_2$ where $A_0 = \{r\}$, $A_1 = \omega$ and $A_2 = \omega \times \omega$. The edges of T are all edges of the form $\{(r, i)\}$, $i \in \omega$ and all pairs of the form $\{i, (i, j)\}$, $i, j \in \omega$.

Let $\eta : V \to \omega \times \omega$ be any function.

Define the graph G_η as follows: the set of vertices of G_η is the union of all graphs $G_{\eta(x)}$, $x \in V$. The edges of G_η are edges of all graphs $G_{\eta(x)}$, $x \in \omega$ together with all edges of the form $\{a, b\}$ where $a \in G_{\eta(x)}$, $b \in G_{\eta(y)}$, $\{x, y\} \in E$ and $\{c(A), c(b)\} \in E(H)$.

We have analogously as in Claim 2:

Claim 3. *Let $\eta : V \to \omega$ be any function which is unbounded on ω and each of the sets $\{i\} \times \omega$, $i \in \omega$. Then $G_\eta \not\leq G^1$.*

Now, consider the graph G again. We have to distinguish two cases:

Case 1. $\chi(H) \geq 5$.

In this case we proceed completely analogously as in the proof of Theorem 1.1 with the only change that we denote by K the minimal number of vertices of a subgraph G' of G such that $G' \nrightarrow H_i$ and $\chi(H_i) \leq 4$ (by compactness it is $K \in \omega$). In this case we derive a contradiction as above. Leaving this at that we have to consider:

Case 2. $\chi(H) < 5$.

In this case we have to continue and we introduce one more construction of the graph G^4.

Let T be an infinite binary tree. Explicitly, $V(T)$ denotes the set of all binary sequences ordered by the initial segment containment. For a sequence $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(p))$ we put $i(\sigma) = \sum_{i=0}^{p} 2^{\sigma(i)}$ ($i(\sigma)$ is a level-preserving enumeration of vertices of T) and $\ell(\sigma) = \max\{i; \sigma(i) \neq 0\}$ ($\ell(\sigma)$ is the level of σ in T).

Assume that the graphs H_n satisfy $H_m < H_n$ and $|V(H_m)| < |V(H_n)|$ for all $m < n$. This can be assumed without loss of generality as we can consider a subset of ω with this property.

Let F_σ, $\sigma \in V(T)$ be a set of disjoint graphs with the following properties:

(i) $F_\sigma \leq H_{i(\sigma)}$.

(ii) $F_\sigma > H_{i(\sigma)-1}$, moreover for every homomorphism $f : F_\sigma \to H$ satisfying $|f(V(F_\sigma))| < |V(F_\sigma)|$ there exist homomorphisms $g : F_\sigma \to H_{i(\sigma)}$ and $h : H_{i(\sigma)} \to H$ such that $f = h \circ g$ (in other words each f with a small image factorizes through $H_{i(\sigma)}$).

(iii) F_σ does not contain odd cycles of length $\leq k_1$ where k_1 denotes the shortest length of an odd cycle in G.

(iv) In each F_σ are given two distinct vertices x_σ and $y = y_\sigma$ such that $\{c(x_\sigma), c(y_\sigma)\} \in E(H)$.

(See [13, 15]; it suffices to put $F_\sigma = H_{i(\sigma)} \times K$ where K is a graph without short odd cycles with sufficiently large chromatic number.)

Denote by G^4 the disjoint union of graphs F_σ with added edges of the form $\{x, y\}$ where $x = x_\sigma$ and $y = y'_\sigma$ and $\{\sigma, \sigma'\}$ form an edge of T.

This concludes the definition of G^4. For G^4 we define $G^3 = G_\eta$ for the following function $\eta : A_0 \cup A_1 \cup A_2 \to \omega$ (see the above definition of the graph G_η for general η):

$$\eta(r) = 1, \qquad \eta(i) = \left(i, \sum(|V(F_\sigma)|; \ell(\sigma) < i)\right),$$
$$\eta(i, j) = \left(j, \sum(|V(F_\sigma)|; \ell(\sigma) < j)\right).$$

This only means we consider graphs with rapidly progressing odd girth.

We know that $G^3 \nrightarrow G^2$ (for any η unbounded on the stars of the corresponding tree).

Thus assume that $f : G^4 \to G^3$. Due to the tree structure of the graph G^3 we see that for each $\sigma \in V(T)$ the image $f(F_\sigma)$ intersects a finite set of graphs G_x, $x \in I \subset A_0 \cup A_1 \cup A_2$ and due to the tree structure of the graph G^3 we see easily that there is a homomorphism $f' : F_\sigma \to H_{i(I)}$ where $i(I)$ is the maximal index appearing among all $i \in I$ and $(j, i) \in I$ and we arrive at a contradiction.

Thus $G^4 \not\leq G^3$ and consequently also $G \leq G^4$.

As G^4 contains odd cycles only in copies of graphs H_σ and as all these cycles have lengths $>k_0$ we conclude that $G \not\leq G^4$. $\square$

7. Concluding remarks

1. The problem to characterize gaps below H is not as isolated as it perhaps seems at the first glance. Put $GRA_H = \{G; G < H\}$. We have the following easy theorem:

Theorem 7.1. *For countable graphs H the following statements are equivalent:*

(i) *There is no gap below H;*
(ii) *For every $G \in GRA_H$ there is $G' \in GRA_H$ such that $G' \not\leq G$;*
(iii) *For every $H' \in GRA_H$ the class GRA'_H has no hom-universal graph.*

Motivated by Theorem 1.3 one is tempted to also include here the following condition:

(iv) For every $G \in GRA_H$ there exists $G' \in GRA_H$ such that $G < G'$ and G' is rigid.

However (iv) is false as shown by the following example:

Let $H = K_3$ and let G be the disjoint union of all odd cycles of length >3. Then any rigid graph G', $G' < H$ which contains G as a subgraph is necessarily a disconnected graph. Let $\{G'_i; i \in \omega\}$ be all the components of G'. Then $\chi(G_i) = 3$ for every $i \in \omega$ and thus let G_i contain an odd cycle $C_{\ell(i)}$ of length $\ell(i)$. Let G_j be the component which maps to $C_{\ell(i)}$ (as a component of G). Clearly $i \neq j$ and thus $G_j \to G_i$, a contradiction.

Note also that the above Theorem 7.1 is true for any fixed infinite cardinality.

2. We say that a set G of countable graphs is *maximal* (or *unextendable*) if there is no graph $G \notin \mathcal{G}$ such that G is independent of all $G' \in \mathcal{G}$.

$\{K_\omega\}$ is maximal but there are other maximal families. For example $\{K_k\} \cup \{G; G$ finite and $\chi(G) > k\}$ is a maximal set and more generally for every finite graph H the following is a maximal set:

$$\{H\} \cup \{G; G \text{ finite and } G > H\}$$

Corollary 1.1 implies existence of finite maximal sets.

The characterization of maximal sets seems to be a difficult problem related to *duality theorems*, see [17]. However no maximal set is known which consists of infinite graphs only.

Acknowledgements

Jaroslav Nešetřil was partially supported by the Project LN00A056 of the Czech Ministry of Education and by GAUK 158 grant. The research of Saharon Shelah was supported by the United States–Israel Binational Science Foundation, Publication 745. The authors wish to thank the Mittag-Leffler Institute, for its support and pleasant environment during September, 2000, when this paper was written.

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