Available online at www.sciencedirect.com
SCIENCE DIRECT®

ELSEVIER

European Journal of Combinatorics 26 (2005) 325–338

European Journal of Combinatorics

www.elsevier.com/locate/ejc

Projective-planar double coverings of graphs

Seiya Negami

Department of Mathematics, Faculty of Education and Human Sciences, Yokohama National University, 79-2 Tokiwadai, Hodogaya-Ku, Yokohama 240-8501, Japan

Received 19 August 2002; accepted 9 February 2004
Available online 7 June 2004

Abstract

We shall show that a connected graph G is projective-planar if and only if it has a projective-planar double covering and that any projective-planar double covering of a 2-connected nonplanar graph is planar.

0. Introduction

Our graphs are simple and finite. A graph $\tilde{G}$ is called an (*n-fold*) *covering* of a graph G with a *projection* $p : \tilde{G} \to G$ if there is an (n-to-one) surjection $p : V(\tilde{G}) \to V(G)$ which sends the neighbors of each vertex $v \in V(\tilde{G})$ bijectively to those of $p(v)$. In particular, $\tilde{G}$ is called a *regular covering* provided that there is a subgroup A in the automorphism group $\mathrm{Aut}(\tilde{G})$ of $\tilde{G}$ such that $p(u) = p(v)$ if and only if $\tau(u) = v$ for some $\tau \in A$. It is easy to see that a 2-fold (or *double*) covering is necessarily a regular one.

A graph is said to be *projective-planar* if it can be embedded in the projective plane. Negami [8] has discussed the relation between planar double coverings and embeddings of graphs in the projective plane, and established the following characterization of projective-planar graphs:

Theorem 1 (Negami [8]). *A connected graph is projective-planar if and only if it has a planar double covering.*

Furthermore, he has proved the following theorem, which extends Theorem 1 with "regular" instead of "double":

E-mail address: negami@edhs.ynu.ac.jp (S. Negami).

0195-6698/$ - see front matter
doi:10.1016/j.ejc.2004.02.011

Theorem 2 (Negami [9]). *A connected graph is projective-planar if and only if it has a planar regular covering.*

These theorems motivated him to propose the following conjecture. This is called "the 1-2-∞ conjecture" or "Negami's planar cover conjecture":

Conjecture 1 (Negami [9], 1986). *A connected graph is projective-planar if and only if it has a planar covering.*

There have been many papers on studies around this conjecture, but the sufficiency is still open.

A graph H is called a *minor* of a graph G if H can be obtained from G by contracting and deleting some edges. It is easy to see that if G has a planar covering, then so does H. Thus, it suffices to show that every minor-minimal graph among those graphs that are not projective-planar does not have a planar covering to solve the conjecture affirmatively. Such minor-minimal graphs have been already identified in [1] and [4]; they are 35 in number and three of them are disconnected.

Let G_Y be a graph with a vertex v of degree 3 and let v_1, v_2 and v_3 be the three neighbors of v. A *Y-Δ transformation* is to add three new edges v_1v_2, v_2v_3 and v_3v_1 after deleting v. Let G_Δ be a graph obtained from G_Y by a Y-Δ transformation. It is easy to see that if G_Y has a planar covering, then so does G_Δ. It has been known that the 32 minor-minimal connected graphs can be classified into 11 families, up to Y-Δ transformations.

Combining the results in [2, 3, 5, 7, 9, 10], we can show that every member in the 10 families not including $K_{1,2,2,2}$ does not have any planar covering and conclude the following theorem at present:

Theorem 3 (Archdeacon, Fellows, Hliněný and Negami). *If $K_{1,2,2,2}$ has no planar covering, then Conjecture 1 is true.*

By Theorem 1, if a connected graph has a projective-planar covering, then it has a planar covering, which covers the latter doubly. Also a planar covering can be embedded in the projective plane. These imply that Conjecture 1 is equivalent to the following conjecture, as observed by Hliněný in [6]:

Conjecture 2. *A connected graph is projective-planar if and only if it has a projective-planar covering.*

Replacing two appearances of "projective-planar" with "Klein bottle" in the above, he has posed another interesting conjecture in [6]. However, it seems to be hardly possible to solve such a conjecture.

So, we shall discuss projective-planar double coverings of nonplanar graphs, turning to Conjecture 2, and prove the following theorem, using the notion of "composite coverings" developed in [11]:

Theorem 4. *A connected graph is projective-planar if and only if it has a projective-planar double covering.*

This theorem might look like one that gives us evidence supporting Conjecture 2. Our arguments in this paper will however suggest that Conjecture 2 presents something vain

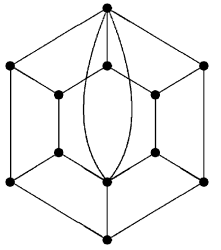

Fig. 1. A double covering of $K_{3,3}$ with a self-loop.

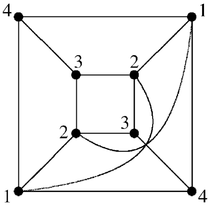

Fig. 2. A double covering of C_4 with multiple edges.

even if it is true. The essential phenomenon on projective-planar double coverings is that:

Theorem 5. *Every projective-planar double covering of a 2-connected nonplanar graph is planar.*

Note that none of the 2-connectedness and the nonplanarity of a graph can be omitted from Theorem 5. For example, consider the graph given in Fig. 1. This is projective-planar but not planar, and covers doubly $K_{3,3}$ with a self-loop attached at one vertex. To get its projective-planar embedding, draw the pair of multiple edges so that they cross together one edge on the inner hexagon and put a crosscap at the edge to clip the two crossings. Subdividing the self-loop to make it simple, we obtain a nonplanar graph which is not 2-connected and which has a nonplanar projective-planar double covering.

On the other hand, Fig. 2 presents a nonplanar graph which is projective-planar and which covers doubly a 2-connected planar graph obtained from the cycle C_4 of length 4 by replacing three edges with multiple edges. Two vertices with the same label project to one of the vertices lying along a cycle 1234. It is clear where we should put a crosscap to embed this nonplanar graph in the projective plane.

In fact, graph-minor arguments work for Theorem 4 and it suffices to prove that $K_{1,2,2,2}$ does not have any projective-planar double covering, although we prove the theorem in Section 1, applying the result in [11]. On the other hand, graph-minor arguments do not work well for Theorem 5. We need to classify the projective-planar double coverings of $K_{3,3}$ and K_5, as in Section 2; those coverings should be planar. We shall give a proof of the theorem in Section 3. Section 4 presents another proof of Theorem 4 as an application of our arguments in Section 3.

Related to Theorem 5 in this paper, the author and Suzuku [12] have shown recently that any projective-planar double covering of a 3-connected graph also is planar. However, such a 3-connected graph is not assumed to be nonplanar.

1. Composite planar coverings

To prove Theorem 4, we shall introduce the notion of "composite coverings", as mentioned in the introduction.

In general, let $p_1 : \tilde{G} \to G'$ and $p_2 : G' \to G$ be two covering projections of graphs. Then the *composition* of these two projections $p = p_2 p_1 : \tilde{G} \to G$ defines another covering projection from $\tilde{G}$ to G. Conversely, a covering $\tilde{G}$ of a graph G with projection $p : \tilde{G} \to G$ is said to be *composite* if its projection can be obtained as a composition $p = p_2 p_1$ of two covering projections $p_1 : \tilde{G} \to G'$ and $p_2 : G' \to G$ via another suitable graph G'. In particular, if p_1 and p_2 are n_1-fold and n_2-fold, respectively, for natural numbers $n_1, n_2 \geq 2$, then $\tilde{G}$ is called an (n_1, n_2)*-composite covering*.

The author has discussed such composite coverings in connection with Conjecture 1 and established the following theorem in [11]:

Theorem 6 (Negami [11]). *A connected graph G is projective-planar if and only if it has an $(n, 2)$-composite planar connected covering for some $n \geq 1$.*

Furthermore, he has proved that every planar connected regular covering of a nonplanar connected graph is $(n, 2)$-composite for some $n \geq 1$. Theorem 4 is just an easy consequence from the above theorem:

Proof of Theorem 4. The necessity is clear since any projective-planar graph has a planar double covering, which is also projective-planar.

Suppose that a connected graph G has a projective-planar double covering $p_2 : G' \to G$. Then G' has a planar connected double covering $p_1 : \tilde{G} \to G'$ by Theorem 1, and $p_2 p_1 : \tilde{G} \to G$ is a $(2, 2)$-composite planar covering of G. By Theorem 6, G must be projective-planar. Thus, the sufficiency follows. □

Here, we shall introduce another formulation on planar coverings to show an easy application of Theorem 4. Let $p_i : G_i \to G_{i-1}$ be a double covering projection from G_i to G_{i-1}. A series of double coverings $G_n \to G_{n-1} \to \cdots \to G_0$ is called a *planar tower* of G_0 (of height n) if the top graph G_n is planar. The composition $p = p_1 p_2 \ldots p_n : G_n \to G_0$ of covering projections is a 2^n-fold covering projection from G_n to G_0 and is said to be obtained by *tower construction*.

Theorem 7. *A connected graph is projective-planar if and only if it has a planar tower.*

Proof. Let G_0 be a nonplanar connected graph. We shall show only the sufficiency, using induction on the height n of a planar tower $G_n \to G_{n-1} \to \cdots \to G_1 \to G_0$. If $n = 1$, then G_0 is projective-planar, by Theorem 1. If $n > 1$, then $G_n \to G_{n-1} \to \cdots \to G_1$ is a planar tower of G_1 of height $n - 1$ and hence G_1 is projective-planar, by the induction hypothesis. Therefore, G_0 is projective-planar by Theorem 4. This completes the induction. □

Since a planar covering obtained by tower construction is not regular in general, this theorem covers a part which Theorem 2 does not. For example, $K_{1,2,2,2}$ does not have a planar tower even if it might have a planar covering.

2. Double coverings of Kuratowski graphs

It is well-known as Kuratowski's theorem that any nonplanar graph contains a subdivision of either $K_{3,3}$ or K_5. Thus, the fact given as Theorem 5 should hold for $K_{3,3}$ and K_5 at least. Lemmas 10 and 11 guarantee it and will play an essential role in our proof of Theorem 5.

Before showing them, we prepare the following lemma, which we shall often use later to decide the projective-planarity of double coverings of graphs. This has been proved in [4], where the subgraphs H_1 and H_2 discussed in the lemma are called "disjoint k-subgraphs".

Lemma 8. *Let G be a connected graph such that:*

(i) *There exist two disjoint subgraphs H_1 and H_2 of G each of which is isomorphic to either K_4 or $K_{2,3}$.*

(ii) *Let $X_i = V(H_i)$(or let X_i be the set of vertices of degree 2) if H_i is isomorphic to K_4(or $K_{2,3}$). Each vertex in X_i is adjacent to a vertex in $G - V(H_i)$ for $i \in \{1, 2\}$.*

(iii) *Both $G - V(H_1)$ and $G - V(H_2)$ are connected.*

Then G is not projective-planar.

Let $\tilde{G}$ be a double covering of a graph G with projection $p : \tilde{G} \to G$ in general. Then there is an automorphism $\tau : \tilde{G} \to \tilde{G}$ of period 2 such that $\tau(u) = v$ and $\tau(v) = u$ for any pair $\{u, v\}$ of vertices in $\tilde{G}$ with $p(u) = p(v)$. This automorphism τ is called the *covering transformation* of a double covering.

It should be noticed that if we can find a subgraph H_1 in a double covering $\tilde{G}$ so that it satisfies three conditions in Lemma 8, then $\tau(H_1)$ can be chosen as H_2 and we can conclude that $\tilde{G}$ is not projective-planar. The following lemma is a restricted form of Lemma 8 but is useful to prove Lemmas 10 and 11:

Lemma 9. *Let G be a connected graph and $p : \tilde{G} \to G$ a double covering of G with covering transformation $\tau : \tilde{G} \to \tilde{G}$. Suppose that:*

(i)′ *There exists a subgraph H of $\tilde{G}$ isomorphic to either K_4 or $K_{2,3}$.*

(ii)′ *Let $X = V(H)$ (or let X be the set of vertices of degree 2) if H is isomorphic to K_4 (or $K_{2,3}$). Each vertex in X is adjacent to a vertex in $\tilde{G} - V(H)$.*

(iii)′ *If $M = G - V(p(H))$ is not empty, then M is connected and each vertex in $p(X)$ is adjacent to a vertex in M.*

Then $\tilde{G}$ is not projective-planar.

Proof. It is easy to see that H projects to $p(H)$ isomorphically and that $\tau(H) \cap H = \emptyset$. Put $H_1 = H$ and $H_2 = \tau(H)$. Then (i) and (ii) in Lemma 8 hold for $\tilde{G}$ and $\tilde{G} - V(H_1)$ is isomorphic to $\tilde{G} - V(H_2)$ via τ. If M is empty, then the connected graph H_{3-i} is a spanning subgraph of $\tilde{G} - V(H_i)$ and hence $\tilde{G} - V(H_i)$ is connected for $i = 1, 2$. Thus, $\tilde{G}$ is not projective-planar by Lemma 8.

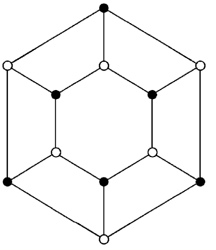

Fig. 3. The unique planar double covering of $K_{3,3}$.

Now suppose that M is nonempty and connected. Then $\tilde{G} - V(H_i)$ consists of $H_{3-i} \cup p^{-1}(M)$ and some edges. If it is not connected, then $p^{-1}(M)$ splits into two components M_1 and M_2 so that there is no edge between M_i and H_{3-i} for $i = 1, 2$. In this case, $\tilde{G}$ is contracted to either the disjoint union $K_{3,3} \cup K_{3,3}$ or $K_5 \cup K_5$. They are not projective-planar and hence $\tilde{G}$ is not, either. On the other hand, if $\tilde{G} - V(H_i)$ is connected, then $\tilde{G}$ is not projective-planar by Lemma 8. □

Lemma 10. *$K_{3,3}$ has exactly one projective-planar double covering given in Fig. 3, up to graph isomorphism, and it is planar.*

Proof. Let $\tilde{G}$ be a projective-planar double covering of $K_{3,3}$ with covering transformation $\tau : \tilde{G} \to \tilde{G}$. Since $K_{3,3} \cup K_{3,3}$ is not projective-planar, $\tilde{G}$ also is connected and is bipartite. We may assume that its vertices are colored by black and white. Let b_1 be any black vertex of $\tilde{G}$ and let w_1, w_2, w_3 be the three neighbors of b_1 in $\tilde{G}$, which are white. Then $\{b_1, w_1, w_2, w_3\}$ induces a subgraph isomorphic to $K_{1,3}$, say T_1, and $\tau(T_1)$ is disjoint from T_1.

Choose any other black vertex $b_2 \notin \{b_1, \tau(b_1)\}$. If b_2 is adjacent to all of w_1, w_2, w_3, then $\{b_1, b_2, w_1, w_2, w_3\}$ induces a subgraph isomorphic to $K_{2,3}$. This works as H in Lemma 9, and hence $\tilde{G}$ would not be projective-planar, a contradiction. Thus, we may assume that b_2 is adjacent to w_1 and w_2, but not to w_3, up to symmetry. Then b_2 is adjacent to $\tau(w_3)$ since $p(b_2)$ must be adjacent to $p(w_3) = p(\tau(w_3))$ in $K_{3,3}$.

Let b_3 be the third black vertex adjacent to w_1, which is different from $b_1, b_2, \tau(b_1)$ and $\tau(b_2)$. If b_3 is adjacent to w_2, then $\{b_1, b_2, b_3, w_1, w_2\}$ induces a subgraph isomorphic to $K_{2,3}$, which can be used as H in Lemma 9. Thus, b_3 is not adjacent to w_2. If b_3 is not adjacent to w_3, then $\tau(b_3)$ must be adjacent to both w_2 and w_3. Replacing w_1, w_2 and b_3 with w_2, w_1 and $\tau(b_3)$ in order in this case, we may assume that b_3 is adjacent to w_3.

Now we can determine the adjacency over all vertices uniquely. There are two cycles $b_1 w_2 \tau(b_3) \tau(w_1) \tau(b_2) w_3$ and $w_1 b_2 \tau(w_3) \tau(b_1) \tau(w_2) b_3$, and each pair of vertices in corresponding positions in these sequences are joined with an edge. Thus, $\tilde{G}$ is isomorphic to a hexagonal prism given in Fig. 3 and is planar. □

Lemma 11. *K_5 has exactly two projective-planar double coverings given in Fig. 4, up to graph isomorphism, and they are planar.*

Proof. Let $\tilde{G}$ be a projective-planar double covering of K_5 with covering transformation $\tau : \tilde{G} \to \tilde{G}$. Then $\tilde{G}$ is connected. Choose a vertex v_0 in $\tilde{G}$ and three of the four neighbors

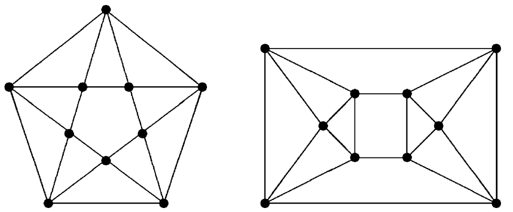

Fig. 4. The two planar double coverings of K_5.

of v_0, say v_1, v_2 and v_3. Let H be the subgraph induced by $\{v_0, v_1, v_2, v_3\}$ in $\tilde{G}$. Let v_4 be one of two vertices not in $\bigcup_{i \le 3}\{v_i, \tau(v_i)\}$. If H is isomorphic to K_4, then it can be used as H in Lemma 9, contrary to $\tilde{G}$ being projective-planar. So we may assume that v_1 and v_3 are not adjacent and that v_4 is adjacent to at least two vertices in H, replacing v_4 with $\tau(v_4)$ if we need.

First suppose that $v_1v_2, v_2v_3 \in E(H)$. If v_4 is adjacent to at least three vertices in H, then they are $\{v_1, v_0, v_3\}$ or $\{v_1, v_2, v_3\}$; otherwise, we would find a subgraph isomorphic to K_4 as H in Lemma 9. In either case, $\{v_0, v_1, v_2, v_3, v_4\}$ induces a subgraph W isomorphic to a wheel with a rim $w_1w_2w_3w_4$ of length 4. Then $\tilde{G}$ consists of two wheels W and $\tau(W)$ with four edges $w_1\tau(w_3)$, $w_2\tau(w_4)$, $w_3\tau(w_1)$ and $w_4\tau(w_2)$ and is planar. This is isomorphic to the right graph in Fig. 4.

If v_4 is adjacent to exactly two vertices, then they are $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$ or $\{v_3, v_0\}$; otherwise, we would find a subgraph isomorphic to $K_{2,3}$ which can be used as H in Lemma 9. In each case, $\tilde{G}$ is isomorphic to the square of C_{10}, which consists of a cycle $c_0c_1 \dots c_9$ of length 10 and 10 edges of form c_ic_{i+2} with scripts taken modulo 10. Since it can be embedded on the sphere so that two cycles $c_0c_2 \dots c_8$ and $c_1c_3 \dots c_9$ bound pentagonal faces, $\tilde{G}$ is planar. This is isomorphic to the left graph in Fig. 4.

Secondly suppose that exactly one of v_1v_2 and v_2v_3 belongs to $E(H)$, say v_1v_2. If v_4 is adjacent to at least two of vertices lying on the triangle $v_0v_1v_2$, then $\{v_0, v_1, v_2, v_4\}$ induces either K_4 or K_4 minus one edge and the previous arguments work for this case with suitable replacement of labels. Thus, the neighborhood of v_4 in H is either $\{v_3, v_0\}$ or $\{v_3, v_1\}$, up to symmetry. In either case, $\tau(v_4)$ is adjacent to two vertices on $v_0v_1v_2$ and we find a subgraph isomorphic to K_4 minus one edge, again.

Finally suppose that there is no edge in $\{v_1, v_2, v_3\}$. If v_4 is adjacent to only two vertices in H, then we find a triangle containing v_4 or $\tau(v_4)$ and this case can be reduced to the previous. Thus, we may assume that v_4 is adjacent to all of v_1, v_2 and v_3, and not to v_0; for, if v_4 is adjacent to v_0, then we find a triangle, again. In this case, $\{v_0, v_1, v_2, v_3, v_4\}$ induces a subgraph isomorphic to $K_{2,3}$, which can be used as H in Lemma 9. Therefore, this is not the case. $\square$

We shall use the planar embeddings of double coverings of $K_{3,3}$ and of K_5 given in Figs. 3 and 4 in the proof of Theorem 5. They should be embedded on the sphere rather than in the plane. Note that the covering transformation $\tau : \tilde{G} \to \tilde{G}$ extends to an auto-homeomorphism over the sphere for each of them.

3. Nonplanarity of double coverings

Let G be a graph and K a subgraph in G. A subgraph B induced by a component of $G - V(K)$ and the edges joining it to K is called a *bridge* for K in G. A subgraph consisting of a single edge $e \notin E(K)$ with both ends in K also is regarded as a bridge for K but it is said to be *singular*. It is clear that G decomposes into K and the bridges for K and that they are mutually edge-disjoint. A vertex of a bridge is called a vertex of *attachment* if it lies in K.

In the following proof, K will be a subdivision of either $K_{3,3}$ or K_5. We call a path in K a *side* if it corresponds to an edge of $K_{3,3}$ or K_5. That is, a side of K is a path in K which joins two vertices of degree not 2 and whose inner vertices all have degree 2. We shall often use Lemma 8 to conclude that graphs in question are not projective-planar. Although we need not only the lemma but also graph-minor arguments logically in some cases, we shall write just "by Lemma 8" to simplify our description below.

Proof of Theorem 5. Let G be a 2-connected nonplanar graph and $p : \tilde{G} \to G$ any projective-planar double covering of G with covering transformation $\tau : \tilde{G} \to \tilde{G}$. We shall prove the theorem by induction on the number of vertices of G. However, if $\tilde{G}$ is 3-connected, we can prove the theorem independently of the number of vertices. Furthermore, the following argument implies that $\tilde{G}$ must be 3-connected in the initial case of induction.

Thus, suppose that $\tilde{G}$ is not 3-connected. Then $\tilde{G}$ decomposes into two connected subgraphs G' and F so that $V(G') \cap V(F)$ forms a 2-cut $\{x, y\}$ of $\tilde{G}$. We may assume that F cannot be decomposed by any 2-cut of $\tilde{G}$. Then either $\tau(F) \cap F = \emptyset$ or $\tau(F) = F$. However, the latter does not happen since it implies that $p(x) = p(y)$ and it becomes a cut vertex of G, contrary to G being 2-connected. Thus, G decomposes into three subgraphs G'', F and $\tau(F)$ so that $\tau(G'') = G''$, $F \cap \tau(F) = \emptyset$ and $V(G'') \cap V(F) = \{x, y\}$.

Corresponding to this decomposition, G also decomposes into two connected subgraphs $p(G'')$ and $p(F)$ with a 2-cut $\{p(x), p(y)\}$ and both $G_0 = p(G'') + p(x)p(y)$ and $p(F) + p(x)p(y)$ are 2-connected; the latter is isomorphic to $F + xy$. It is easy to find a subgraph H in $\tilde{G}$ homeomorphic to either $(F + xy) \cup (\tau(F) + \tau(x)\tau(y))$ or $F \cup \tau(F) + \{x\tau(y), y\tau(x)\}$ and to see that if $F + xy$ were not planar, then H would not be projective-planar, which is contrary to G being projective-planar. Thus, $F + xy$ and $\tau(F) + \tau(x)\tau(y)$ must be planar and hence F can be embedded on the plane so that x and y are incident to the outer region.

On the other hand, $\tilde{G}_0 = G'' + \{xy, \tau(x)\tau(y)\}$ is a projective-planar double covering of G_0 and G_0 must be nonplanar; otherwise, G would be planar. By the induction hypothesis, we can conclude that $\tilde{G}_0$ is planar and can construct a planar embedding of $\tilde{G}$ from any planar embedding of $\tilde{G}_0$ by pasting two copies of the above-mentioned planar embedding of F along the two edges xy and $\tau(x)\tau(y)$. Therefore, $\tilde{G}$ is planar.

Now we shall discuss the case that $\tilde{G}$ is 3-connected. By Kuratowski's theorem, G contains a subdivision K of either $K_{3,3}$ or K_5. First, we shall prove the theorem, assuming the former case. Let $\tilde{K}$ be the pull-back $p^{-1}(K)$ of K in $\tilde{G}$. By Lemma 10, $\tilde{K}$ is a subdivision of the unique planar double covering of $K_{3,3}$ given in Fig. 3.

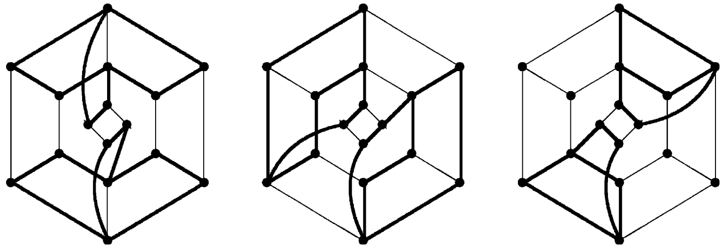

Fig. 5. Nonprojective-planar graphs I.

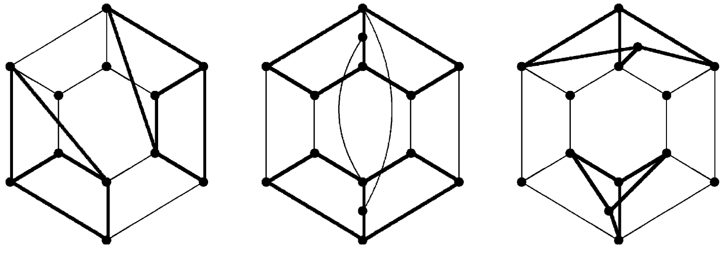

Fig. 6. Nonprojective-planar graphs II.

If G is isomorphic to K, then $\tilde{G}$ must be planar by Lemma 10. Otherwise, there is at least one bridge for K in G. Let B be one of the bridges for K in G. First, suppose that $p^{-1}(B)$ is connected. Then B contains a cycle C disjoint from its vertices of attachment. Since G is 2-connected, B has at least two vertices of attachment, say x and y, and there are two disjoint paths Q and Q' joining x and y to C in B.

Since x and y are distinct, we can contract some edges of K so that they are placed at two distinct vertices of degree 3 in K. Thus, $\tilde{K} \cup p^{-1}(C \cup Q \cup Q')$ is contracted to a union of the planar double covering of $K_{3,3}$ and C_4 with four paths joining them under the symmetry derived by the covering transformation. We can list up those possible structures as given in Fig. 5, focusing the positions of ends of these paths. Thus, one of the three graphs becomes a minor of $\tilde{K} \cup p^{-1}(B)$.

The subgraph indicated by thick lines in each graph works as $H_1 \cup H_2$ in Lemma 8, as well as in Figs. 6 to 10. Thus, they are not projective-planar and $\tilde{G}$ also would not be projective-planar, a contradiction. Therefore, $p^{-1}(B)$ must consist of two components.

Let $\tilde{B}$ be a component of $p^{-1}(B)$. Then $\tilde{B}$ is a bridge for $\tilde{K}$ in $\tilde{G}$ and projects to B isomorphically. Thus, there are no two vertices x and y in $\tilde{B}$ with $p(x) = p(y)$. Take a path Q in $\tilde{B}$ joining two distinct vertices of attachments, say x and y. Let $H = \tilde{K} \cup Q \cup \tau(Q)$. It is easy to see that if Q cannot be embedded in a face of the planar embedding of $\tilde{K}$, then H is isomorphic or contractible to one of the first and second graphs given in Fig. 6. However, H is not projective-planar by Lemma 8 and $\tilde{G}$ would not be projective-planar, either. Therefore, any two vertices x and y of attachment of a bridge $\tilde{B}$ lie along the boundary of a face of the planar embedding of $\tilde{K}$.

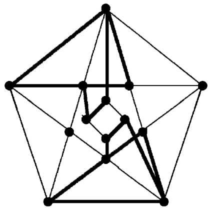

Fig. 7. Nonprojective-planar graphs III.

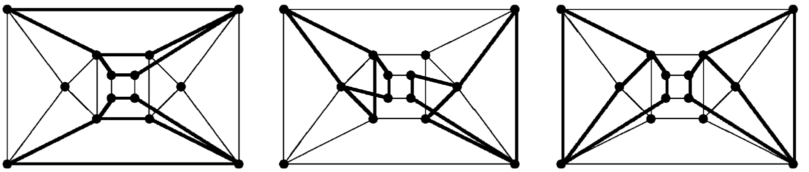

Fig. 8. Nonprojective-planar graphs IV.

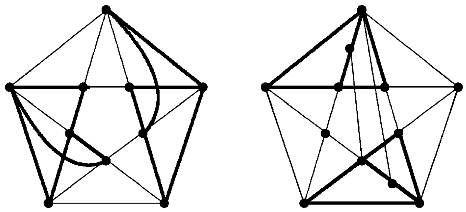

Fig. 9. Nonprojective-planar graphs V.

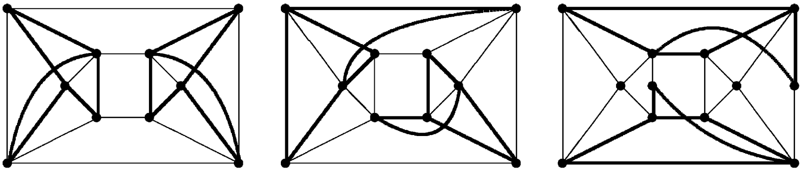

Fig. 10. Nonprojective-planar graphs VI.

Let x_1, x_2 and x_3 be three distinct vertices of attachment of a bridge $\tilde{B}$ for $\tilde{K}$. Then $p(x_1)$, $p(x_2)$ and $p(x_3)$ are all distinct and any pair $\{x_i, x_j\}$ lie along the boundary of a face of the planar embedding of $\tilde{K}$, say A_{ij}. If $A_{ij} \neq A_{ik}$ for $\{i, j, k\} = \{1, 2, 3\}$, then they must meet along a side of $\tilde{K}$ since $\tilde{K}$ is a subdivision of a 3-regular graph. Let e_i be an edge on the side. Then $\{e_1, e_2, e_3\}$ forms a cutset of $\tilde{K}$. It is easy to see that such a cutset cuts off a vertex of degree 3 from $\tilde{K}$ and that H is contractible to the third graph in Fig. 6. However, it is not projective-planar by Lemma 8. Thus, $A_{12} = A_{23} = A_{13}$. This implies that all vertices of attachment of $\tilde{B}$ are contained in the boundary of a face of the planar embedding of $\tilde{K}$ and $\tilde{B} \cap \tau(\tilde{B}) = \emptyset$.

Embed $\tilde{G}$ on the projective plane and consider an embedding of $\tilde{K}$ on the sphere which is a subdivision of the unique planar embedding of $K_{3,3}$ given in Fig. 3. To clear our arguments here, let $f : \tilde{K} \to S^2$ denote the embedding of $\tilde{K}$ on the sphere S^2. Then $\tilde{K}$ can be obtained as the union of four pairs of disjoint cycles $\{C_i, \tau(C_i)\}$ $(i = 1, 2, 3, 4)$ such that $f(C_i)$ and $f(\tau(C_i))$ bound two disjoint faces of $f(\tilde{K})$ on the sphere; they are hexagonal for one pair and are quadrilateral for the others. We shall try to extend this embedding f to that of $\tilde{G}$.

A simple closed curve on a surface is said to be *essential* if it does not bound any 2-cell region and it is well-known that any two essential simple closed curves on the projective plane are not disjoint. This implies that at least one of C_i and $\tau(C_i)$ bounds a 2-cell region on the projective plane. Such a 2-cell region must be a face of $\tilde{K}$ or contain the whole of $\tilde{K}$ since each of C_i and $\tau(C_i)$ does not separate $\tilde{K}$. Thus, we may assume that C_i bounds a face A_i of $\tilde{K}$ in the projective plane for $i = 1, 2, 3, 4$.

Furthermore, we shall assume that $\tilde{K}$ has been chosen to minimize the number of bridges. This assumption excludes those bridges that attach to only one side of $\tilde{K}$, as follows. Let $\tilde{B}$ be such a bridge and let S be the side of $\tilde{K}$ which contains all vertices of attachment of $\tilde{B}$.

First suppose that $\tilde{B}$ lies in a face of $\tilde{K}$ homeomorphic to a 2-cell. Then we can find a path Q along $\tilde{B}$ and a segment Q' of S with the same end $\{x, y\}$ so that $Q \cup Q'$ bounds a 2-cell region containing $\tilde{B}$. Replace the side S with $(S - Q') \cup Q$ in $\tilde{K}$. Since $\tilde{G}$ is 3-connected, there is a path P joining Q' to the outside of Q' and $\tilde{B}$ will be unified with another bridge containing P, which decreases the number of bridges. To preserve the symmetry of $\tilde{K}$ with respect to the covering transformation $\tau : \tilde{G} \to \tilde{G}$, carry out the same deformation for $\tau(\tilde{B})$ and $\tau(S)$ with $\tau(Q)$ and $\tau(Q')$. Then we obtain another subdivision of the planar double covering of $K_{3,3}$ with fewer bridges than $\tilde{K}$, contrary to the assumption on $\tilde{K}$.

Now suppose that $\tilde{B}$ lies in a face A of $\tilde{K}$ which is not homeomorphic to a 2-cell. Then the face A is homeomorphic to a Möbius band. Let C be its boundary cycle. Replacing the face A with a 2-cell yields an embedding of $\tilde{K}$ on the sphere such that C bounds a face there. This implies that $C = \tau(C_i)$ for some $i \in \{1, 2, 3, 4\}$, say $i = 1$, and that each of C_1 to C_4 and $\tau(C_2)$ to $\tau(C_4)$ bounds a 2-cell face of $\tilde{K}$ on the projective plane. Then the union of those seven faces is homeomorphic to a 2-cell and $\tau(C) = C_1$ is contained in the interior of the 2-cell, missing its boundary C. Under such a situation, $\tau(\tilde{B})$ lies in one of the 2-cell faces of $\tilde{K}$ on the projective plane. Thus, we conclude the same contradiction as in the previous case, exchanging $\tilde{B}$ and $\tau(\tilde{B})$.

Now, we have already known that all vertices of attachment of any bridge $\tilde{B}$ for $\tilde{K}$ in $\tilde{G}$ are contained in one of C_1 to C_4 or one of $\tau(C_1)$ to $\tau(C_4)$. Thus, the bridges for $\tilde{K}$ in $\tilde{G}$ can be classified into eight groups, according to which cycles they attach to. (We say that a bridge $\tilde{B}$ *attaches* to a cycle C if C contains all vertices of attachment of $\tilde{B}$. A cycle C to which $\tilde{B}$ attaches is unique since its vertices of attachment do not lie on one side of $\tilde{K}$ under our assumption on $\tilde{K}$.)

If there is a bridge $\tilde{B}$ which attaches to C_i, but which does not lie in the face A_i, then we can find a simple closed curve ℓ on the projective plane so that ℓ passes through $\tilde{B}$

and runs across A_i and meets $\tilde{K}$ in only two points x and y. Since $\tilde{K}$ is 3-edge-connected if we neglect vertices of degree 2, ℓ has to be essential. Since ℓ is disjoint from $\tau(C_i)$, $\tau(C_i)$ must bound a face A'_i and the face A'_i contains all bridges which attach to $\tau(C_i)$; otherwise, we could find another essential simple closed curve ℓ' on the projective plane with $\ell \cap \ell' = \emptyset$ so that ℓ' passes through $\tau(\tilde{B})$ and A'_i. In this case, we replace C_i and A_i with $\tau(C_i)$ and A'_i, respectively. After such replacement, a bridge $\tilde{B}$ attaches to C_i if and only if A_i contains $\tilde{B}$.

Now copy all the bridges lying in A_i to both faces bounded by $f(C_i)$ and by $f(\tau(C_i))$ for $i = 1, 2, 3, 4$. Then we obtain an embedding of $\tilde{G}$ on the sphere and hence $\tilde{G}$ is planar. This completes the proof for graphs containing a subdivision of $K_{3,3}$.

Now we suppose that K is a subdivision of K_5. We may assume that G contains no subdivision of $K_{3,3}$. Then all vertices of attachment of each bridge B for K are contained in a side of K; otherwise, $K \cup B$ would contain a subdivision of $K_{3,3}$. Now $\tilde{K}$ is a subdivision of one of the graphs given in Fig. 4.

Let B be any bridge for K in G. If $p^{-1}(B)$ is connected, one of the graphs in Figs. 7 and 8 will be a minor of $\tilde{K} \cup p^{-1}(B)$ by the same argument as in the previous case with $K_{3,3}$. Since they are not projective-planar by Lemma 8, $\tilde{G}$ would not be projective-planar, a contradiction. Therefore, $p^{-1}(B)$ consists of two components.

Let $\tilde{B}$ be a component of $p^{-1}(B)$ and let Q be a path in $\tilde{B}$ joining two vertices of attachment, say x and y. Suppose that Q cannot be embedded in a face of the planar embedding of $\tilde{K}$. Then $\tilde{K} \cup Q \cup \tau(Q)$ is contractible to one of the graphs shown in Figs. 9 and 10. Since each of them is not projective-planar by Lemma 8, $\tilde{G}$ also would not be projective-planar, either. Therefore, all vertices of attachment of each bridge for $\tilde{K}$ in $\tilde{G}$ are contained in the boundary cycle of a face of the planar embedding of $\tilde{K}$. By the same argument as in the previous case with $K_{3,3}$, we can construct a planar embedding of $\tilde{G}$. This completes the proof. □

4. Decomposing into blocks

Finally, we shall recognize what happens if a graph is not 2-connected. As is well-known, such a graph G decomposes into *blocks* $G_0, G_1, \dots, G_k$ so that any two of them meet in at most one vertex, called a *cut vertex*. Each block G_i is either 2-connected or isomorphic to K_2. The system of those blocks forms a tree-like structure.

Consider a double covering $p : \tilde{G} \to G$ of G. Then $p^{-1}(G_i)$ is a double covering of G_i. If $p^{-1}(G_i)$ is disconnected, then it consists of two components isomorphic to G_i itself. Thus, if $p^{-1}(G_i)$ is disconnected for all blocks G_i but one, say G_0, then Theorem 4 follows immediately from Theorems 1 and 5 since the block decomposition of G induces that of $\tilde{G}$. The double covering $p^{-1}(G_0)$ of G_0 is one of the blocks of $\tilde{G}$ and so is each component of $p^{-1}(G_i)$ for $i = 1, \dots, k$. If $\tilde{G}$ is projective-planar, then all blocks of $\tilde{G}$ are planar by Theorem 5. Thus, each of $G_1, \dots, G_k$ is planar while G_0 is projective-planar by Theorem 1. It is clear that $G = G_0 \cup G_1 \cup \cdots \cup G_k$ also is projective-planar in this case.

However, the above argument does not hold in general. For example, Fig. 1 suggests how to construct a projective-planar double covering $\tilde{G}$ of a connected graph G with two blocks G_0 and G_1 such that both G_0 and G_1 cannot be lifted isomorphically to $\tilde{G}$. So we need more delicate arguments for proving Theorem 4 as a corollary of Theorems 1 and 5.

Lemma 12. *Let G be a connected nonplanar graph and $\tilde{G}$ a planar double covering of G with covering transformation $\tau : \tilde{G} \to \tilde{G}$. Then a pair $\{v, \tau(v)\}$ does not lie on the boundary of a face for each vertex $v \in V(\tilde{G})$.*

Proof. We can observe easily that the lemma holds, adding bridges to the planar double coverings of $K_{3,3}$ and K_5 given in Figs. 3 and 4. □

Theorem 4 implies the following theorem of course. However, the following proof will work for another proof of Theorem 4, not using the arguments in Section 1.

Theorem 13. *If a connected nonplanar graph G has a projective-planar double covering, then exactly one of the blocks of G is nonplanar and projective-planar and the others are planar.*

Proof. Let $\tilde{G}$ be a projective-planar double covering of a connected nonplanar graph G with projection $p : \tilde{G} \to G$ and let $G_0, G_1, \ldots, G_k$ be the blocks of G. Since G is not planar, at least one of them, say G_0, is not planar. Put $\tilde{G}_0 = p^{-1}(G_0)$. Then $\tilde{G}_0$ is projective-planar and covers G_0 doubly. Since G_0 is 2-connected, $\tilde{G}_0$ is planar by Theorem 5. Choose arbitrarily one of $G_1, \ldots, G_k$, say G_i, and let Q be a path in G joining two cut vertices v_0 and x, with possibly $v_0 = x$, such that $v_0 \in V(G_0), x \in V(G_i)$ and $Q \cap (G_0 \cup G_i) = \{v_0, x\}$. Put $p^{-1}(v_0) = \{\tilde{v}_0, \tau(\tilde{v}_0)\}$.

Embed $\tilde{G}$ in the projective plane. If $\tilde{G}_0$ is not 2-cell embedded in the projective plane as a subembedding of $\tilde{G}$, then only one face of $\tilde{G}_0$ is a crosscap and we can construct a planar embedding of $\tilde{G}_0$ so that the boundary cycle of each face in the projective-planar embedding bounds a face in the plane. By Lemma 12, $\tilde{v}_0$ and $\tau(\tilde{v}_0)$ lie on two different boundary cycles separately. This implies that $p^{-1}(G_i \cup Q)$ consists of two components isomorphic to $G_i \cup Q$ and that they are embedded separately in two distinct faces of the projective-planar embedding of $\tilde{G}_0$. Since at least one of the two faces is a 2-cell, this induces a planar embedding of $G_i \cup Q$ and hence G_i is planar.

Now suppose that $\tilde{G}_0$ is 2-cell embedded in the projective plane. If $p^{-1}(G_i \cup Q)$ consists of two components, then we can construct a planar embedding of $G_i \cup Q$ as well as in the previous case and G_i is planar. Otherwise, $p^{-1}(G_i \cup Q)$ is embedded in a 2-cell so that $\tilde{v}_0$ and $\tau(\tilde{v}_0)$ are placed on its boundary together. Since $p^{-1}(G_i \cup Q)$ is a planar double covering of $G_i \cup Q$, $G_i \cup Q$ must be planar by Lemma 12 and hence G_i is planar.

In all cases, we have concluded that G_i is planar. Therefore, G_0 is a unique nonplanar block of G. □

References

[1] D. Archdeacon, A Kuratowski theorem for the projective plane, J. Graph Theory 5 (1981) 243–246.

[2] D. Archdeacon, Two graphs without planar covers, J. Graph Theory 41 (2002) 318–326.

[3] M. Fellows, Planar emulators and planar covers, 1989 (manuscript).

[4] H.H. Glover, J.P. Huneke, C.S. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory Ser. B 27 (1979) 332–370.

[5] P. Hliněný, $K_{4,4} - e$ has no finite planar cover, J. Graph Theory 27 (1998) 51–60.

[6] P. Hliněný, A note on possible extensions of Negami's conjecture, J. Graph Theory 32 (1999) 234–240.
[7] J.P. Huneke, A conjecture in topological graph theory, in: N. Robertson, P. Seymour (Eds.), Graph Structure Theory, Contemp. Math., 147, 1993, pp. 363–379.
[8] S. Negami, Enumeration of projective-planar embeddings of graphs, Discrete Math. 62 (1986) 299–306.
[9] S. Negami, The spherical genus and virtually planar graphs, Discrete Math. 70 (1988) 159–168.
[10] S. Negami, Graphs which have no finite planar covering, Bull. Inst. Math. Acad. Sinica 16 (1988) 377–384.
[11] S. Negami, Composite planar coverings of graphs, Discrete Math. 268 (2003) 207–216.
[12] Y. Suzuku, S. Negami, Projective-planar double coverings of 3-connected graphs, Yokohama Math. J. 50 (2003) 87–95.