A solution to the tennis ball problem

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Abstract

We present a complete solution to the so-called tennis ball problem, which is equivalent to counting lattice paths in the plane that use North and East steps and lie between certain boundaries. The solution takes the form of explicit expressions for the corresponding generating functions.

Our method is based on the properties of Tutte polynomials of matroids associated to lattice paths. We also show how the same method provides a solution to a wide generalization of the problem.

1 Introduction

The statement of the tennis ball problem is the following. There are $2n$ balls numbered $1, 2, 3, \ldots, 2n$. In the first turn balls 1 and 2 are put into a basket and one of them is removed. In the second turn balls 3 and 4 are put into the basket and one of the three remaining balls is removed. Next balls 5 and 6 go in and one of the four remaining balls is removed. The game is played n turns and at the end there are exactly n balls outside the basket. The question is how many different sets of balls may we have at the end outside the basket.

It is easy to reformulate the problem in terms of lattice paths in the plane that use steps $E = (1, 0)$ and $N = (0, 1)$. It amounts to counting the number of lattice paths from $(0, 0)$ to (n, n) that never go above the path $NE \cdots NE = (NE)^n$. Indeed, if $\pi = \pi_1\pi_2 \ldots \pi_{2n-1}\pi_{2n}$ is such a path, a moment's thought shows that we can identify the indices i such that π_{2n-i+1} is a N step with the labels of balls that end up outside the basket. The number of such paths is well-known to be a Catalan number, and this is the answer obtained in [2].

The problem can be generalized as follows [4]. We are given positive integers $t < s$ and sn labelled balls. In the first turn balls $1, \ldots, s$ go into the basket and t of them are removed. In the second turn balls $s + 1, \ldots, 2s$ go into the basket and t among the remaining ones are removed. After n turns, tn balls lie outside the basket, and again the question is how many different sets of balls may we have at the end. Letting $k = t, l = s - t$, the problem is seen as before to be equivalent to counting the number of lattice paths from $(0, 0)$ to (ln, kn) that use N and E steps and never go above the path $N^kE^l \cdots N^kE^l = (N^kE^l)^n$. This is the version of the problem we solve in this paper.

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From now on we concentrate on lattice paths that use N and E steps. To our knowledge, the only cases solved so far are $k = 1$ and $k = l = 2$. The case $k = 1$ is straightforward, the answer being a generalized Catalan number $\frac{1}{(l-1)n+1}\binom{ln}{n}$. The case $k = l = 2$ (corresponding to the original problem when $s = 4, t = 2$) is solved in [4] using recurrence equations; here we include a direct solution. This case is illustrated in Fig. 1, to which we refer next. A path π not above $(N^2E^2)^n$ is "almost" a Catalan path, in the sense that it can raise above the dashed diagonal line only through the dotted points. But clearly between two consecutive dotted points hit by π we must have an E step, followed by a Catalan path of odd semilength, followed by a N step. Thus, π is essentially a sequence of Catalan paths of odd semilength. If $G(z) = \sum_n \frac{1}{n+1}\binom{2n}{n} z^n$ is the generating function for the Catalan numbers, take the odd part $G_o(z) = (G(z) - G(-z))/2$. Then expand $1/(1 - zG_o(z))$ to obtain the sequence $1, 6, 53, 554, 6363, \ldots$, which agrees with the results in [4].

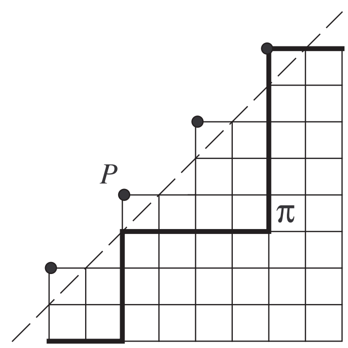

Figure 1: The path $\pi = \underline{EE}NN\underline{N}EEEENNN\underline{NN}EE$ not above $P = (N^2E^2)^4$. It has $i(\pi) = 3$ and $e(\pi) = 2$, corresponding to the steps underlined.

Let P be a lattice path from $(0, 0)$ to (m, r), and let $b(P)$ be the number of paths from $(0, 0)$ to (m, r) that never go above P. If PN denotes the path obtained from P by adding a N step at the end of P, then clearly $b(P) = b(PN)$. However, it is not possible to express $b(PE)$ simply in terms of $b(P)$, where PE has the obvious meaning. As is often the case in counting problems, one has to enrich the objects under enumeration with additional parameters that allow suitable recursive decompositions. This is precisely what is done here: equations (2) and (3) in the next section contain variables x and y, corresponding to two parameters that we define on lattice paths not above a given path P. These equations are the key to our solution.

The basis of our approach is the connection between lattice paths and matroids established in [1], where the link with the tennis ball problem was already remarked. For completeness, we recall the basic facts needed from [1] in the next section. In Section 3 we present our solution to the tennis ball problem, in the form of explicit expressions for the corresponding generating functions; see Theorem 1. In Section 4 we show how the same method can be applied to a more general problem. We conclude with some remarks.

2 Preliminaries

The contents of this section are taken mainly from [1], where the reader can find additional background and references on matroids, Tutte polynomials and lattice path enumeration.

A *matroid* is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection $\mathcal{B}$ of subsets of E, called *bases* of the matroid, that satisfy the following conditions: (1) No set in $\mathcal{B}$ properly contains another set in $\mathcal{B}$, and (2) for each pair of distinct sets B, B' in $\mathcal{B}$ and for each element $x \in B - B'$, there is an element $y \in B' - B$ such that $(B - x) \cup y$ is in $\mathcal{B}$.

Let P be a lattice path from $(0,0)$ to (m,r). Associated to P there is a matroid $M[P]$ on the set $\{1, 2, \ldots, m+r\}$ whose bases are in one-to-one correspondence with the paths from $(0,0)$ to (m,r) that never go above P. Given such a path $\pi = \pi_1\pi_2\ldots\pi_{m+r}$, the basis corresponding to π consists of the indices i such that π_i is a N step. Hence, counting bases of $M[P]$ is the same as counting lattice paths that never go above P.

For any matroid M there is a two-variable polynomial with non-negative integer coefficients, the Tutte polynomial $t(M; x, y)$. It was introduced by Tutte [6] and presently plays an important role in combinatorics and related areas (see [8]). The key property in this context is that $t(M; 1, 1)$ equals the number of bases of M.

Given a path P as above, there is a direct combinatorial interpretation of the coefficients of $t(M[P]; x, y)$. For a path π not above P, let $i(\pi)$ be the number of N steps that π has in common with P, and let $e(\pi)$ be the number of E steps of π before the first N step, which is 0 if π starts with a N step. See Fig. 1 for an illustration.

Then we have (see [1, Th. 5.4])

$$t(M[P]; x, y) = \sum_{\pi} x^{i(\pi)} y^{e(\pi)}, \tag{1}$$

where the sum is over all paths π not above P. A direct consequence is that $t(M[P]; 1, 1)$ is the number of such paths.

Furthermore, for the matroids $M[P]$ there is a rule for computing the Tutte polynomial that we use repeatedly (see [1, Section 6]). If PN and PE denote the paths obtained from P by adding a N step and an E step at the end of P, respectively, then

$$t(M[PN]; x, y) \quad = \quad x\, t(M[P], x, y), \tag{2}$$

$$t(M[PE]; x, y) \quad = \quad \frac{x}{x-1}\, t(M[P], x, y) + \left(y - \frac{x}{x-1}\right) t(M[P]; 1, y). \tag{3}$$

The right-hand side of (3) is actually a polynomial, since $x - 1$ divides $t(M[P]; x, y) - t(M[P]; 1, y)$. The key observation here is that we cannot simply set $x = y = 1$ in (3) to obtain an equation linking $t(M[PE]; 1, 1)$ and $t(M[P]; 1, 1)$.

For those familiar with matroid theory, we remark that $M[PN]$ and $M[PE]$ are obtained from $M[P]$, respectively, by adding an isthmus and taking a free extension; it is known that formulas (2) and (3) correspond precisely to the effect these two operations have on the Tutte polynomial of an arbitrary matroid.

From (1) and the definition of $i(\pi)$ and $e(\pi)$, equation (2) is clear, since any path associated to $M[PN]$ has to use the last N step. For completeness, we include a direct proof of equation (3).

We first rewrite the right-hand side of (3) as

$$
\begin{aligned}
&\frac{x}{x-1}(t(M[P];x,y)-t(M[P];1,y))+yt(M[P];1,y)= \\
&\sum_{\pi}\frac{x}{x-1}y^{e(\pi)}(x^{i(\pi)}-1)+y^{e(\pi)+1}= \\
&\sum_{\pi}y^{e(\pi)}(y+x+x^2+\cdots+x^{i(\pi)}),
\end{aligned}
$$

where the sums are taken over all paths π that do not go above P.

To prove the formula, for each path π not above P we find $i(\pi)+1$ paths not above PE such that their contribution to $t(M[PE];x,y)$ is $y^{e(\pi)}(y+x+x^2+\cdots+x^{i(\pi)})$. Consider first the path $\pi_0=E\pi$; it clearly does not go above PE and its contribution to the Tutte polynomial is $y^{e(\pi)+1}$. Now for each j with $1\le j\le i(\pi)$, define the path π_j as the path obtained from π by inserting an E step after the jth N step that π has in common with P (see Fig. 2). The path π_j has exactly j N steps in common with PE, and begins with $e(\pi)$ E steps.

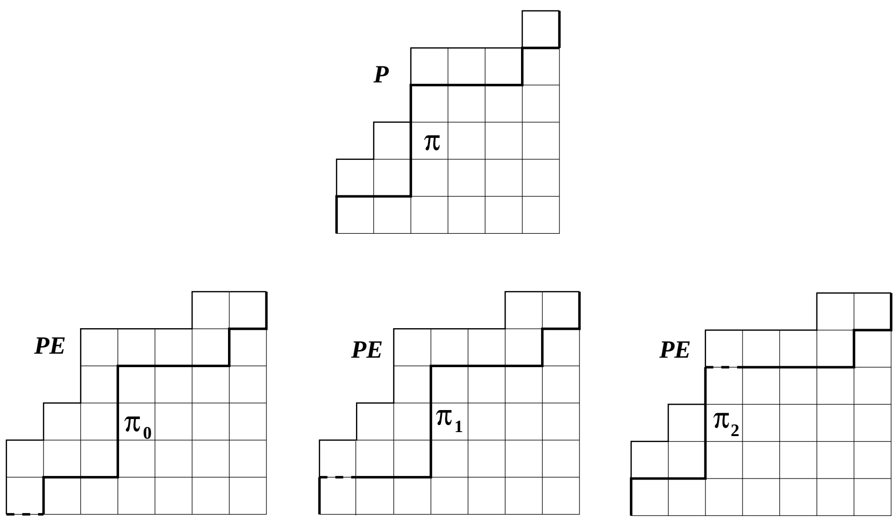

Figure 2: Illustrating the combinatorial proof of formula (3).

It remains only to show that each contribution to the Tutte polynomial of $M[PE]$ arises as described above. Let π' be a path that never goes above PE and consider the last N step that π' has in common with PE; clearly the next step must be E. Let $\widetilde{\pi}$ be the path obtained after removing this E step. Since π' had no N steps in common with PE after the E step removed, the path $\widetilde{\pi}$ does not go above P. Thus the path π' can be obtained from $\widetilde{\pi}$ by adding an E step after the $i(\pi')$-th N step that $\widetilde{\pi}$ has in common with P, and hence π' arises from $\widetilde{\pi}$ as in the previous paragraph. It is easy to show that π' cannot be obtained in any other way by applying the procedure described above, and this finishes the proof.

3 Main result

Let k, l be fixed positive integers, and let $P_n = (N^k E^l)^n$. Our goal is to count the number of lattice paths from $(0,0)$ to (ln, kn) that never go above P_n. From the considerations in the previous section, this is the same as computing $t(M[P_n]; 1, 1)$. Let

$$A_n = A_n(x,y) = t(M[P_n]; x, y).$$

By convention, P_0 is the empty path and $A_0 = 1$.

In order to simplify the notation we introduce the following operator Φ on two-variable polynomials:

$$\Phi A(x,y) = \frac{x}{x-1} A(x,y) + \left(y - \frac{x}{x-1}\right) A(1,y).$$

Then, by equations (2) and (3) we have

$$A_{n+1} = \Phi^l(x^k A_n),$$

where Φ^i denotes the operator Φ applied i times.

For each $n \geq 0$ and $i = 1, \ldots, l$, we define polynomials $B_{i,n}(x,y)$ and $C_{i,n}(y)$ as

$$\begin{aligned} B_{i,n} &= \Phi^i\left(x^k A_n(x,y)\right), \\ C_{i,n} &= B_{i,n}(1,y). \end{aligned}$$

We also set $C_{0,n}(y) = A_n(1,y)$. Notice that $B_{l,n} = A_{n+1}$, and $C_{0,n}(1) = A_n(1,1)$ is the quantity we wish to compute.

Then, by the definition of Φ, we have:

$$\begin{aligned} B_{1,n} &= \tfrac{x}{x-1} x^k A_n &&+ \left(y - \tfrac{x}{x-1}\right) C_{0,n}; \\ B_{2,n} &= \tfrac{x}{x-1} B_{1,n} &&+ \left(y - \tfrac{x}{x-1}\right) C_{1,n}; \\ &\cdots \\ B_{l,n} &= \tfrac{x}{x-1} B_{l-1,n} &&+ \left(y - \tfrac{x}{x-1}\right) C_{l-1,n}; \\ A_{n+1} &= B_{l,n}. \end{aligned}$$

In order to solve these equations, we introduce the following generating functions in the variable z (but recall the coefficients are polynomials in x and y):

$$A = \sum_{n \geq 0} A_n z^n, \qquad C_i = \sum_{n \geq 0} C_{i,n} z^n, \quad i = 0, \ldots, l.$$

We start from the last equation $A_{n+1} = B_{l,n}$ and substitute repeatedly the value of $B_{i,n}$ from the previous equation. Taking into account that $\sum_n A_{n+1} z^n = (A-1)/z$, a simple computation yields

$$\frac{A-1}{z} = \frac{x^{k+l}}{(x-1)^l} A + (yx - y - x) \sum_{i=1}^{l} \frac{x^{i-1}}{(x-1)^i} C_{l-i}.$$

We now set $y = 1$ and obtain

$$A\left((x-1)^l - zx^{k+l}\right) = (x-1)^l - z\sum_{i=1}^{l} x^{i-1}(x-1)^{l-i}\,C_{l-i}, \tag{4}$$

where it is understood that from now on we have set $y = 1$ in the series A and C_i.

By Puiseux's theorem (see [5, Chap. 6]), the algebraic equation in w

$$(w-1)^l - zw^{k+l} = 0 \tag{5}$$

has $k+l$ solutions in the field $\mathbb{C}^{\mathrm{fra}}((z)) = \{\sum_{n \geq n_0} a_n z^{n/N}\}$ of fractional Laurent series. Proposition 6.1.8 in [5] tells us that exactly l of them are fractional power series (without negative powers of z); let them be $w_1(z), \ldots, w_l(z)$.

We substitute $x = w_j$ in (4) for $j = 1, \ldots, l$, so that the left-hand side vanishes, and obtain a system of l linear equations in $C_0, C_1, \ldots, C_{l-1}$, whose coefficients are expressions in the w_j, namely

$$\sum_{i=1}^{l} w_j^{i-1}(w_j-1)^{l-i} zC_{l-i} = (w_j-1)^l, \qquad j = 1, \ldots, l. \tag{6}$$

Notice that, since the C_i are (ordinary) power series, the solutions w of (5) that we substitute in (4) cannot have negative powers of z, hence they must be $w_1, \ldots, w_l$. We remark the similarity of this technique with the one devised by Tutte for counting rooted planar maps (see, for instance, [7]).

It remains only to solve (6) to obtain the desired series $C_0 = \sum_n A_n(1,1)z^n$. The system (6) can we written as

$$\sum_{i=0}^{l-1} \left(\frac{w_j}{w_j-1}\right)^i zC_{l-i-1} = w_j - 1, \qquad j = 1, \ldots, l.$$

The left-hand sides of the previous equations can be viewed as the result of evaluating the polynomial $\sum_{i=0}^{l-1}(zC_{l-i-1})X^i$ of degree $l-1$ at $X = w_j/(w_j-1)$, for j with $1 \leq j \leq l$. Using Lagrange's interpolation formulas, we get that the coefficient of X^{l-1} in this polynomial is

$$zC_0 = \sum_{j=1}^{l} \frac{w_j - 1}{\prod_{i \neq j}\left(\frac{w_j}{w_j-1} - \frac{w_i}{w_i-1}\right)}.$$

By straightforward manipulation this last expression is equal to

$$-\prod_{j=1}^{l}(1-w_j)\sum_{j=1}^{l}\frac{(w_j-1)^{l-1}}{\prod_{i\neq j}(w_j-w_i)} = -\prod_{j=1}^{l}(1-w_j),$$

where the last equality follows from an identity on symmetric functions (set $r = 0$ in Exercise 7.4 in [5]).

Thus we have proved the following result.

Theorem 1. *Let k, l be positive integers. Let q_n be the number of lattice paths from $(0,0)$ to (ln, kn) that never go above the path $(N^k E^l)^n$, and let $w_1, \dots, w_l$ be the unique solutions of the equation*

$$(w-1)^l - z w^{k+l} = 0$$

that are fractional power series. Then the generating function $Q(z) = \sum_{n \geq 0} q_n z^n$ is given by

$$Q(z) = \frac{-1}{z}(1-w_1) \cdots (1-w_l).$$

Remark that, by symmetry, the number of paths not above $(N^l E^k)^n$ must be the same as in Theorem 1, although the algebraic functions involved in the solution are roots of a different equation.

In the particular case $k = l$ the solution can be expressed directly in terms of the generating function $G(z) = \sum_n \frac{1}{n+1}\binom{2n}{n} z^n$ for the Catalan numbers, which satisfies the quadratic equation $G(z) = 1 + zG(z)^2$. Indeed, (5) can be rewritten as

$$w = 1 + z^{1/k} w^2,$$

whose (fractional) power series solutions are $G(\zeta^j z^{1/k})$, $j = 0, \dots, k-1$, where ζ is a primitive k-th root of unity. For instance, for $k = l = 3$ (corresponding to $s = 6, t = 3$ in the original problem), $\zeta = \exp(2\pi i/3)$ and we obtain the solution

$$\frac{-1}{z}(1 - G(z^{1/3}))(1 - G(\zeta z^{1/3}))(1 - G(\zeta^2 z^{1/3})) = \\ 1 + 20z + 662z^2 + 26780z^3 + 1205961z^4 + 58050204z^5 + \cdots .$$

In the same way, if l divides k and we set $p = (k+l)/l$, the solution can be expressed in terms of the generating function $\sum_n \frac{1}{(p-1)n+1}\binom{pn}{n} z^n$ for generalized Catalan numbers; the details are left to the reader. As an example, for $k = 4$, $l = 2$, we obtain the series

$$\frac{-1}{z}(1 - H(z^{1/2}))(1 - H(-z^{1/2})) = \\ 1 + 15z + 360z^2 + 10463z^3 + 337269z^4 + 11599668z^5 + \cdots ,$$

where $H(z) = \sum_n \frac{1}{2n+1}\binom{3n}{n}$ satisfies $H(z) = 1 + zH(z)^3$.

4 A further generalization

In this section we solve a further generalization of the tennis ball problem. Given fixed positive integers $s_1, t_1, \dots, s_r, t_r$ with $t_i < s_i$ for all i, let $s = \sum s_i, t = \sum t_i$. There are sn labelled balls. In the first turn we do the following: balls $1, \dots, s_1$ go into the basket and t_1 of them are removed; then balls $s_1 + 1, \dots, s_1 + s_2$ go into the basket and among the remaining ones t_2 are removed; this goes on until we introduce balls $s - s_r + 1, \dots, s$, and remove t_r balls. After n turns there are tn balls outside the basket and the question is again how many different sets of tn balls may we have at the end.

The equivalent path counting problem is: given $k_1, l_1, \dots, k_r, l_r$ positive integers with $k = \sum k_i$, $l = \sum l_i$, count the number of lattice paths from $(0,0)$ to (ln, kn) that never go above the path $P_n = (N^{k_1}E^{l_1} \cdots N^{k_r}E^{l_r})^n$. The solution parallels the one presented in Section 3. We keep the notations and let $A_n = t(M[P_n]; x, y)$, so that

$$A_{n+1} = \Phi^{l_r}(x^{k_r} \cdots \Phi^{l_1}(x^{k_1}A_n) \cdots).$$

As before, we introduce l polynomials $B_{i,n}(x, y)$ and $C_{i,n}(y) = B_{i,n}(1, y)$, but the definition here is a bit more involved:

$$\begin{aligned}
B_{i,n} &= \Phi^i(x^{k_1}A_n), & i &= 1, \ldots, l_1; \\
B_{l_1+i,n} &= \Phi^i(x^{k_2}B_{l_1,n}), & i &= 1, \ldots, l_2; \\
B_{l_1+l_2+i,n} &= \Phi^i(x^{k_3}B_{l_1+l_2,n}), & i &= 1, \ldots, l_3; \\
&\cdots \\
B_{l-l_r+i,n} &= \Phi^i(x^{k_r}B_{l-l_r,n}), & i &= 1, \ldots, l_r.
\end{aligned} \tag{7}$$

We also set $C_{0,n}(y) = A_n(1, y)$. Again, from the definition of Φ, we obtain a set of equations involving A_n, $A_{n+1} = B_{l,n}$, the $B_{i,n}$ and $C_{i,n}$. We define generating functions A and C_i ($i = 0, \ldots, l$) as in Section 3.

Starting with $A_{n+1} = B_{l,n}$, we substitute repeatedly the values of the $B_{i,n}$ from previous equations and set $y = 1$. After a simple computation we arrive at

$$A\left((x-1)^l - zx^{k+l}\right) = (x-1)^l + z\,U(x, C_0, \ldots, C_{l-1}), \tag{8}$$

where U is a polynomial in the variables $x, C_0, \ldots, C_{l-1}$. Observe that the difference between (8) and equation (4) is that now U is not a concrete expression but a certain polynomial that depends on the particular values of the k_i and l_i.

Let $w_1, \ldots, w_l$ be again the power series solutions of (5). Substituting $x = w_j$ in (8) for $j = 1, \ldots, l$, we obtain a system of linear equations in the C_i. Since the coefficients are rational functions in the w_j, the solution consists also of rational functions; they are necessarily symmetric since the w_j, being conjugate roots of the same algebraic equation, are indistinguishable.

Thus we have proved the following result.

Theorem 2. *Let $k_1, l_1, \ldots, k_r, l_r$ be positive integers, and let $k = \sum k_i$, $l = \sum l_i$. Let q_n be the number of lattice paths from $(0,0)$ to (ln, kn) that never go above the path $(N^{k_1}E^{l_1} \cdots N^{k_r}E^{l_r})^n$, and let $w_1, \ldots, w_l$ be the unique solutions of the equation*

$$(w-1)^l - zw^{k+l} = 0$$

that are fractional power series. Then the generating function $Q(z) = \sum_{n \geq 0} q_n z^n$ is given by

$$Q(z) = \frac{1}{z} R(w_1, \ldots, w_l),$$

where R is a computable symmetric rational function of $w_1, \ldots, w_l$.

As an example, let $r = 2$ and $(k_1, l_1, k_2, l_2) = (2, 2, 1, 1)$, so that $k = l = 3$. Solving the corresponding linear system we obtain

$$R = \frac{(1-w_1)(1-w_2)(1-w_3)}{2w_1w_2w_3 - (w_1w_2 + w_1w_3 + w_2w_3)},$$

and

$$Q(z) = \frac{1}{z} R = 1 + 16z + 503z^2 + 19904z^3 + 885500z^4 + 42298944z^5 + \cdots .$$

It should be clear that for any values of the k_i and l_i the rational function R can be computed effectively. In fact, a simple computer program could be written that on input $k_1, l_1, \ldots, k_r, l_r$, outputs R.

5 Concluding Remarks

It is possible to obtain an expression for the generating function of the full Tutte polynomials $A_n(x, y)$ defined in Section 3. We have to find the values of $C_0, C_1, \ldots, C_{l-1}$ satisfying the system (6) and substitute back into (4). After some algebraic manipulation, the final expression becomes

$$\sum_{n \geq 0} A_n(x, y) z^n = \frac{-(x - w_1) \cdots (x - w_l)}{(zx^{k+l} - (x-1)^l)(y - w_1(y-1)) \cdots (y - w_l(y-1))}.$$

Taking $x = y = 1$ we recover the formula stated in Theorem 1.

On the other hand, references [3] and [4] also study a different question on the tennis ball problem, namely to compute the sum of the labels of the balls outside the basket for all possible configurations. For a given lattice path P_n, this amounts to computing the sum of all elements in all bases of the matroid $M[P_n]$. We remark that this quantity does not appear to be computable from the corresponding Tutte polynomials alone.

Finally, as already mentioned, the technique of forcing an expression to vanish by substituting algebraic functions was introduced by Tutte in his landmark papers on the enumeration of planar maps. Thus the present paper draws in more than one way on the work of the late William Tutte.

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