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Highly arc transitive digraphs: reachability, topological groups

Aleksander Malnič[a], Dragan Marušič[a], Rögnvaldur G. Möller[b], Norbert Seifter[c], Vladimir Trofimov[d], Boris Zgrablič[a]

[a] *IMFM, University of Ljubljana, 1111 Ljubljana, Slovenia*
[b] *Science Institute, University of Iceland, IS-107 Reykjavik, Iceland*
[c] *Institut für Mathematik und Angewandte Geometrie, Montanuniversität Leoben, A-8700 Leoben, Austria*
[d] *Institute of Mathematics and Mechanics, Russian Academy of Sciences, Ekaterinburg, Russia*

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Abstract

Let D be a locally finite, connected, 1-arc transitive digraph. It is shown that the reachability relation is not universal in D provided that the stabilizer of an edge satisfies certain conditions which seem to be typical for highly arc transitive digraphs. As an implication, the reachability relation cannot be universal in highly arc transitive digraphs with prime in- or out-degree.

Two different aspects of the connection between highly arc transitive digraphs and the theory of totally disconnected locally compact groups are also considered.

1. Introduction

Let $D(V, E)$ denote a digraph with vertex-set $V(D)$ and edge-set $E(D) \subseteq V(D) \times V(D)$. If not stated otherwise, digraphs considered in this paper are connected, infinite, locally finite (i.e. the in-degree $d_D^-(v)$ and the out-degree $d_D^+(v)$ of each vertex $v \in V(D)$ are both finite) and have no loops. Here "connected" means that for any pair of vertices there is a path joining them in the underlying undirected graph. Denote by $\mathrm{Aut}(D)$ the automorphism group of D. If $\mathrm{Aut}(D)$ acts transitively on $V(D)$ then all vertices have the same in-degree and the same out-degree, which are then denoted by d_D^- and d_D^+,

E-mail address: seifter@unileoben.ac.at (N. Seifter).

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respectively. For simplicity reasons the subscript D is omitted whenever the digraph in question is clear from the context. Furthermore, for a vertex $v \in V(D)$ define $N_D^+(v) = \{u \in V(D) \mid (v, u) \in E(D)\}$ and $N_D^-(v) = \{u \in V(D) \mid (u, v) \in E(D)\}$.

Let $s \geq 0$ be an integer. An *s-arc* in a digraph D is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices such that $(v_i, v_{i+1}) \in E(D)$ for each i, $0 \leq i \leq s - 1$, and $v_{i-1} \neq v_{i+1}$ for each i, $1 \leq i \leq s - 1$. If Aut(D) acts transitively on the set of s-arcs, then D is called *s-arc transitive*. A digraph D is said to be *highly arc transitive* if Aut(D) is s-arc transitive for all finite $s \geq 0$.

Highly arc transitive digraphs were considered in [1] from several different viewpoints, leading to, among others, constructions of highly arc transitive digraphs and a characterization of universal covering digraphs of highly arc transitive digraphs. Moreover, a number of interesting problems were posed in [1], some of which have already been partially or completely solved (see [3, 6, 7, 9]). An additional motivation for the study of these digraphs is their connection to totally disconnected locally compact groups (see [8]).

In Section 3 we consider the reachability relation. An *alternating walk* in a digraph D is a sequence $(v_0, v_1, \ldots, v_n)$ of vertices such that, for each i, $0 < i < n$, either (v_{i-1}, v_i) and (v_{i+1}, v_i) are edges, or (v_i, v_{i-1}) and (v_i, v_{i+1}) are edges in D. If e and e' are edges in D then we say that e' is *reachable* from e, written $e\mathcal{A}e'$, if there exists an alternating walk in D whose initial and terminal edges are e and e'. Of course $\mathcal{A}$ is an equivalence relation, and the equivalence class containing e is denoted by $\mathcal{A}(e)$.

The relation $\mathcal{A}$ is closely related to the so-called property **Z**. (A digraph D has *property* **Z** if there exists a homomorphism from D onto the directed infinite line which is a Cayley digraph of the additive group of integers. Obviously, the reachability relation in such a digraph is not universal.) In [1] the problem of constructing highly arc transitive digraphs without property **Z** was posed and in [6] the first examples of such digraphs with finite in- and out-degrees were given. On the other hand it was shown in [9] that highly arc transitive digraphs with finite and unequal in- and out-degrees always have property **Z**. (In [3] an example is constructed of a highly arc transitive digraph that does not have property **Z** and has infinite in-degree and finite out-degree.) Hence, when looking for examples of highly arc transitive digraphs with finite in- and out-degrees and with universal reachability relation—a problem also posed in [1]—digraphs with equal in- and out-degrees are the only candidates. In Proposition 3.2 a connection between alternating walks in a given digraph and a certain somewhat technical condition placed on the stabilizer of an edge is established. As an immediate consequence it is shown that the reachability relation in a 1-arc transitive digraph D is not universal provided the in- and out-degrees of both are equal to a prime integer (Theorem 3.3). Moreover, the same holds provided the stabilizer of an edge $e \in E(D)$, with respect to its action on edges adjacent to e, induces a group of a special form which, nevertheless, seems to be typical in the case of highly arc transitive digraphs D (Theorem 3.4).

Section 4 deals with the connection between highly arc transitive digraphs and totally disconnected locally compact groups (see [8]). The first result is that if g is an element of a totally disconnected locally compact group G and the conjugacy class of g has compact closure in G then $s(g) = 1$ where s is the scale function on G. The proof of this relies on a simple result about bounded automorphisms of highly arc transitive digraphs. (An automorphism g of a connected digraph is said to be *bounded* if there is a constant C

such that $\mathrm{dist}(v, g(v)) \leq C$ for all vertices v, where $\mathrm{dist}(-, -)$ denotes the usual digraph distance.) Next is a short proof of a theorem of Willis ([14, Theorem 2]).

In Section 2 we discuss connections between the concept of out-spread (see definition below and [1, Definition 3.5]) and results from [8]. A new characterization of connected highly arc transitive digraphs that have out-spread 1 is also given. Furthermore we present a result on bounded automorphisms of highly arc transitive digraphs which plays an important role in the proofs of the results in Section 4 but also seems to be of some interest on its own. We also show that for a highly arc transitive digraph D already the stabilizer of a vertex together with one additional automorphism generates a subgroup of $\mathrm{Aut}(D)$ which acts highly arc transitively on D.

2. Definitions, bounded automorphisms and out-spread

A *line* L in a digraph D is an infinite sequence $(\ldots v_{-1}, v_0, v_1, v_2 \ldots)$ of vertices such that $(v_i, v_{i+1}) \in E(D)$ for every $i \in \mathbf{Z}$. Since infinite connected highly arc transitive digraphs contain no directed cycles, the vertices of a line in such a graph are pairwise different. If L is as above, then $L[v_i, \infty)$ and $L[v_i, -\infty)$ denote the sequences $(v_i, v_{i+1}, v_{i+2}, \ldots)$ and $(v_i, v_{i-1}, v_{i-2}, \ldots)$, respectively. We also say that $L[v_i, \infty)$ is a *positive halfline* and $L[v_i, -\infty)$ is a *negative halfline*.

Two positive (negative) halflines P and Q in D are *equivalent* if in the underlying undirected graph there are infinitely many disjoint paths connecting vertices in P to vertices in Q. The equivalence classes of all infinite paths (not necessarily directed) with respect to this relation are called the *ends* of D. The concept of ends can be defined in several different ways; this form of the definition is due to Halin [5].

The set of *descendants* $\mathrm{desc}(v)$ of a vertex v of a digraph D is the set of those vertices of D which can be reached from v by a directed path such that its first edge (if such an edge exists) is of the form (v, w) for some $w \in V(D)$. The set of *ancestors* ($\mathrm{anc}(v)$) of a vertex v is the set of those vertices of D for which v is a descendant. For a subset $A \subseteq V(D)$ we define the sets $\mathrm{desc}(A) = \bigcup_{v \in A} \mathrm{desc}(v)$ and $\mathrm{anc}(A) = \bigcup_{v \in A} \mathrm{anc}(v)$. If L is a line in D, then the subdigraph induced by $\mathrm{desc}(L)$ is denoted by F_L, and E_L denotes the subdigraph induced by $\mathrm{anc}(L)$.

For a connected digraph D, define $\mathrm{dist}(u, v)$ as the distance between vertices u and v in the underlying undirected graph ($\mathrm{dist}(u, v)$ is the length of the shortest possible path between u and v).

Let $v \in V(D)$ and n a positive integer. By $B(v, n) = \{u \in V(D) \mid \mathrm{dist}(v, u) \leq n\}$ we denote the ball of radius n with center v. In the case D admits a vertex transitive group of automorphisms, the cardinality of $B(v, n)$ does not depend on the vertex v.

In [7] the descendants of lines are investigated. Some of the results in [7] are repeatedly used in the proofs later on. To facilitate reading of this paper we list those results below:

Lemma 2.1 ([7, Lemma 1]). *Let D be an infinite locally finite highly arc transitive digraph. Then $\mathrm{Aut}(D)$ is transitive on the set of directed lines. Furthermore, if $L = (\ldots, v_{-1}, v_0, v_1, \ldots)$ is a directed line in D then there exists, for every $k \in \mathbf{Z}$, an element $g \in \mathrm{Aut}(D)$ such that $g(v_i) = v_{i+k}$ for all $i \in \mathbf{Z}$.*

Lemma 2.2 ([7, Lemma 3]). *Let D be an infinite locally finite highly arc transitive digraph. Suppose L is a line in D. Then*

(1) $d^+_{F_L} = d^+_D$.
(2) *F_L is highly arc transitive.*
(3) *The undirected graph corresponding to F_L has more than one end.*

Theorem 2.3 ([7, Theorem 1]). *Let D be a locally finite highly arc transitive digraph. Suppose that there is a line $L = (\dots, v_{-1}, v_0, v_1, \dots)$ such that $V(D) = \mathrm{desc}(L)$. Then there exists a surjective homomorphism $\phi : D \to T$ where T is a directed tree with $d^-_T = 1$ and finite d^+_T. The automorphism group of D has a natural action on T as a group of automorphisms such that $\phi(g(v)) = g(\phi(v))$ for every $g \in \mathrm{Aut}(D)$ and every $v \in V(D)$. The action of $\mathrm{Aut}(D)$ on T is highly arc transitive. Furthermore, the fibers $\phi^{-1}(v)$, $v \in V(T)$ are finite and all have the same number of elements.*

Remark. These results can clearly be formulated analogously for E_L.

Bounded automorphisms of graphs have been investigated for example in [4, 10]. In [4, 16] the connections between bounded automorphisms, and FC elements (elements with finite conjugacy class) and FC^- elements (elements in a topological group which have a conjugacy class with compact closure) are pointed out. In Section 4 we use the following result on bounded automorphisms of highly arc transitive digraphs to establish a result about FC^- elements in topological groups.

Proposition 2.4. *Let L be a line in the locally finite highly arc transitive infinite digraph D. Suppose that g is a bounded automorphism of D. If L_1 and L_2 are two positive halflines in F_L, and $L_2 = g(L_1)$ then L_1 and L_2 are in the same end of F_L.*

If g is a bounded automorphism of D such that $g(v) \in \mathrm{desc}(v)\backslash\{v\}$ for some vertex v then D has precisely two ends.

Proof. Since $L_1 = (v_0, v_1, \dots)$ and $L_2 = (w_0, w_1, \dots)$ are both contained in F_L the two halflines will have a common ancestor $v \in V(F_L)$. Suppose g is a bounded automorphism of D which maps L_1 onto L_2. We can without loss of generality assume that some vertex w_n is in $\mathrm{desc}(v_0)$. If necessary take a directed path P from v to v_0 and replace L_1 with $P \cup L_1$ and L_2 with $g(P \cup L_1)$. Of course a finite part of $g(P \cup L_1)$, namely $g(P)$, may not be contained in F_L. But since this would not change the arguments of the following two paragraphs we do not take care of this case separately and assume in the sequel that $g(P \cup L_1) \subset F_L$ holds.

Since g is bounded we can assume that there is an integer $k > 0$ such that for infinitely many $v_i \in V(L_1)$, $i \in \mathcal{I}$, we can find infinitely many pairwise distinct $w_{j_i} \in V(L_2)$ such that $w_{j_i} \in B(v_i, k)$. By 2.1, there is an $a \in \mathrm{Aut}(D)$ such that $a^i(v_0) = v_i$ for all integers i and therefore $B(v_i, k) = a^i(B(v_0, k))$ also holds. Since $B(v_0, k)$ is finite, there is a vertex $y \in B(v_0, k)$ and infinitely many positive integers p_i such that $a^{p_i}(y) \in L_2[w_n, \infty)$. For simplicity we renumber the p_i such that $p_1, p_2, p_3, \dots$ are these numbers and $p_{l+1} > p_l$ for all $l \geq 1$.

Let $a^{p_1}(y) = w_r$ for some $w_r \in V(L_2)$. Then there is a directed path R from v_0 to w_r. Under each $a^{p_l - p_1}$ the vertex w_r is mapped to a vertex of L_2 and v_0 is mapped to a vertex of L_1. Hence R is mapped onto a directed path from L_1 to L_2 under each $a^{p_l-p_1}$. But this implies that L_1 and L_2 are in the same end of F_L.

Now we come to the proof of the latter statement in the lemma. Define K as the line $\ldots g^{-1}(Q)Qg(Q)g^2(Q)\ldots$ where Q is a directed path with initial vertex v and terminal vertex w such that $g(v) \in N_D^+(w)$. Let $\phi : F_K \to T$ be the map described in Theorem 2.3. If F_K has more than two ends then there is a directed positive halfline K_1 that starts at v but does not belong to the same end of F_K as $K[v, \infty)$. Then $g(K_1)$ cannot belong to the same end as $g(K[v, \infty)) = K[g(v), \infty)$. Looking at the action of g on the tree T it is clear that $\phi(K_1)$ and $\phi(g(K_1))$ belong to different ends of T and hence K_1 and $g(K_1)$ belong to different ends of F_K. This contradicts the first part of the Lemma. We conclude that F_K has only two ends. Similarly E_K has only two ends and we can now conclude that D has only two ends. □

Let D be a digraph with finite out-degree and vertex transitive automorphism group. Choose a vertex v in D and define p_k as the number of vertices u in D such that there is a directed path of length k from v to u. Because of the vertex transitivity of the automorphism group, the value of p_k does not depend on the choice of v. The *out-spread* of D is defined as

$$\limsup_{k\to\infty} (p_k)^{1/k}.$$

The *in-spread* of a digraph D with finite in-degree is defined as the out-spread of the digraph one gets by reversing the direction of all the arcs in D.

In [7, Theorem 2] it is shown that the out-spread of a highly arc transitive digraph is always an integer. Furthermore, in [7] the emphasis is on the digraphs F_L and E_L. It is thus of particular interest to look at highly arc transitive locally finite digraphs D such that $D = F_L$ or $D = E_L$. The theorem below gives a characterization of such digraphs in terms of the out-spread of D.

Lemma 2.5. *Let D be a locally finite connected highly arc transitive digraph and L a line in D. If F_L has only two ends then $D = E_L$. Similarly, if E_L has only two ends then $D = F_L$.*

Proof. Let $L = (\ldots, v_{-1}, v_0, v_1, v_2, \ldots)$. Suppose F_L has only two ends. We know that $N^-_{E_L}(v) = N^-_D(v)$ for all vertices v in E_L. In order to prove the lemma we must show that $N^+_{E_L}(v) = N^+_D(v)$ for all vertices v in E_L and then the result follows because D is connected. Let $\phi : F_L \to T$ be a map analogous to the map defined in Theorem 2.3. Set $V_i = \phi^{-1}(\phi(v_i))$. Because F_L has only two ends we know that T is just a line. Hence, if $g \in \mathrm{Aut}(F_L)$ and v and $g(v)$ are in the same ϕ-fiber then all the ϕ-fibers are invariant under g. We now show that we can find a number k such that all the vertices in V_k are descendants of v_1.

By Lemma 2.1 there exists an automorphism h of D such that $h(v_i) = v_{i+1}$ holds for all $i \in \mathbf{Z}$. Note that then $h(V_i) = V_{i+1}$ for all $i \in \mathbf{Z}$. Clearly F_L is invariant under h. In addition each $w \in V_0$ is the descendant of some vertex $v_{n(w)} \in L$. Let $-m = \min\{n(w) \mid w \in V_0\}$. Then all vertices of V_0 are descendants of $v_{-m} \in V_{-m}$ and clearly all vertices of $h^{m+1}(V_0)$ are descendants of $h^{m+1}(v_{-m}) \in V_1$. Hence, if $k \geq m$, each vertex of V_k is contained in $\mathrm{desc}(v_1)$.

If v is any vertex in V_1 then clearly every vertex in V_k is a descendant of v. Note that every vertex in $N_D^+(v_0)$ is contained in V_1 and thus every vertex in $N_D^+(v_0)$ is an ancestor of V_k. Thus $N_D^+(v_0) \subseteq \mathrm{anc}(L)$. Hence the out-degree of the vertices in D is the same as the out-degree of the vertices in E_L and thus $E_L = D$. □

Theorem 2.6. *Let D be a locally finite connected highly arc transitive digraph. The in-spread of D is 1 if and only if there is a line L such that $D = F_L$. Similarly, the out-spread of D is 1 if and only if there is a line L such that $D = E_L$.*

Proof. By Lemma 2.2 and the remark following Theorem 2.3 E_L has more than one end. Since the in-spread of D and therefore also of E_L is equal to 1, E_L has exactly two ends for every line L. Then it follows from the lemma above that $D = F_L$. The other statement of the proposition follows in the same way. □

Remark. Lemma 2.1 states that in a locally finite highly arc transitive digraph the automorphism group acts transitively on the set of lines. Hence we could say in the above theorem that D has in-spread 1 if and only if $D = F_L$ for every line L in D and analogously for the out-spread.

We end this section with the following result about automorphism groups of highly arc transitive digraphs.

Proposition 2.7. *Let D be an infinite connected highly arc transitive digraph and let a group H act highly arc transitively on D. Furthermore, let $v \in V(D)$, $(v, u) \in E(D)$, and let g be an automorphism of D with $g(v) = u$. Then the group $G = \langle H_v \cup g \rangle$ acts highly arc transitively on D.*

Proof. For each $s \geq 0$, the group H_v acts transitively on the set of s-arcs in D with initial vertex v. At the same time, the group G acts transitively on $V(D)$, since D is connected and H_v acts transitively on $N_D^+(v)$ and $N_D^-(v)$, where $g(v) \in N_D^+(v)$ and $g^{-1}(v) \in N_D^-(v)$. The result follows. □

3. Reachability

If a digraph D is 1-arc transitive, then the subdigraphs $\langle \mathcal{A}(e) \rangle$ induced by $\mathcal{A}(e)$ are isomorphic to a fixed digraph which will be denoted by $\Delta(D)$. In [1] the following result about $\Delta(D)$ was shown:

Proposition 3.1 ([1, Proposition 1.1]). *Let D be a connected 1-arc transitive digraph. Then $\Delta(D)$ is 1-arc transitive and connected. Furthermore, either*

(1) *$\mathcal{A}$ is universal and $\Delta(D) = D$, or*
(2) *$\Delta(D)$ is bipartite.*

There are examples of highly arc transitive infinite digraphs for which the reachability relation is universal (see [1, p. 378]), but no example of an infinite locally finite highly arc transitive digraph with universal reachability relation is known. As mentioned in the introduction all highly arc transitive digraphs with finite and unequal in- and out-degree have property **Z** which implies that they cannot have universal reachability relation. Hence the following results are formulated only for digraphs with equal in- and out-degrees.

We have chosen a quite general approach to the question of universality of the reachability relation. It is based on the analysis of the action of the stabilizer of an edge $(u, v) \in E(D)$ on the set $N_D^+(v)$ in 1-arc transitive digraphs D. The results of this section can be seen as illustrations of this approach.

Let a group G act on a set V and let $\Omega \subseteq V$ be setwise fixed by G. By G_Ω we denote the pointwise stabilizer in G of Ω, and by $G^\Omega \cong G/G_\Omega$ the restriction of G to Ω. For elements $u, v \in V$ we define $G_{u,v} = \{g \in G \mid g(u) = u \text{ and } g(v) = v\}$.

Proposition 3.2. *Let D be a digraph such that in-degree and out-degree of any vertex are equal to a fixed integer d. Let $G = \mathrm{Aut}(D)$, $e = (u, v) \in E(D)$ and $\Omega \subseteq V(D)\backslash\{u, v\}$. Furthermore, let H be a subgroup of $G_{u,v}$ which fixes Ω setwise and stabilizes no vertex from Ω. If all proper subgroups of H^Ω have index at least d, then there is no alternating walk with initial edge e and terminal vertex in Ω.*

Proof. Let f be an edge adjacent to e such that $e \cup f$ determines an alternating walk, and let $w \notin \{u, v\}$ denote a vertex of f. Since the orbit of w under H has length less than d, the stabilizer H_w of w has index less than d in H. Therefore $|H^\Omega : H_w^\Omega| < d$ also holds. But this implies $H_w^\Omega = H^\Omega$ by our assumption. Since H stabilizes no vertex from Ω, H_w also has this property. By induction this property extends to the stabilizer of all vertices contained in an alternating walk with initial edge e.

In particular let P be such an alternating walk with its terminal vertex $x \in \Omega$. Then the pointwise stabilizer $H_{V(P)}$ satisfies $H_{V(P)}^\Omega = H^\Omega$ and fixes x, contradicting the fact that H stabilizes no vertex from Ω. □

Theorem 3.3. *Let p be a prime and let D be a 2-arc transitive digraph with in-degree and out-degree equal to p. Then the reachability relation in D is not universal.*

Proof. Let (u, v) be an edge of D. As D is 2-arc transitive, the stabilizer $G_{u,v}$, where $G = \mathrm{Aut}(D)$, acts transitively on $N_D^+(v)$. Since $|N_D^+(v)| = p$ is a prime there is a group $H \le G_{u,v}$ which restriction to $N_D^+(v)$ is isomorphic to $\mathbf{Z}_p$. By replacing Ω with $N_D^+(v)$ in Proposition 3.2, the result follows. □

We mention that the examples of those highly arc transitive digraphs without property **Z** which were constructed in [6] do not have a universal reachability relation. This follows also from Theorem 3.3 since those digraphs have in-degree and out-degree equal to 2.

Similar arguments can be used to prove the following more general result. We mention that also Theorem 3.3 can be deduced from the following one by putting $d = p$ and $K \cong \mathbf{Z}_p$.

Theorem 3.4. *Let D be a 1-arc transitive digraph with $d_D^+ = d_D^- = d > 1$. Let $(u, v) \in E(D)$ be such that the restriction $G_{u,v}^{N_D^+(v)}$ of the stabilizer $G_{u,v}$ to $N_D^+(v)$ contains a subgroup $K \neq 1$ which has no nontrivial permutation representation of degree less than d. (For example, let K be a simple group in a nontrivial permutation representation of smallest degree.) Then the reachability relation in D is not universal.*

Proof. By replacing Ω with $N_D^+(v)$ and H with the preimage of K in $G_{u,v}$ in Proposition 3.2, the result follows. □

We mention that the nontrivial permutation representations of smallest degree of all finite simple groups are known. See [12], where the determination of such representations of finite simple groups modulo their classification was completed. In connection with the results of this section it would also be interesting to know which permutation groups can arise as $\mathrm{Aut}_{u,v}^{N_D^+(v)}$ where (u, v) is an edge of a locally finite highly arc transitive digraph D.

4. Highly arc transitive digraphs and topological groups

The theory of locally compact groups is that part of the theory of topological groups that has widest appeal and most applications. When looking at locally compact groups there are the connected groups on one end of the spectrum and the totally disconnected groups on the other end of the spectrum. The automorphism group of a locally finite connected graph with the topology of pointwise convergence is an example of a totally disconnected locally compact group [11, 16].

An important result in the theory of locally compact totally disconnected groups is the theorem of van Dantzig [2] that such a group must always contain a compact open subgroup. The applications that follow involve concepts and ideas from the structure theory developed by Willis, see [13, 15]. The important concepts of Willis's theory are the scale function and tidy subgroups.

Definition 4.1. (A) Let G be a locally compact totally disconnected group and g an element in G. For a compact open subgroup U in G define

$$U_+ = \bigcap_{i=0}^{\infty} g^i U g^{-i} \qquad \text{and} \qquad U_- = \bigcap_{i=0}^{\infty} g^{-i} U g^i.$$

Say U is *tidy* for g if

(1) $U = U_+U_- = U_-U_+$ and
(2) $\bigcup_{i=0}^{\infty} g^i U_+ g^{-i}$ and $\bigcup_{i=0}^{\infty} g^{-i} U_- g^i$ are both closed in G.

(B) Let G be a locally compact totally disconnected group. The *scale function* on G is defined by the formula

$$s(g) = \min\{|U : U \cap g^{-1}Ug| : U \text{ a compact open subgroup of } G\}.$$

The connection between the scale function and tidy subgroups is described in the following theorem due to Willis.

Theorem 4.2 ([15, Theorem 3.1], see [8, Theorem 6.1]). *Let G be a locally compact totally disconnected group and $g \in G$. Then $s(g) = |U : U \cap g^{-1}Ug|$ if and only if U is tidy for g.*

Willis's theory can be understood in terms of graphs and automorphism groups of graphs, see [8]. In this approach to Willis's theory a fundamental role is played by highly arc transitive digraphs. In the following Lemma several results from [8, Sections 2–4] are collected together.

Theorem 4.3. *Let G be a locally compact totally disconnected group, g an element in G and U a compact open subgroup of G. Put $\Omega = G/U$. Let v_0 be a point in Ω. Set $v_i = g^i(v_0)$. Define a digraph D such that the vertex set of D is Ω and the edge set is the orbit $G(v_0, v_1)$. Then:*

(1) *If U satisfies condition* (1) *in Definition* 4.1 (A) *then the digraph D is highly arc transitive.*
(2) *If U is tidy for g then D is not only highly arc transitive but the subgraph in D spanned by* $\mathrm{desc}(v_0)$ *is a tree.*

We now turn to bounded automorphisms and the scale function.

Definition 4.4. Let G be a totally disconnected locally compact group. An element g in G is said to be an FC^- *element* if the conjugacy class of g has compact closure in G.

Suppose now G is a totally disconnected locally compact group and that G acts as a group of automorphism on some connected locally finite graph X. Furthermore suppose that the stabilizer in G of a vertex in X is a compact open subgroup. Then [16, Lemma 4] says that g is an FC^- element of G if and only if g acts as a bounded automorphism on X.

Theorem 4.5. *Let G be a totally disconnected locally compact group and $s : G \to \mathbf{R}$ the scale function on G. If g is an FC^- element in G then $s(g) = 1 = s(g^{-1})$.*

Proof. Note that if g is an FC^- element of G, then g^{-1} is also an FC^- element. Let U be a compact open subgroup of G that is tidy for g. Set $\Omega = G/U$. Let v_0 be a point in Ω such that $U = G_{v_0}$ and $v_i = g^i(v_0)$. Define a digraph D such that the vertex set of D is Ω and the edge set of D is the orbit $G(v_0, v_1)$. It follows from Theorem 4.3 that the digraph D is highly arc transitive. Let D' be the connected component of D that contains v_0. Furthermore, define G' as the subgroup of G that leaves D' invariant. Clearly g is contained in G' and g is an FC^- element of G'. By [16, Lemma 4] we know that g acts on D' as a bounded automorphism. But $v_1 = g(v_0) \in \mathrm{desc}(v_0)$ and therefore g leaves invariant a line L. Hence Proposition 2.4 implies that g can only be a bounded automorphism if D' has precisely two ends. Since U is assumed to be tidy, the subgraph induced by $\mathrm{desc}(v_0)$ is a tree. This implies that D' is just an infinite directed line and hence G_{v_0} fixes v_{-1}. By Theorem 4.2

$$s(g) = |U : U \cap g^{-1}Ug| = |G_{v_0} : G_{v_0} \cap G_{v_{-1}}| = |G_{v_0}v_{-1}| = 1. \quad \square$$

We now turn to periodic elements and cycles in digraphs.

Definition 4.6. An element g in a topological group G is said to be *periodic* if the closure of the cyclic subgroup $\langle g \rangle$ is compact in G. Define $P(G)$ as the set of all periodic elements in G.

Suppose now that G is a totally disconnected locally compact group and U is a compact open subgroup of G. Considering the action of G on the coset space G/U we can recognize the periodic elements because they are the only elements generating cyclic subgroups that have all their orbits finite (follows from [16, Lemma 2]). The following proof of [14, Theorem 2] further illustrates the use of highly arc transitive digraphs in the theory of totally disconnected locally compact groups.

Theorem 4.7 ([14, Theorem 2]). *Let G be a totally disconnected locally compact group. The set $P(G)$ of periodic elements in G is closed.*

Proof. The trick is to use the fact that a connected infinite highly arc transitive digraph has no directed cycles.

Suppose g is in the closure of $P(G)$ but g is not periodic. Let U be a compact open subgroup of G that is tidy for g. Define the digraph D' in the same way as in the proof of Theorem 4.5. If g is not periodic then the orbit of v_0 under $\langle g \rangle$ is infinite and therefore D' is infinite. The set gU is an open neighborhood of g and must therefore contain some periodic element h. The fact that $h \in gU = gG_{v_0}$ implies $h(v_0) = g(v_0) = v_1$. The element h is periodic so the orbit of v_0 under $\langle h \rangle$ is finite and therefore there is an integer n such that $h^n(v_0) = v_0$. The sequence $v_0, v_1 = h(v_0), v_2 = h^2(v_0), \ldots, v_n = h^n(v_0) = v_0$ is a directed cycle in D'. This contradicts the result mentioned above. Hence we conclude that it is impossible that the closure of $P(G)$ contains any elements that are not periodic. Thus $P(G)$ is closed. □

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