Packing Triangles in a Graph and Its Complement

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Abstract: How few edge-disjoint triangles can there be in a graph G on n vertices and in its complement $\overline{G}$? This question was posed by P. Erdős, who noticed that if G is a disjoint union of two complete graphs of order $n/2$ then this number is $n^2/12 + o(n^2)$. Erdős conjectured that any other graph with n vertices together with its complement should also contain at least that many edge-disjoint triangles. In this paper, we show how to use a fractional relaxation of the above problem to prove that for every graph G of order n, the total number of edge-disjoint triangles contained in G and $\overline{G}$ is at least $n^2/13$ for all sufficiently large n. This bound improves some earlier results. We discuss a few related questions as well.
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1. INTRODUCTION

How few triangles can there be in a graph and its complement? Equivalently, one can ask for the minimum number of monochromatic triangles in any 2-coloring of the edges of K_n, the complete graph of order n. From the well-known result of Goodman [5] (see also [10]), it follows that this number is at least $n^3/24 + o(n^3)$ and the disjoint union of two complete graphs of order $n/2$ shows that this estimate is tight.

Motivated by this result Erdős posed a naturally related question. What is the minimum number of *edge-disjoint* monochromatic triangles in any 2-coloring of the edges of the complete graph K_n? Denote this minimum by $f_3(n)$. To obtain an upper bound on $f_3(n)$, consider again the coloring in which the edges of the first color form two disjoint cliques of size $n/2$ each. Then the second color forms a complete bipartite graph and thus there are no monochromatic triangles in this color. On the other hand, there are approximately

$$2\frac{\binom{n/2}{2}}{3} = \frac{n^2}{12} + o\left(n^2\right)$$

edge-disjoint triangles in the first color. This example shows that $f_3(n) \leq n^2/12 + o(n^2)$ and led to the following conjecture of Erdős (see, e.g., Problem 14 in [2]).

Conjecture 1.1. *If n is sufficiently large, then*

$$f_3(n) = \frac{n^2}{12} + o\left(n^2\right).$$

This conjecture was studied recently by Erdős et al. in [3] where they computed $f_3(n)$ for small values of n and showed that $f_3(11) = 6$. They use this result, together with the special case of Wilson's theorem that guarantees the existence of nearly optimal packing of the edges of K_n by edge-disjoint copies of K_{11}, to prove that $f_3(n) \geq (3n^2/55) + o(n^2)$.

In this paper, we show that the conjecture of Erdős can be studied more profitably using the fractional version of the original problem. Using this approach, we are able to obtain an improvement on the above bound for $f_3(n)$. Our main result is as follows.

Theorem 1.1. *If n is sufficiently large, then*

$$f_3(n) \geq \frac{n^2}{12.89} + o\left(n^2\right).$$

In [3] the authors also considered the more general question of finding the minimum number of monochromatic copies of K_k ($k=3$ is triangle case) in a 2-coloring of the edges of K_n. As before, let $f_k(n)$ denote the value of this minimum. They proved that the sequence $f_k(n)/n(n-1)$ converges to the limit, which equals its supremum. Erdős et al. also obtain upper and lower bounds on $f_k(n)$. In particular, they proved that

$$\frac{n^2}{204}+o(n^2)\le f_4(n)\le\frac{n^2}{36}+o(n^2).$$

Using similar ideas as in the case of triangles, we are able to improve this bound and obtain the following result.

Theorem 1.2. *If n is sufficiently large, then*

$$f_4(n)\ge\frac{n^2}{106}+o(n^2).$$

Our methods are easily generalizable and can also be used to improve known estimates for $f_k(n)$ when $k\ge 5$. We discuss this briefly in concluding remarks.

The rest of this short paper is organized as follows. In the next section, we discuss the fractional version of the triangle-packing problem and use a result from [8] to show that it is equivalent to the original question. We also prove a useful averaging lemma that allows us to iteratively improve the density of fractional triangle packings. In Section 3, we determine the minimum values of fractional triangle packings for 2-colorings of complete graphs of order n for $n\le 10$. These values (even the result for $n=7$) together with ideas from Section 2 already give an improvement of the best previously known bound for $f_3(n)$. In this section, we also prove some basic results about the possible values of fractional triangle packings, which may be of independent interest. In Section 4, we discuss the computational aspects of the problem for $n\ge 11$ and describe our computer-aided proof of Theorem 1.1. In Section 5, we briefly extend our methods to the problem of packing edge-disjoint copies of K_4 and prove Theorem 1.2. The last section contains some concluding remarks.

2. FRACTIONAL VERSION

In this section, we discuss the fractional version of Conjecture 1.1 and observe that it is equivalent to the original problem. Then we will use this fact to obtain a procedure which allows us iteratively improve lower bounds on $f_3(n)$.

Our starting point is a theorem of Haxell and Rödl relating values of integer and fractional packings of graphs. Let H be any fixed graph. For a graph G, we define $\nu_H(G)$ to be the maximum size of a set of pairwise edge-disjoint copies of

H in G. We say that a function w from the set of copies of H in G to $[0,1]$ is a *fractional H-packing* of G if $\sum_{H\ni e} w(H) \le 1$ for every edge e of G. Then $\nu^*_H(G)$ is defined to be the maximum value of $\sum_H w(H)$ over all fractional H-packings w of G. Clearly, $\nu_H(G) \le \nu^*_H(G)$ for any two graphs H and G. On the other hand, the following result of Haxell and Rödl [8] shows that for sufficiently large graphs G the values of the optimal integer and fractional H-packings differ only by a relatively small additive factor.

Theorem 2.1. *Let H be a fixed graph and G be a graph on n vertices. Then* $\nu^*_H(G) - \nu_H(G) = o(n^2)$.

We will simplify the notation to deal with the special case when H is a triangle, writing $\nu_\triangle(G) = \nu_{K_3}(G)$, $\nu^*_\triangle(G) = \nu^*_{K_3}(G)$. We will often refer to the latter as the number of *fractional triangles* in G. Let $f_3^*(n)$ be the minimum possible value of $\nu^*_\triangle(G) + \nu^*_\triangle(\overline{G})$ over all 2-colorings $G \cup \overline{G}$ of the edges of complete graph of order n. The following corollary follows immediately from the above theorem together with the fact that $f_3^*(n) \ge f_3(n) = \Omega(n^2)$. It shows that to prove the conjecture of Erdős, it is enough to study $f_3^*(n)$.

Corollary 2.1. *If n is sufficiently large, then* $f_3(n) = \big(1 - o(1)\big) f_3^*(n)$.

In fact this corollary (which we found independently) has a considerably simpler proof than Theorem 2.1. It can be deduced in a fairly straightforward manner from Szemerédi's Regularity Lemma (see, e.g., [9]) and a result of Pippenger and Spencer [11]. The simplications are due particularly to the fact that we are considering G and $\overline{G}$ simultaneously, and also that we only need the case when H is a triangle. Next we need to establish a few properties of $f_3^*(n)$.

Lemma 2.1. *The sequence* $\frac{f_3^*(n)}{n(n-1)}$ *is increasing in n.*

Proof. Consider 2-coloring the edges of complete graph K_{n+1}. Let $C_1, \ldots, C_{n+1}$ be its complete subgraphs induced by all possible subsets of vertices of size n. By definition, for every $1 \le i \le n+1$ we can find a fractional packing w_i of monochromatic triangles of C_i with total value at least $f_3^*(n)$. Note that each edge of K_{n+1} belongs to $n-1$ of the C_i, so $w = \frac{1}{n-1}\sum_i w_i$ is a valid fractional packing of the monochromatic triangles of K_{n+1}. Its value is at least $\frac{n+1}{n-1} f_3^*(n)$. Therefore

$$\frac{f_3^*(n+1)}{(n+1)n} \ge \frac{\frac{n+1}{n-1} f_3^*(n)}{(n+1)n} = \frac{f_3^*(n)}{n(n-1)}. \qquad \blacksquare$$

Note that, by definition, $f_3^*(n)/n(n-1)$ is bounded by $1/2$ for every n. Therefore by the above lemma, this sequence converges to a constant which we denote by c_3. From the example in the Introduction, we have that $c_3 \le 1/12$, and Conjecture 1.1 suggests that it equals $1/12$. We adopt the idea of the previous lemma to give a procedure which iteratively improves lower bounds on c_3 starting with some particular value of $f_3^*(n)$.

Lemma 2.2. $f_3^*(3n) \geq 9f_3^*(n) + n - 1$.

Proof. Consider a 2-edge-coloring of K_{3n}. Since any six vertices contain a monochromatic triangle, we can greedily find $n-1$ vertex disjoint monochromatic triangles $T_1, \ldots, T_{n-1}$. Denote by T_n the remaining three vertices and consider an n-partite subgraph of K_{3n} with the parts $T_1, \ldots, T_n$. By definition, for any of 3^n distinct copies of K_n with exactly one vertex in every T_j, we can find a fractional packing w_i of monochromatic triangles with total value at least $f_3^*(n)$. Note also that every edge of the n-partite subgraph is contained in precisely 3^{n-2} such copies of K_n. Therefore $3^{-(n-2)}\sum_i w_i$ is a valid fractional packing of monochromatic triangles of this n-partite subgraph. Hence this subgraph contains at least $3^{-(n-2)} \cdot 3^n f_3^*(n) = 9f_3^*(n)$ fractional monochromatic triangles. All of them are edge-disjoint from $T_1, \ldots, T_{n-1}$. This implies that $f_3^*\big(3n\big) \geq 9f_3^*(n) + n - 1$ and completes the proof of the lemma. ■

Corollary 2.2. *$c_3 \geq \frac{f_3^*(n)}{n^2} + \frac{1}{6n} - \frac{1}{8n^2}$ for every $n \geq 1$.*

Proof. Iterating the result of Lemma 2.2, we obtain that $f_3^*\big(3^{k+1}n\big) \geq 9^{k+1}f_3^*(n) + \sum_{i=0}^{k} 9^{k-i}\big(3^i n - 1\big)$ for every $k \geq 0$. This implies

$$c_3 \geq \frac{f_3^*(3^{k+1}n)}{3^{k+1}n(3^{k+1}n-1)} > \frac{f_3^*(3^{k+1}n)}{9^{k+1}n^2} \geq \frac{f_3^*(n)}{n^2} + \frac{1}{n}\sum_{i=0}^{k}\frac{1}{9}3^{-i} - \frac{1}{n^2}\sum_{i=0}^{k}9^{-i-1}.$$

Taking the limiting value of this expression as k tends to infinity gives the result of the lemma. ■

It is easy to verify that the last result beats the simple bound $c_3 \geq f_3^*(n)/n(n-1)$ for every n. Moreover, we can immediately improve on the bound of $3/55$ from [3]. Indeed, it is known and easy to show that $f_3^*(7) \geq f_3(7) = 2$, that is, every 2-coloring of the edges of K_7 contains two edge-disjoint monochromatic triangles. By substituting the value of $f_3^*(7)$ in Corollary 2.2, we deduce $c_3 \geq \frac{2}{49} + \frac{1}{42} - \frac{1}{392} = \frac{73}{1176} > \frac{1}{16.11}$.

3. VALUES OF $f_3^*(n)$ FOR SMALL n

In this section, we will calculate $f_3^*(n)$ for a few small values of n. These values combined with Corollary 2.2 will give further improvements on the lower bound for c_3. In the course of doing so, we prove some results about the values of $\nu_\triangle^*(G)$ which may be of some independent interest. We start with a lemma which gives an upper bound on $f_3^*(n)$.

Lemma 3.1. *For $6 \leq n \leq 10$ we have $f_3^*(n) \leq n - 5$.*

Proof. It is easy to see that the size of the maximum fractional packing of monochromatic triangles in 2-edge-coloring of K_n is always bounded by the size of a set of edges which covers all monochromatic triangles. Therefore to prove the lemma, it is sufficient to exhibit a coloring of K_n and a set of $n-5$ edges that cover all monochromatic triangles in this coloring. For $n=6$, let one color be the graph which is obtained by joining the new vertex to the three consecutive vertices of 5-cycle. For future reference, we denote it by G_6. In this coloring, there are two monochromatic triangles which share one edge and we are done. Suppose that we already constructed graph G_{n-1} such that $\overline{G}_{n-1}$ is triangle-free and all the triangles in G_{n-1} can be covered by the set of edges E_{n-1} of size $n-1-5$ (it is true for $n=7$). We obtain G_n from G_{n-1} as follows. Let v be a vertex in G_{n-1} that is not an endpoint of any edge in E_{n-1}. Add a new vertex v' which has the same neighborhood as v in G_{n-1} together with the edge (v, v'). From our construction, it follows that $\overline{G}_n$ remains triangle-free and $E_n = E_{n-1} \cup (v, v')$ is a set of edges of size $n-5$ which covers all triangles in G_n. This completes the proof of the lemma. ■

Lemma 3.2. $f_3^*(6) = f_3(6) = 1$.

Proof. The well-known fact that the Ramsey number $R(3,3)=6$ implies that both values are at least 1, and the corresponding upper bound follows from Lemma 3.1. ■

Lemma 3.3. $\nu_\triangle^*(G) = 1$ *iff all triangles of* G *share a common edge. If* $\nu_\triangle^*(G) > 1$ *then* $\nu_\triangle^*(G) \geq 2$.

Proof. For $\nu_\triangle^*(G) < 2$, we have that every pair of triangles must share an edge. If all the triangles contain the same edge, then $\nu_\triangle^*(G) = 1$ and we are done. Otherwise G contains three triangles which are pairwise edge-intersecting, but which do not have an edge which belongs to all of them. Clearly such triangles form a K_4, and therefore $\nu_\triangle^*(G) \geq \nu_\triangle^*(K_4) = 2$, since we can assign weight $1/2$ to all four triangles in K_4. ■

Corollary 3.1. (i) $f_3^*(7) = 2$.
(ii) G_6 *and its complement is the only coloring achieving* $f_3^*(6) = 1$.
(iii) $f_3(7) = 2$.

Proof. (i) As we already proved in Lemma 2.1, $f_3^*(7) \geq \frac{7}{5} f_3^*(6) \geq \frac{7}{5} > 1$. Therefore, from Lemma 3.1, we have that $f_3^*(7) \geq 2$, and the equality holds by Lemma 3.1.

(ii) Let G be a graph on six vertices with $\nu_\triangle^*(G) = 1$ and $\nu_\triangle^*(\overline{G}) = 0$. Let e be the edge which belongs to all triangles of G (such e exists by Lemma 3.3). Deleting either endpoint of e leaves a graph on five vertices with no triangles in it or in its complement, that is, an induced 5-cycle. This uniquely determines G_6. Note also that $\overline{G_6} \cong G_6 - e$.

(iii) If $f_3(7) = 1$ then there is a graph G on seven vertices with triangle-free complement that does not contain 2 edge-disjoint triangles. By the proof of

Lemma 3.3, it must contain a K_4. Let X be the set of vertices of this K_4 and let Y be the set of the remaining three vertices. Clearly Y cannot form a triangle, and hence contains a pair u, v of non-adjacent vertices. If u or v have two neighbors in X, this creates two edge-disjoint triangles. Therefore each of them has at most one neighbor in X and thus there is a vertex $w \in X$ not adjacent to both u and v. This contradicts the assumption that $\overline{G}$ is triangle-free and completes the proof. ■

Lemma 3.4. *If* $2 < \nu^*_\triangle(G) < 3$ *then* $\nu^*_\triangle(G) = 5/2$ *and the triangles of G form a wheel with five spokes.*

Proof. First note that there is no optimal packing which assigns weight 1 to some triangle T. For such a packing, the graph formed by the triangles edge-disjoint from T would have a value of $\nu^*_\triangle$ strictly between 1 and 2, in contradiction to Lemma 3.3. In particular, for every triangle, at least two of its edges are shared with another triangles. Indeed, suppose T is a triangle and e is the only edge of it shared with another triangle. For any valid assignment of weights to the triangles of G consider changing the weight of T to 1 and the weight of all other triangles containing e to 0. This gives a valid weight assignment which has the same total weight, so the optimum can be achieved while giving T weight 1, contradiction.

Next we note that there cannot be two vertex disjoint triangles A,B. By the above discussion, two edges of A are shared with other triangles, which must be edge disjoint from B. Thus we have two triangles or possibly a K_4 edge-disjoint from B. This implies $\nu^*_\triangle \geq 3$. So every two triangles share a vertex.

Case 1. G contains K_4. Let X be such a copy of K_4. Since $2 < \nu^*_\triangle(G) < 3$, G contains an additional triangle T, which must share some edge with X. As noted above, T has another edge also shared with a triangle T'. Note that T' cannot be edge-disjoint from X, so in fact G contains K_5 with a deleted edge. But $\nu^*_\triangle(K_5 - e) = 3$, as can be seen by assigning weight $1/2$ to the six triangles that share a vertex with the missing edge e.

Case 2. G is K_4-free. Note that any triangle shares exactly two of its edges with other triangles, as $\nu_\triangle < 3$ and in this case triangles which share different edges of the same triangle are edge-disjoint. Consider a triangle A, and triangles B, C that share two different edges with A. Let x be their common vertex. Then any other triangle must contain x. Indeed, any other triangle must share a vertex with each of A, B, and C, and if this vertex is not x we form K_4.

Let b, c be the vertices belonging to B, C but not A. There must be triangles $T_b \neq B$ containing xb and $T_c \neq C$ containing xc. T_b and T_c cannot be edge-disjoint, or together with A we have $\nu_\triangle(G) \geq 3$. If T_b and T_c share exactly one edge, we have a wheel with five spokes, which has $\nu^*_\triangle = 5/2$. Since the addition of any triangle to this wheel will cause $\nu^*_\triangle \geq 3$, we conclude that G is the wheel with five spokes. Otherwise $T_b = T_c$, and we have a wheel with four spokes, which has $\nu^*_\triangle = 2$. In this case, there is another triangle T in G which contains x and cannot be edge-disjoint from the wheel, so it must contain a spoke. Also, we know that since $\nu^*_\triangle(G) > 2$ we cannot cover all the triangles in the graph by two edges. Therefore, there are two triangles T and T' outside the wheel which contain two neighboring spokes. Note that in this case, T and T' must be edge-disjoint to avoid K_4. In addition, one of the four triangles in the wheel is also edge-disjoint from both T and T'. This implies that $\nu_\triangle(G) \geq 3$ and completes the proof. ■

Corollary 3.2. $f_3^*(8) = 3$.

Proof. Similarly as before, $f_3^*(8) \geq \frac{8}{6} f_3^*(7) \geq \frac{8}{3} > 5/2$. Therefore, by the previous result, we obtain that if all the triangles are in the same color then $\nu^*_\triangle$ of this color is at least 3. On the other hand, if there are triangles of both colors then one color should have $\nu^*_\triangle > 1$. Then Lemma 3.3 implies that $\nu^*_\triangle$ of this color is at least 2. Since the other color also contains a triangle, we again obtain three fractional monochromatic triangles and therefore $f_3^*(8) \geq 3$. Now the equality follows by Lemma 3.1. ■

It is worth mentioning that $f_3(8) = 2$ (see [3]), and thus $n = 8$ is the first value of n for which $f_3^*(n)$ is larger than $f_3(n)$.

Lemma 3.5. $f_3^*(9) = 4$.

Proof. Note that $f_3^*(9) \geq \frac{9}{7} f_3^*(8) \geq \frac{27}{7}$. So by Lemmas 3.3 and 3.4, we can assume that all triangles are in one color. Consider a graph G on nine vertices such that $\overline{G}$ is triangle-free, that is, the size of the maximum independent set $\alpha(G) \leq 2$. We need to show that $\nu^*_\triangle(G) \geq 4$.

First we make the simple observation that, since $\alpha(G) \leq 2$, any four vertices of G induce a subgraph that contains a triangle or two vertex disjoint edges. Now we consider a few simple cases according to the degrees of vertices in G. First, suppose that there is a vertex v of degree at most 2. Then six non-neighbors of v form a K_6 and therefore $\nu^*_\triangle(G) \geq \nu^*_\triangle(K_6) = 5 > 4$.

If there is a vertex v of degree 3 then, since $\alpha(G) \leq 2$, the five non-neighbors of v form a K_5, and the three neighbors of v must contain an edge. This forms a triangle which is edge-disjoint from K_5 and implies that $\nu^*_\triangle(G) \geq \nu^*_\triangle(K_5) + 1 = 10/3 + 1 > 4$.

If there is a vertex v of degree 4 then the four non-neighbors of v form a K_4. As we already pointed out above the four neighbors of v contain either a triangle, which together with v forms another K_4, or two vertex disjoint edges, which form two edge-disjoint triangles with v. In both cases, $\nu^*_\triangle(G) \geq 2 + 2 = 4$.

If there is a vertex v of degree 6 then, since $R(3,3) = 6$ and $\alpha(G) \leq 2$, the neighbors of v contain a triangle. The other three vertices of $N(v)$ span an edge, so we have an edge-disjoint K_4 and triangle in $\{v\} \cup N(v)$. Also, the two non-neighbors of v must be adjacent. By the above discussion, they both have degree at least 5, so each of them have at least four neighbors in $N(v)$. Since $|N(v)| = 6$, they must have a common neighbor in $N(v)$. This forms an additional triangle, which is edge-disjoint from the previous configuration, and again implies that $\nu^*_\triangle(G) \geq 4$.

If there is a vertex v of degree at least 7 then the neighbors of v contain a triangle, and four other neighbors contain either a triangle or two independent edges. Together with v, this forms two edge-disjoint K_4's or edge-disjoint copies of K_4 and two triangles. Therefore $\nu^*_\triangle(G) \geq 4$. Finally, since G has an odd number of vertices, it cannot be 5-regular. Therefore, we cover all the possible cases and the proof is complete. ■

From this result together with Lemma 3.1, it follows that $f_3^*(10) = 5$, since $5 \geq f_3^*(10) \geq \frac{10}{8} f_3^*(9) = 5$. Also substituting the value of $f_3^*(9)$ into Corollary 2.2, we can further improve the bound on c_3 and show $c_3 \geq \frac{4}{81} + \frac{1}{54} - \frac{1}{648} > \frac{1}{15.07}$.

4. COMPUTER-AIDED PROOFS

Up to this point, the evaluation of $f_3^*(n)$ has been relatively painless, but it seems that to proceed further by the above methods would require excessive case checking. It also might be discouraging to note that evaluating $f_3^*(10)$ did not improve the bound for c_3 we got from $f_3^*(9)$. But at $n = 11$, the pattern of Lemma 3.1 breaks: $f_3^*(11) \geq \frac{11}{9} f_3^*(10) \geq \frac{55}{9} > 6$, so there is a jump of more than one. In fact, we will see that $f_3^*(11) = 7$, a jump of two, which gives us another significant improvement on the bound. This gives us one incentive for continuing our efforts.

We also want to demonstrate the computational advantages that the fractional method offers, in addition to the improvement that Corollary 2.2 gives on any result obtained from the integer method. The most important is the use of averaging to prune the search space: instead of checking all colorings on n vertices, we will see it is enough to only check colorings that are one vertex extensions of the colorings on $n-1$ vertices with the lowest values of $\nu^*_\triangle = \nu^*_\triangle(G) + \nu^*_\triangle(\overline{G})$. In fact, this reduction of complexity from 2^{n^2} to 2^n is necessary to make the problem computationally feasible at the values of n we consider.

A secondary factor is the speed of computing $\nu^*_\triangle(G)$ compared to $\nu_\triangle(G)$. Very efficient algorithms have been developed for solving linear programs, whereas integer programs are much harder. We now give a description of our algorithm. A magma program implementing it can be downloaded from www.math.princeton.edu/~bsudakov/papers.html.

Initialization. Choose a starting value for n, the number of vertices, and d, the 'search depth.' Check all graphs G on n vertices. Find the d smallest values of $\nu^*_\triangle = \nu^*_\triangle(G) + \nu^*_\triangle(\overline{G})$ that occur. Build a list L_n of all graphs for which $\nu^*_\triangle$ is one of the $d-1$ smallest values. (Our lists will always be 'reduced': containing no isomorphic or complementary pairs.)

Iteration. Check all one-vertex extensions of the graphs in L_n. By averaging, any other graph on $n+1$ vertices must have $\nu^*_\triangle$ at least $(n+1)/(n-1)$ times the

dth smallest value for graphs on n vertices. We refer to this lower bound as the *level* at $n+1$. The smallest values below the level that we find by checking these extensions are indeed the smallest values for all graphs on $n+1$ vertices. We find as many of the smallest values as possible up to a maximum of d.

If there are d values below the level, then we let L_{n+1} be the graphs with the $d-1$ smallest values and proceed with the iteration. However, there may not be d values below the level, so then we have to reduce the depth of the search: we let L_{n+1} be all the graphs we have found below the level, and calculate the level for $n+2$ to be $(n+2)/n$ times the level for $n+1$. Then we proceed with the iteration.

Termination. There may come a point at which no values are found below the level, at which point nothing further can be said. To get results for larger n, it is then necessary to start again with a larger search depth.

The values in the table below were obtained by running the algorithm for only a few days starting at $n=6$ with search depths 4 and 5. Each column begins with the number of vertices n, and the ith row contains the ith smallest value of $\nu^*_\triangle(G)+\nu^*_\triangle(\overline{G})$ for graphs G on n vertices.

$i\backslash n$	6	7	8	9	10	11	12	13	14	15
1	1	2	3	4	5	7	9	11	13	15
2	2	3	4	5	6	8	10	12	*	*
3	5/2	10/3	13/3	16/3	20/3	25/3	31/3	38/3	*	*
4	3	7/2	9/2	17/3	7	26/3	32/3	*	*	*
5	10/3	4	14/3	6	22/3	*	*	*	*	*

Note that the first row of the table gives $f^*_3(n)$, which is all that concerns us. However, it is necessary to run the algorithm at search depth at least 4 to obtain it. Also, it is worth mentioning that the above table shows that to compute $f^*_3(n)$ for $n\le 15$, it is enough to check only the one-vertex extensions of the colorings which achieve the value $f^*_3(n-1)$.

The construction achieving these values is just a continuation of Lemma 3.1. As in the proof of this lemma we denote by G_n one of the colors. Note that we have already constructed G_n for $n\le 10$. To get G_n from G_{n-1}, we pick an edge of E_{10} (these are the five edges which cover all the triangles in G_{10}) that we have not yet used and *duplicate* (see Lemma 3.1 for description) one of its vertices. This allows us to assert that $f^*_3(15)=15$ without actually computing it. The above construction gives an upper bound, and the lower bound follows from the fact that $f^*_3(15)\ge\frac{15}{13}\cdot f^*_3(14)=15$.

***Proof of Theorem* 1.1.** Substituting the value of $f^*_3(14)$ in Corollary 2.2, we obtain $c_3\ge\frac{13}{196}+\frac{1}{84}-\frac{1}{1568}>\frac{1}{12.888}$. ∎

Finally, we present some interesting facts discovered in the course of our computations, which may suggest some further topics of investigation.

First, there are comparatively few graphs that achieve the minimum value of $\nu^*_{\triangle}$. Most strikingly there are only four graphs (excluding isomorphs and complements) on ten vertices that achieve $\nu^*_{\triangle}(G) + \nu^*_{\triangle}(\overline{G}) = 5$, and these are given by the duplication procedure of Lemma 3.1. We can also describe these graphs as those obtained by taking $C_5(2)$ (the two point blowup of the 5-cycle) and possibly adding edges that connect two points that represent the same point of C_5.

It is natural to ask for a classification of the colorings with minimum value of $\nu^*_{\triangle}(G) + \nu^*_{\triangle}(\overline{G})$. It seems that a change in structure should occur at $n = 20$, because here both $K_{10,10}$ and $C_5(4)$ give the conjectured minimum value of 30 fractional triangles. So we might think that for $n \leq 20$ the extremal graphs are derived by the duplication procedure, and for $n \geq 20$ are complete bipartite graphs.

Next, in our computation, we have observed a value of $\nu^*_{\triangle}(G)$ with denominator k for all $1 \leq k \leq 17$. This suggest the following two natural questions. Can all denominators be achieved? How does the maximum denominator depend on the size of the graph?

5. EDGE-DISJOINT K_4's

In this section, we study packings of edge-disjoint cliques of size 4 and present the proof of Theorem 1.2. To do so, we will adapt the ideas and methods of Section 2. Similarly as before we define $f_4^*(n)$ to be the minimum possible value of $\nu^*_{K_4}(G) + \nu^*_{K_4}(\overline{G})$ over all possible 2-colorings $G \cup \overline{G}$ of the edges of K_n. The abovementioned result of Haxell and Rödl (Theorem 2.1) implies that $f_4(n) = \big(1 - o(1)\big)f_4^*(n)$. Also, essentially the same proof as in Lemma 2.1 shows that $f_4^*(n)/n(n-1)$ is increasing. Since this sequence is bounded, it converges to a limit which we denote by c_4. Next we prove the analogs of Lemma 2.2 and Corollary 2.2.

Lemma 5.1. $f_4^*(4n) \geq 16 f_4^*(n) + n - 4$.

Proof. Consider 2-edge-coloring of K_{4n}. By the well-known bound on the Ramsey number $R(4,4)$ (see, e.g., [6]) any 18 vertices of K_{4n} contain a monochromatic copy of K_4. So by a greedy procedure, we can find $n-4$ vertex disjoint monochromatic K_4's $C_1, \ldots, C_{n-4}$. Partition the remaining 16 vertices into 4 sets $C_{n-3}, \ldots, C_n$ of size 4 each and consider an n-partite subgraph of K_{4n} with the parts $C_1, \ldots, C_n$. By definition, for any of 4^n distinct copies of K_n with exactly one vertex in every C_j, we can find a fractional packing w_i of monochromatic K_4's with total value at least $f_4^*(n)$. Note also that every edge of the n-partite subgraph is contained in precisely 4^{n-2} such copies of K_n. Therefore $4^{-(n-2)} \sum_i w_i$ is a valid fractional packing of monochromatic K_4's of this n-partite subgraph. Hence this subgraph contains at least $4^{-(n-2)} \cdot 4^n f_3^*(n) = 16 f_4^*(n)$ fractional monochromatic K_4's. All of them are edge-disjoint from $C_1, \ldots, C_{n-4}$.

This implies that $f_4^*(4n) \geq 16\, f_4^*(n) + n - 4$ and completes the proof of the lemma. ∎

Corollary 5.1. $c_4 \geq \frac{g_4(n)}{n^2} + \frac{1}{12n} - \frac{4}{15n^2}$.

Proof. Iterating the result of Lemma 5.1, we obtain that for every $k \geq 0$

$$f_4^*\left(4^{k+1}n\right) \geq 16^{k+1} f_4^*(n) + \sum_{i=0}^{k} 16^{k-i}(4^i n - 4).$$

This implies

$$c_4 \geq \frac{f_4^*(4^{k+1}n)}{4^{k+1}n(4^{k+1}n-1)} > \frac{f_4^*(4^{k+1}n)}{16^{k+1}n^2} \geq \frac{f_4^*(n)}{n^2} + \frac{1}{16n}\sum_{i=0}^{k} 4^{-i} - \frac{4}{n^2}\sum_{i=0}^{k} 16^{-i-1}.$$

Taking the limiting value of this expression as k tends to infinity gives the result of the lemma. ∎

Finally we will need a result of Gyárfás, who proved in [7] that $f_4(19) \geq 2$.

Proof of Theorem 1.2. First, note that $f_4^*(22) \geq f_4(22) \geq 3$. Indeed, in any 2-edge-coloring of K_{22} there is a monochromatic copy of K_4. Delete any three vertices from this K_4. By the result of Gyárfás, we can find two edge-disjoint monochromatic K_4's in the remaining 19 vertices. Altogether, we obtain three monochromatic copies of K_4, which by our construction, are obviously edge-disjoint. Now, by substituting the value of $f_4^*(22)$ in Corollary 5.1, we conclude that $c_4 \geq \frac{3}{484} + \frac{1}{264} - \frac{4}{7260} = \frac{137}{14520} > 1/106$. ∎

6. CONCLUDING REMARKS

The conjecture of Erdős concerning edge-disjoint triangles remains open. We consider its resolution to be the main outstanding problem, and so have not attempted a more detailed analysis of the question for copies of larger complete graphs. We merely remark that our general method extends in a straightforward manner, but that the main difficulty would be establishing good bounds for small graphs. It seems that such bounds will depend on the Ramsey numbers $R(k,k)$, which are unknown for $k \geq 5$. In the triangle case, our result is sufficiently close to $1/12$ to provide strong evidence for the truth of the conjecture. However, it seems that a method based on analysis of small cases will get ever closer to the answer without reaching it, so new ideas are required.

There are various ways of weakening the statement that may lead to more manageable but still interesting problems. For instance, in discussion with N. Alon and N. Linial, they suggested that a simpler question one could ask is to show that the smallest number of edge-disjoint triangles in the complement of a triangle-free graphs G of order n is at least $(1+o(1))n^2/12$. It can be proved that in this problem, it is enough to consider graph G with $\Omega(n^2)$ edges. Perhaps some structure of dense triangle free graphs can be exploited here?

Another question raised in [3] asks for the minimum number of edge-disjoint monochromatic triangles all of the same color. A construction based on the blowup of C_5 shows that there can be as few as $n^2/20$ in each color, and Jacobson conjectures that this is best possible. We have nothing to add to the trivial observation that halving a lower bound for the Erdős conjecture gives a lower bound for Jacobson's conjecture. Indeed, one half of the edge-disjoint monochromatic triangles must be of the same color. Thus Theorem 1.1 gives a bound of roughly $n^2/26$ for this problem.

Our investigations of the possible values of $\nu^*_\triangle(G)$ suggest the following problem. It is known (see e.g., [1] and [4]) that the fractional part of the linear programming value of a fractional matching of a 3-uniform hypergraph can take any rational value between 0 and 1, but is this true for the hypergraph of triangles? It is not even obvious that any denominator can be achieved, but our computations so far suggest that this is the case.

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