Random cubic graphs are not homomorphic to the cycle of size 7.

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Abstract

We prove that a random cubic graph almost surely is not homomorphic to a cycle of size 7. This implies that there exist cubic graphs of arbitrarily high girth with no homomorphisms to the cycle of size 7.

KEYWORDS: homomorphism, cubic graph, random cubic graph, circular chromatic number.

1 Introduction

For a graph G, we denote its vertex set by $V(G)$. Suppose G and H are graphs. A homomorphism from G to H is a mapping h from $V(G)$ to $V(H)$ such that for each edge xy of G, $h(x)h(y)$ is an edge of H. We say that G is homomorphic to H, if there exists a homomorphism from G to H. A homomorphism from G to K_n is equivalent to an n-coloring of G. So graph homomorphism is a generalization of coloring. Since every even cycle is bipartite, it is homomorphic to a single edge. So for every even $n > 0$, a graph G is homomorphic to C_n, if and only if G is bipartite.

Suppose that $G \neq K_4$ is a cubic graph. Then G is 3-colorable, and so it is homomorphic to C_3. Since a graph containing C_i for i odd has no homomorphism onto C_j for any $j > i$, if G contains a triangle, then it is not homomorphic to C_5. The Peterson graph is triangle-free, but it is not homomorphic to C_5. The following question is first asked in [3] (see also [1]).

Question 1.1 *Is it true that any cubic graph G with sufficiently large girth g is homomorphic to C_5?*

It is shown in [1] that the answer is negative when C_5 is replaced with C_{11}. In [4] Wanless and Wormald improved this result to C_9 by studying the problem on random cubic graphs. In this note we improve their result to C_7 by showing that a random cubic graph almost surely is not homomorphic to C_7, where we say that a property $P(n)$ a.s. (almost surely) holds, if $\Pr[P(n)] = 1 - o(1)$. It is easy to see that there exist nonbipartite cubic graphs of arbitrarily high girth. So we leave C_5 as the only remaining open case.

We consider the following probability space. Choose three random perfect matchings independently and uniformly on n (n even) vertices. Let G be the multigraph obtained by taking the union of these three perfect matchings. If G is restricted to having no multiple edges, then any property holds for G a.s., if and only if it holds a.s. for a random cubic graph (see [5]). The probability that G has no multiple edges is asymptotic to $e^{-\frac{3}{2}} < 1$ (see [4]). Thus we can conclude that if a property holds for G a.s., then it holds a.s. for a random cubic graph.

Circular chromatic number is a generalization of the chromatic number. It is not known if there exists a cubic graph with arbitrarily large girth whose circular chromatic number is exactly 3. Our result shows that there exist cubic graphs with arbitrarily large girth whose circular chromatic number is more than $7/3 = 2.33\ldots$.

In Section 2 we prove that a random cubic graph a.s. is not homomorphic to the cycle of size 7. In Section 3 we introduce the relation between our result and the circular chromatic number of cubic graphs, and we pose some open problems.

2 Main result

We take C_k with vertex set being the congruence classes modulo k, and edges joining i to $i+1$, where in such notation the integers represent their congruence classes. Let h be a homomorphism from a graph G to C_k. We say that h is tight, if for every vertex v where $h(v) \neq 0$, there exists a vertex u adjacent to v such that $h(u) = h(v) + 1$.

Lemma 2.1 *Suppose that G is a graph on n vertices which has a homomorphism to C_{2k+1}. Then there is a tight homomorphism from G to C_{2k+1}.*

Proof. Begin with a homomorphism h_0 from G to C_{2k+1} which maps every isolated vertex of G to 0. For every $i \geq 0$, if h_i is not tight, then the homomorphism h_{i+1} is defined recursively as in the following. Let v be a vertex in G such that $h_i(v) \neq 0$, and $h_i(u) = h_i(v) - 1$ for every vertex u adjacent to v. Define h_{i+1} as follows: $h_{i+1}(v) = h_i(v) - 2$, and for every other vertex w, $h_{i+1}(w) = h_i(w)$. Observe that h_{i+1} is a homomorphism, and $h_i(v) = 0$ implies that $h_j(v) = 0$ for every $j \geq i$. Since for every $0 \leq x \leq 2k$ there exists $0 \leq i \leq 2k$ such that $x - 2i = 0 \pmod{2k+1}$, there is some t such that h_t is a tight homomorphism. In fact $t < 2kn$. ∎

In [4] the expected number of homomorphisms of a random cubic graph to C_9 is studied. They showed that this expected number is $o(1)$ conditioned on some property which is true, almost surely. For C_7 this value is exponentially high, so instead of the expected number of homomorphisms, we focus on the expected number of tight homomorphisms. This allows us to improve the previous result.

Theorem 2.2 *A random cubic graph a.s. is not homomorphic to C_7.*

Proof. Let M_1, M_2, and M_3 be three random perfect matchings chosen independently and uniformly on n vertices, and G be the multigraph obtained by taking the union of M_1, M_2, and M_3. Let A denote the event that the size of the largest independent set of a graph is less than $0.4554n$. McKay [2] showed that A holds a.s. for every cubic graph which implies that A holds a.s. for G. Let I_A be the indicator variable of A. We will bound $\mathbf{E}[X(G)I_A(G)]$, where $X(G)$ is the number of tight homomorphisms from G to C_7. If we can show that $\mathbf{E}[X(G)I_A(G)] = o(1)$, then since a.s. $I_A(G) = 1$, we can conclude that $X(G) = 0$ almost surely. Then Lemma 2.1 implies that a.s. G does not have any homomorphism to C_7.

Let $\mathcal{T}$ be the set of ordered triples of perfect matchings on n vertices, and $\mathcal{H}$ be the set of all mappings from n vertices to C_7. For every $h \in \mathcal{H}$, let $t(h)$ be the number of triples $T \in \mathcal{T}$ such that their corresponding graph G has property A and h is a tight homomorphism from G to C_7. We have

$$\mathbf{E}[X(G)I_A(G)] = \frac{1}{|\mathcal{T}|} \sum_{h \in \mathcal{H}} t(h).$$

Consider a triple T which contributes to $t(h)$ for some mapping h. For each i, let n_i be the cardinality of $h^{-1}(i)$. Consider a particular matching M in T, and let m_i be the number of edges between $h^{-1}(i-1)$ and $h^{-1}(i)$ in M. Then the following trivial equation shows that the values of m_i are equal for all of the three matchings in T.

$$m_i = \frac{n}{2} - (n_{i+1} + n_{i+3} + n_{i+5}). \tag{1}$$

Note that since $h^{-1}(i+1) \cup h^{-1}(i+3) \cup h^{-1}(i+5)$ is an independent set in G, we have $n_{i+1} + n_{i+3} + n_{i+5} \leq 0.4554n$ which implies that $m_i \geq 0.0446n$.

Let us focus on $h^{-1}(i)$ for some $i > 0$. Each perfect matching in T partitions $h^{-1}(i)$ into two sets of size m_i and m_{i+1} which are the vertices that are adjacent to $h^{-1}(i-1)$ and $h^{-1}(i+1)$, respectively in that perfect matching. Since h is a tight homomorphism, no element of $h^{-1}(i)$ is in the partition of size m_i in all of the three perfect matchings. So the number of different possible ways to partition the sets is

$$\binom{n_0}{m_0}^3 \prod_{i=1}^{6} \sum_{j=0}^{m_i} \binom{n_i}{m_i}\binom{m_i}{j}\binom{n_i - m_i}{m_i - j}\binom{n_i - j}{m_i}.$$

Note that $\binom{a}{b}$ is defined to be 0, for $b > a$. When the partitions are determined, there are $\prod_{i=0}^{6} m_i!^3$ different ways to arrange matching edges on them. Since there are $\frac{n!}{n_0!n_1!\ldots n_6!}$ different ways to partition the vertices into groups of size $n_0, n_1, \ldots, n_6$, we have

$$\sum_{h\in\mathcal{H}} t(h) \le \sum_{n_0+\ldots+n_6=n} \frac{n!}{n_0!\ldots n_6!}\left(\prod_{i=0}^{6} m_i!^3\right)\binom{n_0}{m_0}^3 \prod_{i=1}^{6}\sum_{j=0}^{m_i}\binom{n_i}{m_i}\binom{m_i}{j}\binom{m_{i+1}}{m_i - j}\binom{n_i - j}{m_i}.$$

The outer sum has less than n^6 summands, and each one of the six inner sums are taken over at most n terms. So by substituting $n_i = m_i + m_{i+1}$ and converting sums to maximums we have:

$$\mathbf{E}[X(G)I_A(G)] \le \frac{n!}{|\mathcal{T}|} \times n^{12} \max_{m_0,\ldots,m_6} f(m_0, m_1, \ldots, m_6), \tag{2}$$

where

$$f(m_0, m_1, \ldots, m_6) = \binom{m_0 + m_1}{m_0}^2 \left(\prod_{i=0}^{6} m_i!\right)\prod_{i=1}^{6}\max_{0\le j\le m_i}\binom{m_i}{j}\binom{m_{i+1}}{m_i - j}\binom{m_i + m_{i+1} - j}{m_i}.$$

Note that if $j > m_{i+1}$ or $j < m_i - m_{i+1}$, then $\binom{m_i}{j}\binom{m_{i+1}}{m_i-j}\binom{m_i+m_{i+1}-j}{m_i} = 0$. Define $y_i = m_i/n$ and

$$B_i = [0, y_i] \cap [0, y_{i+1}] \cap [y_i - y_{i+1}, \infty].$$

Let $g(x) = x^x$, for $x > 0$; and $g(0) = 1$. Then

$$f(m_0, \ldots, m_6)^{1/n} = \sqrt{\frac{n}{e}}\left(\frac{g(y_0 + y_1)^2}{g(y_0)g(y_1)^2}\prod_{i=1}^{6}\max_{z\in B_i}\frac{g(y_i)g(y_{i+1})g(y_i + y_{i+1} - z)}{g(z)g(y_i - z)^2 g(y_{i+1} - y_i + z)g(y_{i+1} - z)} + o(1)\right). \tag{3}$$

Note that if $y_i > 2y_{i+1}$ for some $0 < i \le 6$, then B_i is empty and $f(m_0, \ldots, m_6) = 0$. By substituting $|\mathcal{T}| = \left(\frac{n!}{2^{n/2}(n/2)!}\right)^3$ in Equation (2), we have

$$\mathbf{E}[X(G)I_A(G)]^{1/n} \le \max \frac{g(y_0 + y_1)^2}{g(y_0)g(y_1)^2}\prod_{i=1}^{6}\max_{z\in B_i}\frac{g(y_i)g(y_{i+1})g(y_i + y_{i+1} - z)}{g(z)g(y_i - z)^2 g(y_{i+1} - y_i + z)g(y_{i+1} - z)} + o(1), \tag{4}$$

where the outer maximum is taken with respect to the following conditions,

$$\begin{array}{rcll} \sum_{i=0}^{6} y_i & = & \frac{1}{2} & \\ y_i & \le & 2y_{i+1} & (1 \le i \le 6) \\ y_i & \ge & 0.0446 & (0 \le i \le 6) \end{array} \tag{5}$$

Suppose that for $0 \le i \le 6$, $0 \le a_i \le b_i$ are given, where $b_i - a_i \le 0.1$. We want to examine Inequality (4) when $a_i \le y_i \le b_i$, for every $0 \le i \le 6$. To satisfy Condition (5) we have the following restrictions on a_i and b_i.

$$\begin{array}{rl} \sum_{i=0}^{6} a_i \le \frac{1}{2} \le \sum_{i=0}^{6} b_i & \\ a_i \le 2b_{i+1} & (1 \le i \le 6) \\ b_i \ge 0.0446 & (0 \le i \le 6) \end{array} \tag{6}$$

By considering the derivatives of $\ln\left(\frac{g(x+y)^2}{g(x)g(y)^2}\right)$ with respect to x and y, we conclude that

$$\frac{g(y_0+y_1)^2}{g(y_0)g(y_1)^2} \le \frac{g(y_0+b_1)^2}{g(y_0)g(b_1)^2} \le \max\left(\frac{g(a_0+b_1)^2}{g(a_0)g(b_1)^2}, \frac{g(b_0+b_1)^2}{g(b_0)g(b_1)^2}\right) = f_0(a_0,b_0,b_1).$$

If Condition (5) is satisfied, then obviously for every $0 \le i \le 6$, we have $y_i \le 0.5 - 6 \times 0.0446 = 0.2324$. So $b_{i+1} - a_i + z < \frac{1}{e} = 0.367\ldots$, which implies that for $z \in B_i$,

$$\frac{g(y_i)g(y_{i+1})g(y_i+y_{i+1}-z)}{g(z)g(y_i-z)^2 g(y_{i+1}-y_i+z)g(y_{i+1}-z)} \le \frac{g(b_i)g(b_{i+1})g(b_i+b_{i+1}-z)}{g(z)g(b_i-z)^2 g(b_{i+1}-z)} \times \frac{1}{g(b_{i+1}-a_i+z)}.$$

Since $B_i \subseteq B_i'$, where B_i' is defined as

$$B_i' = [0,b_i] \cap [0,b_{i+1}] \cap [a_i - b_{i+1}, \infty],$$

we conclude that

$$\max_{z\in B_i} \frac{g(y_i)g(y_{i+1})g(y_i+y_{i+1}-z)}{g(z)g(y_i-z)^2 g(y_{i+1}-y_i+z)g(y_{i+1}-z)} \le \max_{z\in B_i'} \frac{g(b_i)g(b_{i+1})g(b_i+b_{i+1}-z)}{g(z)g(b_i-z)^2 g(b_{i+1}-a_i+z)g(b_{i+1}-z)}. \quad (7)$$

The derivative with respect to z of the natural logarithm of the expression following max on the right hand side of Inequality (7) is

$$\ln\left(\frac{(b_{i+1}-z)(b_i-z)^2}{z(b_i+b_{i+1}-z)(b_{i+1}-a_i+z)}\right).$$

The only critical value of $z \in B_i'$ is $z_i = (-B - \sqrt{B^2-4AC})/(2A)$, where

$$A = b_{i+1} + b_i - a_i,$$

$$B = b_{i+1}a_i + b_i a_i - 3b_{i+1}b_i - b_i^2 - b_{i+1}^2,$$

$$C = b_i^2 b_{i+1}.$$

Since the derivative tends to infinity as z tends from above to $\max(0, a_i - b_{i+1})$, and it tends to minus infinity as z tends from below to $\min(b_i, b_{i+1})$, the function is maximized at z_i. So finally we have

$$\max_{z\in B_i'} \frac{g(y_0+y_1)^2}{g(y_0)g(y_1)^2} \prod_{i=1}^{6} \max_{z\in B_i} \frac{g(y_i)g(y_{i+1})g(y_i+y_{i+1}-z)}{g(z)g(y_i-z)^2 g(y_{i+1}-y_i+z)g(y_{i+1}-z)} \le$$

$$f_0(a_0,b_0,b_1) \prod_{i=1}^{6} \frac{g(b_i)g(b_{i+1})g(b_i+b_{i+1}-z_i)}{g(z_i)g(b_i-z_i)^2 g(b_{i+1}-a_i+z_i)g(b_{i+1}-z_i)} = h(a_0,\ldots,a_6,b_0,\ldots,b_6).$$

Fix some $0 < \epsilon < 0.1$, and let $a_i = \lfloor \frac{y_i}{\epsilon} \rfloor \epsilon$, and $b_i = a_i + \epsilon$. Since a_i is the product of ϵ and some integer, there are finite number of different possible values for each a_i. A computer is used to compute $h(a_0,\ldots,a_6,b_0,\ldots,b_6)$, for these values when Condition (6) is satisfied. It turned out that for $\epsilon = 0.00125$ we have $h(a_0,\ldots,a_6,b_0,\ldots,b_6) < 0.99$ for all possible values of a_i and b_i which satisfy Condition (6). This implies that

$$\mathbf{E}[X(G)I_A(G)]^{1/n} < 0.99 + o(1),$$

which is sufficient to complete the proof. ∎

Remark. For $\epsilon = 0.00125$, the number of possible values of a_i and b_i is very large, and it takes a long time for a computer to do the required computations. To avoid this problem, we started with $\epsilon_1 = 0.01$. Every time that some a_i and b_i, $b_i = a_i + \epsilon_k$ $(0 \le i \le 6)$ satisfying Condition (6) and

$h(a_0,\ldots,a_6,b_0,\ldots,b_6) > 0.99$ are found, we computed h for all $a_i \leq a'_i \leq b'_i \leq b_i$ $(0 \leq i \leq 6)$ where a'_i is the product of ϵ_{k+1} and some integer, $\epsilon_{k+1} = \frac{\epsilon_k}{2}$, $b'_i = a'_i + \epsilon_{k+1}$ and Condition (6) is satisfied. It turned out that $\epsilon_4 = 0.00125$ is always sufficient.

For every integer g there is a constant ϵ such that the probability that the girth of a random cubic graph is greater than g is more than ϵ (see for example [5], Section 2.3). So we have the following corollary.

Corollary 2.3 *For every integer g, there is a cubic graph G with girth at least g which is not homomorphic to the cycle of size 7.*

3 Applications to the circular chromatic number

For a pair of integers p and q such that $p \geq 2q$, let $K_{p/q}$ be the graph that has vertices $\{0, 1, \ldots, p-1\}$ and in which x and y are adjacent if and only if $q \leq |x-y| \leq p-q$. For a graph G, the circular chromatic number $\chi_c(G)$ is defined as the infimum of p/q where there is a homomorphism from G to $K_{p/q}$. It is known that for every graph G the infimum in the definition is always attained and $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ (see for example [6]). Since $K_{7/3}$ is a cycle of size 7, Theorem 2.2 shows that the circular chromatic number of a random cubic graph is a.s. greater than $7/3$ which implies the following corollary.

Corollary 3.1 *There exist cubic graphs of arbitrary large girth whose circular chromatic number is greater than $7/3$.*

We ask the following question.

Question 3.2 *Determine the supremum of r^* such that there exist cubic graphs of arbitrary large girth whose circular chromatic number is at least r^*.*

We know that $7/3 \leq r^* \leq 3$. Since the circular chromatic number of C_5 is $5/2$, the answer of Question 1.1 determines whether $r^* \geq 5/2$ or $r^* \leq 5/2$. If the answer to Question 1.1 is affirmative, then it would be easier to first answer the following question.

Question 3.3 *Is it true that the circular chromatic number of any cubic graph G with sufficiently large girth is less than 3?*

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References

[1] A. Kostochka, J. Nešetřil, and P. Smolíková, *Colorings and homomorphisms of degenerate and bounded degree graphs*, Discrete Math., 233 (2001), pp. 257–266.

[2] B. D. McKay, *Independent sets in regular graphs of high girth*, Ars. Combin., A 32 (1987), pp. 179–185.

[3] J. Nešetřil, *Aspects of structural combinatorics (graph homomorphisms and their use)*, Taiwanese J. of Mathematics, 3,4 (1999), pp. 381–423.

[4] I. M. Wanless and N. C. Wormald, *Regular graphs with no homomorphisms onto cycles*, J. Combin. Theory Ser. B, 82 (2001), pp. 155–160.

[5] N. C. Wormald, *Models of random regular graphs*, in Surveys in Combinatorics, 1999, J. D. Lamb and D. E. Preece, eds., Cambridge University Press, Cambridge, 1999, pp. 239–298.

[6] X. Zhu, *Circular chromatic number: a survey*, Discrete Math., 229 (2001), pp. 371–410.