Improved Bounds for the Chromatic Number of a Graph

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Abstract: After giving a new proof of a well-known theorem of Dirac on critical graphs, we discuss the elegant upper bounds of Matula and Szekeres-Wilf which follow from it. In order to improve these bounds, we consider the following fundamental coloring problem: given an edge-cut (V_1, V_2) in a graph G, together with colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$, what is the least number of colors in a coloring of G which ''respects'' the colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$? We give a constructive optimal solution of this problem, and use it to derive a new upper bound for the chromatic number of a graph. As easy corollaries, we obtain several interesting bounds which also appear to be new, as well as classical bounds of Dirac and Ore, and the above mentioned bounds of Matula and Szekeres-Wilf. We conclude by considering two algorithms suggested by our results.
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1. INTRODUCTION

We consider only finite, undirected graphs without loops or multiple edges. Good references for undefined terms are [3] and [11].

Given a graph G, we will use $\delta(G), \Delta(G), \lambda(G), \chi(G)$ and $\chi'(G)$ to denote respectively the minimum vertex degree, maximum vertex degree, edge-connectivity, chromatic number, and chromatic index of G. If $X \subseteq V(G)$, we will use $\langle X \rangle$ to denote the subgraph of G induced by X. If $V_1 \cup V_2$ is a partition of $V(G)$ with $E(V_1, V_2) \neq \emptyset$, we call the partition an *edge-cut* in G and denote it by (V_1, V_2). We use $e(V_2, V_2)$ to denote $|E(V_1, V_2)|$, the number of edges in the edge-cut (V_1, V_2). We call G *critical* if $\chi(G - v) = \chi(G) - 1$, for all $v \in V(G)$.

In the next section (Section 2), we begin with a new and simpler proof of a wellknown theorem of Dirac [4] on critical graphs, and then derive from it the upper bound $\chi(G) \leq 1 + \max_{H \subseteq G} \lambda(H)$, which is due to Matula [6] and strengthens the classical inequality of Szekeres and Wilf [10].

In the following section (Section 3), we develop a strengthening of Matula's bound. We begin by considering the following problem: let (V_1, V_2) be an edge-cut in a graph G, and suppose we are given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$. What is the minimum number of colors needed in a coloring of G which respects the given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$ (that is, so each color class in $\langle V_j \rangle$ belongs in its entirety to a single color class of G, and so distinct color classes in $\langle V_j \rangle$ occur in distinct color classes of G, for $j = 1, 2$)? After giving a constructive optimal solution of this problem, we use the solution to derive an upper bound for $\chi(G)$ which appears to be new. As easy corollaries of this new upper bound, we obtain the above mentioned bounds of Matula and Szekeres-Wilf, two classical bounds of Dirac and Ore, and several interesting upper bounds which also appear to be new.

In the final section (Section 4), we will consider two algorithms suggested by two of the results in Section 3. The first algorithm computes an upper bound for $\chi(G)$, with no attempt to color G. The bound it returns is never worse than Matula's bound and has the same time complexity. By contrast, the second algorithm actually colors G. It uses no more colors than the bound in the first algorithm and has the same time complexity.

2. PREVIOUS RESULTS

We first present a new proof of a wellknown theorem of Dirac [4] on critical graphs, which seems simpler than the proofs that have appeared previously [2, 3, 9, 11].

Theorem 2.1. *If G is critical, then $\lambda(G) \geq \chi(G) - 1$.*

Proof. Let $\chi(G) = k + 1$, and suppose that G contains an edge-cut (V_1, V_2) with $e(V_1, V_2) \leq \chi(G) - 2 = k - 1$. Since G is critical, $\langle V_i \rangle$ is k-colorable, for $i = 1, 2$. Properly color $\langle V_1 \rangle$ and $\langle V_2 \rangle$ independently with colors $\{1, 2, \ldots, k\}$. If all edges in $E(V_1, V_2)$ have their endvertices colored differently, we would have a k-coloring of G, a contradiction. But, if some edge $e \in E(V_1, V_2)$ has both endvertices colored say 1, then since $e(V_1, V_2) \leq k - 1$, there is a color $c \in \{2, 3, \ldots, k\}$ such that no edge in $E(V_1, V_2)$ has endvertices colored 1 and c. Exchange colors 1 and c on $\langle V_1 \rangle$, thereby coloring the endvertices of e differently, without creating additional edges in $E(V_1, V_2)$ with endvertices colored the same. Iterating this procedure as long as there remain edges in $E(V_1, V_2)$ with endvertices colored the same, we eventually color G with k-colors, a contradiction. ■

As an immediate corollary, we have

Corollary 2.1. *If G is critical, then* $\delta(G) \geq \chi(G) - 1$.

Theorem 2.1 can also be used to derive a nice upper bound for $\chi(G)$, which does not seem to be as well known as it should. This noteworthy result is due to Matula [6].

Theorem 2.2. *For any graph G,* $\chi(G) \leq 1 + \max_{H \subseteq G} \lambda(H)$.

Proof. By removing a finite number of vertices, if necessary, we obtain a critical subgraph $G_c \subseteq G$ with $\chi(G_c) = \chi(G)$. Applying Theorem 2.1, we have $\chi(G) = \chi(G_c) \leq 1 + \lambda(G_c) \leq 1 + \max_{H \subseteq G} \lambda(H)$.

The above proof of Theorem 2.2 does not, however, suggest an efficient way to color G with $1 + \max_{H \subseteq G} \lambda(H)$ colors. In [7], Matula gave an interesting and elaborate algorithm to accomplish this. We also note that our proof of Theorem 2.1 can easily be adapted to give a direct proof of Theorem 2.2, leading to a simple algorithm to color G with $1 + \max_{H \subseteq G} \lambda(G)$ colors.

An immediate corollary of Theorem B is the classical bound of Szekeres and Wilf [10].

Corollary 2.2. *For any graph G,* $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$.

Of course, Corollary 2.2 could be obtained from Corollary 2.1 in exactly the way Theorem 2.2 was obtained from Theorem 2.1.

Later, we will construct a family of graphs showing that the difference in the bounds in Theorem 2.2 and Corollary 2.2 (namely, $d(G) \doteq \max_{H \subseteq G} \delta(H) - \max_{H \subseteq G} \lambda(H)$) can be arbitrarily large. However, the relative differences $d(G)/\max_{H \subseteq G} \delta(H)$ within this family approach 0 as $\max_{H \subseteq G} \delta(H)$ increases. We do not yet know if a positive relative difference as $\max_{H \subseteq G} \delta(H)$ approaches infinity is attainable.

Regarding the complexity of the bounds in Theorem 2.2 and Corollary 2.2, it is wellknown that $\max_{H \subseteq G} \delta(H)$ (the *degeneracy* of G) can be computed by the interative removal of a vertex of minimum degree. Thus, the bound in Corollary 2.2 takes $O(|V|^2)$ time. On the other hand, Matula [8] has shown that $\max_{H \subseteq G} \lambda(H)$ can be computed by the interative removal of at most $|V(G)| - 1$ minimum edge-cuts. Matula has also shown a minimum edge-cut can be found in $O(|V||E|)$ time [1, p. 273ff], and thus the time complexity of the bound in Theorem 2.2 is $O(|V|^2|E|)$.

3. OPTIMAL RESPECTING COLORINGS AND APPLICATIONS

Let (V_1, V_2) be an edge-cut in a graph G, and suppose a k_j-coloring of $\langle V_j \rangle$ is given for $j = 1, 2$. Let $k = \max\{k_1, k_2\}$. A coloring of G is called a *respecting coloring* of G (relative to the given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$) if each color class in $\langle V_j \rangle$ belongs in its entirety to a single color class of G, and if distinct color classes in $\langle V_j \rangle$ occur in distinct color classes of G, for $j = 1, 2$.

Our first goal in this section is to give an easy formula for the minimum number of colors in a respecting coloring of G (relative to the given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$), as well as a simple method to obtain such a coloring. Let I_j denote the set of vertices in V_j incident to the edges in $E(V_1, V_2)$ and set $m = \max\{|I_1|, |I_2|\}$. Assume that β_j colors occur in I_j in the k-coloring of $\langle V_j \rangle$, and without loss of generality let these colors be $\{1, 2, \ldots, \beta_j\}$, for $j = 1, 2$. Form the bipartite graph B by taking $V(B) = \{x_1, x_2, \ldots, x_{\beta_1}\} \cup \{y_1, y_2, \ldots, y_{\beta_2}\}$, and letting $(x_i, y_j) \in E(B)$ precisely if there is an edge $(v_1, v_2) \in E(V_1, V_2)$ with v_1 colored i in $\langle V_1 \rangle$, and v_2 colored j in $\langle V_2 \rangle$. The complementary bipartite graph $\overline{B}$ is defined (on the same vertex partition sets as B) by $(x_i, y_j) \in E(\overline{B})$ if and only if $(x_i, y_j) \notin E(B)$. Let $\overline{M}$ denote a maximum matching in $\overline{B}$. If $(x_i, y_j) \in \overline{M}$, we will say informally that color (class) i in $\langle V_1 \rangle$ and color(class) j in $\langle V_2 \rangle$ are *matched* under $\overline{M}$.

We then have the following key result.

Theorem 3.1. *G has a respecting coloring (relative to the given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$) with $\max\{k, \beta_1 + \beta_2 - |\overline{M}|\}$ colors, and no fewer.*

Proof. Without loss of generality, suppose the colors occuring in I_1 which are matched under $\overline{M}$ are $1, 2, \ldots, |\overline{M}|$, the remaining colors occuring in I_1 are $|\overline{M}| + 1, \ldots, \beta_1$, and the colors in $\langle V_1 \rangle$ not occuring in I_1 are $\beta_1 + 1, \ldots, k_1$. Label (relabel if necessary) the colors in $\langle V_2 \rangle$ as follows. The colors in I_2 which are matched under $\overline{M}$ are labeled with the color to which there are matched in I_1, the $\beta_2 - |\overline{M}|$ colors occuring in I_2 which are unmatched under $\overline{M}$ are labeled $\beta_1 + 1, \ldots, \beta_1 + \beta_2 - |\overline{M}|$, and the colors in $\langle V_2 \rangle$ not occuring in I_2 are labeled with the smallest available (that is, unused so far in $\langle V_2 \rangle$) color in $\{1, 2, \ldots, k\}$. It is easily seen that this gives a respecting coloring of G using just $\max\{k, \beta_1 + \beta_2 - |\overline{M}|\}$ colors.

On the other hand, coloring B so that $\{x_1, \ldots, x_{\beta_1}\}$ are colored differently and $\{y_1, \ldots, y_{\beta_2}\}$ are colored differently clearly requires at least $\beta_1 + \beta_2 - |\overline{M}|$ colors.

Thus, the number of colors in a respecting coloring of G (relative to the given colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$) cannot be lower then the bound in Theorem 3.1. ■

In Section 4, we describe a coloring algorithm based on Theorem 3.1. In order to measure the goodness of this algorithm, we derive from Theorem 3.1 an upper bound for $\chi(G)$ (Theorem 3.2), which appears not to have been noted previously. We then obtain, as easy corollaries of Theorem 3.2, several classical bounds for $\chi(G)$, as well as some interesting new ones. Taken together, the bounds on $\chi(G)$ which follow easily from Theorem 3.2 will indicate the quality of Theorem 3.1 and the coloring algorithm based on it.

Before proving Theorem 3.2, we need several preliminary results. For any graph H, let e_H denote $|E(H)|$.

Lemma 3.1. *If $\beta_1 \geq \beta_2$, then*

$$(\beta_1 - 1)(\beta_2 - |\overline{M}|) \leq e_B - \beta_2 \leq (\beta_1 - 1)\tfrac{e_B}{\beta_1} \leq (\beta_1 - 1)\tfrac{e(V_1, V_2)}{\beta_1}.$$

Proof. The last inequality is trivial, since $e_B \leq e(V_1, V_2)$. The second inequality is equivalent to $e_B \leq \beta_1\beta_2$, which is also clear. The first inequality is equivalent to $e_{\overline{B}} = \beta_1\beta_2 - e_B \leq (\beta_1 - 1)|\overline{M}|$. If $|\overline{M}| = 0$, then $e_{\overline{B}} = 0$ and the inequality holds. Otherwise, the inequality is equivalent to $\beta_1 - 1 \geq \frac{e_{\overline{B}}}{|\overline{M}|}$. But $\chi'(\overline{B}) = \triangle(\overline{B})$ since $\overline{B}$ is bipartite, and thus $\beta_1 - 1 \geq \triangle(\overline{B}) = \chi'(\overline{B}) \geq \frac{e_{\overline{B}}}{|\overline{M}|}$. ■

Lemma 3.2. *If $\beta_1 \geq \beta_2$, then $\beta_1 + \beta_2 - |\overline{M}| \leq \beta_1 + \frac{e(V_1, V_2)}{\beta_1}$.*

Proof. If $\beta_1 = \beta_2 = 1$, the inequality holds since $e(V_1, V_2) \geq 1$. Otherwise, dividing through by $\beta_1 - 1 > 0$ in Lemma 3.1 gives $\beta_1 + \beta_2 - |\overline{M}| \leq \beta_1 + \frac{e(V_1, V_2)}{\beta_1}$, as asserted. ■

In the sequel, we will set $\beta = \max\{\beta_1, \beta_2\}$.

Lemma 3.3. *If $\beta \leq \sqrt{e(V_1, V_2)}$ then $\beta_1 + \beta_2 - |\overline{M}| \leq 2\sqrt{e(V_1, V_2)}$.*

Proof. The desired inequality is equivalent to $|\overline{M}| \geq \beta_1 + \beta_2 - 2\sqrt{e(V_1, V_2)}$. But $\beta_j \leq \beta \leq \sqrt{e(V_1, V_2)}$ for $j = 1, 2$. So $\beta_1 + \beta_2 - 2\sqrt{e(V_1, V_2)} \leq 0 \leq |\overline{M}|$. ■

We are now in a position to prove the following.

Theorem 3.2. *Let (V_1, V_2) be an edge-cut in a graph G. Suppose $\langle V_j \rangle$ is k_j-colored, for $j = 1, 2$, and let k and β be as above. Then for any $x \geq \beta$, we have*

$$\chi(G) \leq \max\left\{k, x + \tfrac{e(V_1, V_2)}{x}\right\}.$$

Proof. By Theorem 3.1, it suffices to show

$$\beta_1 + \beta_2 - |\overline{M}| \leq \min_{x \geq \beta} \left\{x + \tfrac{e(V_1, V_2)}{x}\right\}.$$

But a consideration of the function $x + \frac{e(V_1,V_2)}{x}$ shows

$$\min_{x \geq \beta}\left\{x + \frac{e(V_1,V_2)}{x}\right\} = \begin{cases} 2\sqrt{e(V_1,V_2)}, & \text{if } \beta \leq \sqrt{e(V_1,V_2)} \\ \beta + \frac{e(V_1,V_2)}{\beta}, & \text{if } \beta > \sqrt{e(V_1,V_2)} \end{cases}$$

Lemmas 3.2 and 3.3 now give $\beta_1 + \beta_2 - |\overline{M}| \leq \min_{x \geq \beta}\left\{x + \frac{e(V_1,V_2)}{x}\right\}$, as required. ∎

From Theorem 3.2, we can easily derive an interesting and apparently new upper bound for $\chi(G)$. This bound will be the basis for the first algorithm in Section 4. Recall that $m = \max\{|I_1|, |I_2|\}$.

Corollary 3.1. *Let (V_1, V_2) be an edge-cut in a graph G. Then*

$$\chi(G) \leq \max\left\{\chi(\langle V_1\rangle), \chi(\langle V_2\rangle), m + \frac{e(V_1,V_2)}{m}\right\}.$$

Proof. In any $\chi(\langle V_j\rangle)$-coloring of $\langle V_j\rangle$, the number of colors β_j used in I_j cannot exceed $|I_j| \leq m$. Thus, $m \geq \max\{\beta_1, \beta_2\} = \beta$. Theorem 3.2 now gives the desired bound. ∎

Corollary 3.1 is occasionally much better than Theorem 2.2. Consider the graph G formed from the disjoint union of graphs $G_1, \ldots, G_4$ by adding all possible edges between G_i and G_{i+1}, for $i = 1, 2, 3$, where $G_1 = G_4 = K_{2n}$, and $G_2 = G_3 = \overline{K}_n$. Theorem 2.2 gives only $\chi(G) \leq 3n + 1$, while Corollary 3.1 (starting with edge-cut $V_1 = V(G_1 \cup G_2)$ and $V_2 = V(G_3 \cup V_4)$) gives $\chi(G) \leq \max\{\chi(K_{2n} + \overline{K_n}), n + \frac{n^2}{n}\} = 2n + 1$, the chromatic number of G. Of course, the initial edge-cut (V_1, V_2) is not a minimum edge-cut in G, and finding such favorable edge-cuts is almost surely an intractable problem.

However, Corollary 3.1 can be arbitrarily better than Theorem 2.2 even when we limit the edge-cuts used to minimum edge-cuts. Consider the graph G formed as follows: let n be a multiple of 4. Start with the disjoint union of $G_1, \ldots, G_4$, where $G_1 = G_4$ is any $(n^2 - n/4)$-regular graph on $2(n-1)^2 + n/2$ vertices, and $G_2 = G_3 = K_n$. Join each vertex in G_1 to half the vertices in G_2 in such a way that each vertex in G_2 is also joined to exactly half the vertices in G_1. Make a similar join between G_3 and G_4. Finally, add all possible edges between G_2 and G_3. The resulting graph G is $(n^2 + n/4)$-regular. It is easily verified that Corollary 2.2 gives only $\chi(G) \leq 1 + (n^2 + n/4)$, while Theorem 2.2 (starting with the minimum edge-cut $V_1 = V(G_1 \cup G_2)$ and $V_2 = V(G_3 \cup G_4)$) gives $\chi(G) \leq 1 + n^2$. But Corollary 3.1 (starting with the same minimum edge-cut (V_1, V_2)) gives $\chi(G) \leq 1 + n^2 - n/4$.

On the other hand, since $\sqrt{e(V_1, V_2)} \leq m \leq e(V_1, V_2)$ and $f(x) = x + \frac{e(V_1,V_2)}{x}$ is increasing between $\sqrt{e(V_1, V_2)}$ and $e(V_1, V_2)$, it follows that $m + \frac{e(V_1,V_2)}{m}$

$\leq f(e(V_1, V_2)) = 1 + e(V_1, V_2)$. Thus, the following result of Ore is an immediate consequence of Corollary 3.1.

Corollary 3.2 (Ore [9]). *Let (V_1, V_2) be an edge-cut in a graph G. Then, $\chi(G) \leq \max\{\chi(\langle V_1\rangle), \chi(\langle V_2\rangle), 1 + e(V_1, V_2)\}$.*

We can, in turn, obtain Theorem 2.1 as an easy consequence of Corollary 3.2.

Corollary 3.3. (Dirac [4]). *If G is critical, then $\lambda(G) \geq \chi(G) - 1$.*

Proof. Suppose a critical graph G contains an edge-cut (V_1, V_2) with $e(V_1, V_2) \leq \chi(G) - 2$. Since G is critical, we have $\chi(\langle V_j\rangle) \leq \chi(G) - 1$, for $j = 1, 2$. So by Corollary 3.2, we have $\chi(G) \leq \max\{\chi(\langle V_1\rangle), \chi(\langle V_2\rangle), 1 + e(V_1, V_2)\} \leq \chi(G) - 1$, a contradiction. ■

As we saw above, Theorem 2.2 (Matula) and Corollary 2.2 (Szekeres and Wilf) both follow easily from Corollary 3.3.

Another immediate consequence of Corollary 3.2 is the following result, first noted by Dirac [4].

Corollary 3.4. *Let (V_1, V_2) be an edge-cut in a graph G. Suppose $\langle V_1\rangle$ and $\langle V_2\rangle$ are k-colorable. If $e(V_1, V_2) \leq k - 1$, then $\chi(G) \leq k$.*

Corollary 3.4 has a simple generalization, which appears not to have been noted before.

Corollary 3.5. *Let (V_1, V_2) be an edge-cut in a graph G. If $\langle V_1\rangle$ and $\langle V_2\rangle$ are k-colorable, then $\chi(G) \leq k + \frac{e(V_1,V_2)}{k}$.*

Proof. Simply take $x = k \geq \beta$ in Theorem 3.2.

There is also a direct proof of Corollary 3.5, which actually proves somewhat more than Corollary 3.5 when k-colorings of $\langle V_1\rangle$ and $\langle V_2\rangle$ are given. It is based on a proof idea of Paul Kainen [11, p. 211].

Alternate Proof of Corollary 3.5. Let $V_{j,1} \cup \cdots \cup V_{j,k}$ be a k-coloring of $\langle V_j\rangle$, for $j = 1, 2$. Define a bipartite graph H by $V(H) = \{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_k\}$, with $(x_i, y_j) \in E(H)$ iff $E(V_{1,i}, V_{2,j}) \neq \emptyset$. Let $\overline{H}$ denote the complement bipartite graph on the same vertex partition sets. We find $e_{\overline{H}} = k^2 - e_H = k(k - \frac{e_H}{k})$. Thus, any vertex cover of $\overline{H}$ contains at least $k - \frac{e_H}{k}$. vertices, since each vertex of $\overline{H}$ covers at most k edges. By the Koenig-Egervary Theorem [11, p. 112], a maximum matching M in $\overline{H}$ satisfies $|M| \geq k - \frac{e_H}{k}$. So $\chi(G) \leq 2k - |M| \leq 2k - (k - \frac{e_H}{k}) = k + \frac{e_H}{k} \leq k + \frac{e(V_1,V_2)}{k}$. ■

The above proof shows $\chi(G) \leq k + \frac{e_H}{k}$ (and thus $\chi(G) \leq k$, if $e_H \leq k - 1$) if we are given k-colorings of $\langle V_1\rangle$ and $\langle V_2\rangle$. Since $e_H \leq e(V_1, V_2)$, this strengthens

Corollaries 3.4 and 3.5 when colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$ are given. (These stronger bounds also follow immediately from Theorem 3.1 and Lemma 3.1. Since $\sqrt{e_B} \leq \beta \leq k$ and $e_B = e_H$, we have $\beta + \frac{e_B}{\beta} \leq k + \frac{e_B}{k} = k + \frac{e_H}{k}$. But then $\chi(G) \leq \max\{k, \beta_1 + \beta_2 - |\overline{M}|\} \leq \max\left\{k, \beta + \frac{e_B}{\beta}\right\} \leq \max\left\{k, k + \frac{e_H}{k}\right\} = k + \frac{e_H}{k}$.)

4. ALGORITHMIC IMPLICATIONS OF OUR RESULTS

We now present two algorithms based on the above results. The first algorithm, based on Corollary 3.1, computes an upper bound for $\chi(G)$ with no attempt to color G. The bound it returns is never worse than the bound in Theorem 2.2, and is just as easy to compute. By contrast, the second algorithm, which is based on Theorem 3.1, actually colors G using no more colors than the bound returned by the first algorithm (often far less) and essentially no more time than the first algorithm. In both of these algorithms, we do not try to find edge-cuts (V_1, V_2) which optimize the bounds in Corollary 3.1 or Theorem 3.1, since finding such optimal cuts is almost certainly intractable. Instead, merely selecting minimum edge-cuts at every step will yield the good algorithmic results described above.

The first algorithm is easy to describe. Iteratively select and remove a minimum edge-cut (V_1, V_2) in any remaining component, until all the edges have been removed. As we obtain each such cut (V_1, V_2) we compute the value $m + \frac{e(V_1,V_2)}{m}$. At the end, we return the maximum of these values over all the cuts.

It is easy to see this algorithm is correct. If $\chi(\langle V_1 \rangle), \chi(\langle V_2 \rangle) < \chi(G)$, then Corollary 3.1 gives $\chi(G) \leq m + \frac{e(V_1,V_2)}{m}$. Otherwise, $\chi(G) = \chi(\langle V_j \rangle)$, for $j = 1$ or 2, and the correctness follows by induction on $|V|$, since $|V_j| < |V(G)|$.

Like Theorem 2.2, the above algorithm simply finds $|V| - 1$ minimum edge cuts, and so its time complexity is exactly the same as Theorem 2.2, namely $O(|V|^2|E|)$.

Finally, the bound returned by this algorithm is never worse than the bound in Theorem 2.2. If $m + \frac{e(V_1,V_2)}{m}$ is returned, then (V_1, V_2) is a minimum edge-cut for some induced subgraph H' of G, and so $e(V_1, V_2) = \lambda(H')$. But as noted above, $m + \frac{e(V_1,V_2)}{m} \leq 1 + e(V_1, V_2) = 1 + \lambda(H') \leq 1 + \max_{H \subseteq G} \lambda(H)$.

Our second algorithm is based on Theorem 3.1, and actually colors G using no more colors than the bound in the first algorithm. Like the first algorithm, it first iteratively removes minimum edge-cuts until no edges remain (this takes $O(|V|^2|E|)$ time). The vertices (now independent) are all colored 1. The minimum edge-cuts are then restored in reverse order to their removal (a stack will facilitate this). As each cut (V_1, V_2)-say from subgraph H-is restored, we give H the optimal coloring which respects the inherited colorings of $\langle V_1 \rangle$ and $\langle V_2 \rangle$. Since there are just $|V| - 1$ cuts to restore, and finding an optimal respecting coloring takes $O(|V|^{\frac{1}{2}}|E|)$ time (essentially to find the maximum matching $\overline{M}$ in $\overline{B}$ [1, p. 469]), this rebuilding-coloring phase takes just $O(|V|^{\frac{3}{2}}|E|)$ time. Thus the entire algorithm uses no more time asymptotically than is needed to compute the bound in Theorem 2.2.

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