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Double covers of cubic graphs with oddness 4

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Abstract

We prove that a cubic 2-connected graph which has a 2-factor containing exactly 4 odd cycles has a cycle double cover.

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1. Introduction

For a graph G we let $v(G)$ denote the number of vertices in G. We let E_v denote the set of edges incident with a vertex v, and we let $N_G(v)$ be the set of vertices which are neighbours to v. For a subset $X \subseteq V(G)$, or a subgraph $X \subseteq G$ we let ∂X be the set of edges with one end in X and the other in $V(G)\backslash X$ and we let $d_G(X)$ be the number of edges in this set. For $l \geqslant 0$ we let $v_l(G)$ be the number of vertices of degree l, and we let $v_{\geqslant l}(G)$ (resp., $v_{\leqslant l}(G)$) be the number of vertices of degree at least l(resp., at most l).

A *bridge* in a graph is an edge whose deletion results in a graph with more components. We say that a cubic graph is *cyclically k-edge connected* if for any separating subset $A \subset E(G)$ where $|A| < k$, it holds that at most one component of $G\backslash A$ is not a tree. A *snark* is defined to be a cubic, cyclically 4-edge connected graph G having girth at least 5 and chromatic index 4; that is, $\chi'(G) = 4$. Here the chromatic index of a graph G, denoted $\chi'(G)$, is the smallest number of colours which can be assigned to the edges of G so that no 2 edges of

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the same colour meet at a vertex. The smallest snark is known to be the Petersen graph P_{10}, which has 10 vertices. It is also known that there are no snarks with 12, 14, or 16 vertices, but there are 2 snarks with 18 vertices, 6 snarks with 20 vertices, and 20 snarks with 22 vertices(see [1,3]).

We shall refer to a subgraph all of whose degrees are even as *circuit*. On the other hand, a connected, 2-regular subgraph will be called a *cycle*. A collection of cycles (resp., circuits) which covers the edges of a graph exactly twice will be called a *cycle double cover* (resp., *circuit double cover*). A k-cycle (resp., k-circuit) double cover is a cycle (resp., circuit) double cover with at most k cycles (resp., circuits).

For a cubic bridgeless graph G, we can partition the vertices by a set of vertices X (possibly empty) and a set of disjoint cycles $\mathcal{C}$. We call the pair $(X, \mathcal{C})$ a *pseudo 2-factor* of G. We define the *oddness* of G, denoted $o(G)$, to be the minimum k such that there is a pseudo 2-factor $(X, \mathcal{C})$ where $|X|$ plus the number of odd cycles in $\mathcal{C}$ equals k. This definition extends the one given by Huck and Kochol [8] who proved the following:

Theorem 1.1 (*Huck and Kochol [8]*). *Let G be a cubic, bridgeless graph. If G has a* 2-*factor with at most* 2 *odd cycles, then G has a* 5-*circuit double cover*.

As a consequence of this theorem, any cubic bridgeless graph having a hamilton path (a path traversing all vertices) has a double cover. This was also shown in [5]. In this paper, we extend Huck and Kochol's result by showing that for graphs with oddness at most 4, there is a cycle double cover.

Theorem 1.2. *Let G be a cubic bridgeless graph. If $o(G) = 4$, then G has a cycle double cover*.

Suppose that G is a cubic, bridgeless graph and $(X, \mathcal{C})$ is a pseudo 2-factor of G. We form the graph $G_{\mathcal{C}}$ by contracting every cycle of $\mathcal{C}$ so that they become vertices. We call a bridgeless subgraph $A_{\mathcal{C}} \subset G_{\mathcal{C}}$ a *degree-compatible subgraph* of $G_{\mathcal{C}}$ if the odd vertices of $A_{\mathcal{C}}$ are exactly the odd vertices of $G_{\mathcal{C}}$. Given $v \in V(G_{\mathcal{C}})\backslash X$, we let $C(v) \in \mathcal{C}$ denote the corresponding cycle in G. Any subgraph of $G_{\mathcal{C}}$ will be given the subscript $\mathcal{C}$, and given a subgraph $J_{\mathcal{C}} \subseteq G_{\mathcal{C}}$, we let J be the subgraph of G by taking the union of $C(v),\ v \in V(J_{\mathcal{C}})\backslash X$ together with the vertices of X belonging to $J_{\mathcal{C}}$ and edges of G corresponding to edges in $J_{\mathcal{C}}$. We let $h(J)$ denote the graph obtained from J by suppressing all vertices of degree 2. If $J_{\mathcal{C}}$ is a subgraph of $G_{\mathcal{C}}$, then for $v \in V(J_{\mathcal{C}})\backslash X$ we let $C_{h(J)}(v)$ be the cycle in $h(J)$ corresponding to v.

For each subgraph $J_{\mathcal{C}}$ of $G_{\mathcal{C}}$ we let $p_{h(J)} : E(h(J)) \to \{1, 2\}$ be a weighting for $h(J)$ where

$$p_{h(J)}(e) = \begin{cases} 1 & \text{if } e \in \bigcup_{v \in V(J_{\mathcal{C}})\backslash X} C_{h(J)}(v), \\ 2 & \text{otherwise.} \end{cases}$$

If there is a collection of cycles in $h(J)$ which covers each edge $e \in E(h(J))$ exactly $p_{h(J)}(e)$ times, then we say that $h(J)$ is $\mathcal{C}$-*compatible*.

Huck [7] proved independently the above theorem, showing not only that G has a double cover, but also showing that it has a 5-circuit double cover. His proof is long and complicated.

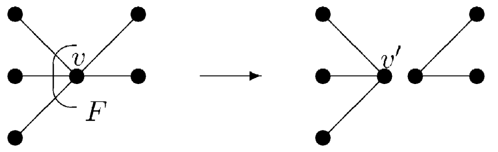

Fig. 1. Splitting the edges of F away from v.

This paper presents a more cohesive approach which utilizes splitting and expansion operations to show the following (Theorem 6.1): for a cubic, bridgeless graph G, if $o(G) \leqslant 4$, then either one can find a degree compatible subgraph $H_{\mathcal{C}}$ of $G_{\mathcal{C}}$ such that $h(H)$ is $\mathcal{C}$-compatible, or G has a non-trivial 3-edge cut.

Note: With some extra work, one can show that the theorem stated above is still true even if we replace the condition "$h(H)$ is $\mathcal{C}$-compatible" with $\chi'(h(H)) = 3$. Using this, one can strengthen Theorem 1.2 to yield Huck's result.

The initial steps in the proof of Theorem 1.2 use a "splitting" operation for vertices. Let G be a graph and suppose $v \in V(G)$ and $F \subset E_v$. We define a new graph $G_{[v;F]}$ by splitting the edges of F away from v and creating a new vertex v' whose incident edges are those of F.

We call this operation a *splitting* of F at v (see Fig. 1). The following theorem (see [4] or [9]) tells us when splitting is possible without creating bridges.

Theorem 1.3. *Let G be a connected bridgeless graph. Suppose $v \in V(G)$ where $d_G(v) \geqslant 4$ and let $e_0, e_1, e_2 \in E_v$. Then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is connected and bridgeless or $G_{[v;\{e_1,e_2,e_3\}]}$ has more components than G.*

The above theorem has the immediate corollary:

Corollary 1.4. *Suppose G is a connected bridgeless graph and $v \in V(G)$ where $d_G(v) = 4$ and $e_0, e_1, e_2 \in E_v$. Then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is connected and bridgeless.*

Let G be a cubic graph and suppose $C = v_0e_0v_1e_1v_2e_2v_3e_3v_0$ is a 4-cycle. We create a new cubic graph by deleting the edges e_1 and e_3 and suppressing the resulting vertices of degree 2. Such a graph we denote by $G \oplus_{\square} \{e_1, e_3\}$. We call the corresponding operation a $\square$-reduction.

Lemma 1.5. *Suppose G is a 2-connected cubic graph and $C = v_0e_0v_1e_1v_2e_2v_3e_3v_0$ is a 4-cycle. Then either $G \oplus_{\square} \{e_1, e_3\}$ or $G \oplus_{\square} \{e_0, e_2\}$ is 2-connected.*

Proof. Let G and C be as in the statement of the lemma. If C contains a chord, then the result is clear. We suppose therefore that C has no chords and we contract the edges of C so that it becomes a single vertex v which has degree 4. Let G' be the resulting graph and suppose v has incident edges f_0, f_1, f_2, f_3. Here f_i corresponds to an edge in G incident with v_i. By Corollary 1.4, either $G_{[v;\{f_0,f_3\}]}$ or $G_{[v;\{f_0,f_1\}]}$ is connected and bridgeless. This in turn implies that either $G \oplus_{\square} \{e_0, e_2\}$ or $G \oplus_{\square} \{e_1, e_3\}$ is 2-connected. $\square$

The next lemma is a basic observation about $\square$-reductions and colourings. The proof is left to the reader.

Lemma 1.6. *Suppose G is a cubic graph and let H be a cubic graph obtained from G via a $\square$-reduction. Then $\chi'(G) = 3$ if $\chi'(H) = 3$.*

Combining Lemmas 1.5 and 1.6 we obtain:

Lemma 1.7. *Suppose G is a 2-connected cubic graph having disjoint 4-cycles $C_1, \ldots, C_k$. There exist $\square$-reductions on each 4-cycle $C_1, \ldots, C_k$ such that after performing these reductions, we obtain a 2-connected cubic graph H. Moreover, if $\chi'(H) = 3$, then$\chi'(G) = 3$.*

Corollary 1.8. *Suppose G is a 2-connected cubic graph having disjoint 4-cycles $C_1, \ldots, C_k$. If $G \backslash (C_1 \cup \ldots \cup C_k)$ has at most 8 vertices, then $\chi'(G) = 3$.*

Proof. By Lemma 1.7 there exist $\square$-reductions on each 4-cycle $C_i,\ i = 1, 2, \ldots, k$ such that after performing these reductions we obtain a 2-connected cubic graph H. Since $G \backslash (C_1 \cup \ldots \cup C_k)$ has at most 8 vertices, we have that $v(H) \leqslant 8$. This means that $\chi'(H) = 3$, since the smallest 2-connected cubic graph with chromatic index 4 is P_{10}. Now Lemma 1.7 implies that $\chi'(G) = 3$. $\square$

2. Reductions and extensions

Let G be a 2-connected cubic graph having a 2-edge cut $\partial X = \{e, f\}$ where $e = uu'$, $f = vv'$, and $u, v \in X$. We define a new graph by deleting e and f and adding new edges $e' = uv$ and $f' = u'v'$, and we denote this graph by $G \oplus \{e, f\}$. We call the corresponding operation a 2-*edge reduction*. If G has a 3-edge cut $\partial X = \{e_1, e_2, e_3\}$ where $e_i = u_i v_i,\ u_i \in X,\ i = 1, 2, 3$, then we can define a new graph by deleting $e_i,\ i = 1, 2, 3$ and adding new vertices u and v together with edges uu_i and $vv_i,\ i = 1, 2, 3$. We denote this graph by $G \oplus \{e_1, e_2, e_3\}$ We call the corresponding operation a 3-*edge reduction*.

Suppose G has a triangle $T = v_1e_1v_2e_2v_3e_3v_1$. We define a new cubic graph by contracting (ie. identifying) T with a single vertex. Such a graph we denote by $G \oplus_{\triangle} (T)$. We call the corresponding operation a Δ-*reduction*.

Let u_1 and u_2 be the endvertices of a digon D in G. By *digon* we mean a pair of edges inducing a 2-cycle. Let $N_G(D) = \{u'_1, u'_2\}$ (here we allow for $u'_1 = u'_2$). We define a new graph $G \oplus_{\circ} (D) = (G \backslash D) \cup \{u'_1 u'_2\}$. Such an operation we call a $\circ$-*reduction*.

Suppose v is a vertex of degree 2 which is not incident with a loop. Given $N_G(v) = \{v_1, v_2\}$ (with possibly $v_1 = v_2$) we define a new graph $G \oplus_{\vee} (v) = (G \backslash v) \cup \{v_1 v_2\}$. We call the corresponding operation a $\vee$-*reduction*.

Suppose G and H are graphs. We say that G and H are *homeomorphic* if one graph can be obtained from the other via $\vee$-reductions and subdividing edges.

If G is a graph having no components which are cycles, then we can perform successive $\vee$-reductions on G to obtain a graph with no vertices of degree 2. This graph is seen to be

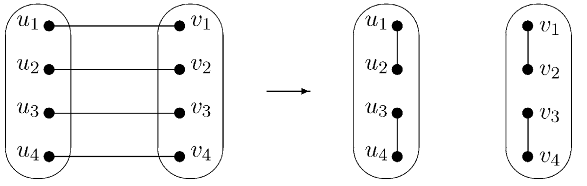

Fig. 2. 4-edge reduction.

the unique graph homeomorphic to G which has no vertices of degree 2. We denote such a graph by $h(G)$. We define the *homeomorph chromatic index* of G, denoted $\chi'_h(G)$, by $\chi'_h(G) = \chi'(h(G))$.

Let G be a cubic graph, and let $e \in E(G)$ be such that e is not incident with any loops. We define a graph $G \oplus e = h(G \backslash e)$. We obtain $G \oplus e$ from G via an operation which we refer to as an *edge-reduction*. The following is a standard result and we refer the reader to [6], chapter 3.

Theorem 2.1. *Suppose G is a bridgeless cubic graph.*

(a) *If G' is obtained from G via a* 2- *or* 3-*edge reduction, or via a* $\vee$-, $\circ$-, *or* $\triangle$-*reduction, then* $\chi'(G') = \chi'(G)$.
(b) *If G' is obtained from G via a* $\square$-*reduction, then* $\chi'(G') = 4$ *if* $\chi'(G) = 4$.

Corollary 2.2. *Let G be a cubic graph and let G' be a cubic graph obtained from G via a sequence of* 2-, 3-, $\vee$-, $\circ$-, *or* $\triangle$-*reductions. If* $\chi'(G') = 3$, *then* $\chi'(G) = 3$.

Suppose we are given a 4-edge cut ∂X where we order the edges as e_1, e_2, e_3, e_4, and $e_i = u_i v_i,\ u_i \in X,\ i = 1, 2, 3, 4$. We define a new graph, denoted $G \oplus (e_1, e_2, e_3, e_4)$, where we delete the edges $e_i,\ i = 1, 2, 3, 4$ and add the edges $u_1u_2, u_3u_4, v_1v_2, v_3v_4$. We call the corresponding operation a 4-*edge reduction* (see Fig. 2).

Similarly, given a 5-edge cut ∂X, if we order the edges of ∂X as e_1, e_2, e_3, e_4, e_5 where $e_i = u_i v_i,\ u_i \in X,\ i = 1, \ldots, 5$, we define a new graph, denoted $G \oplus (e_1, e_2, e_3, e_4, e_5)$ by first deleting $e_1, \ldots, e_5$, and then adding edges u_4u_5, v_4v_5, and 2 new vertices u and v together with the edges $u_i u, v_i v,\ i = 1, 2, 3$. The corresponding operation we call a 5-*edge reduction* (see Fig. 3).

We define an insertion operation in the following way: we subdivide an edge of a graph G inserting a vertex u, and then subdivide a new edge in the resulting graph, inserting another vertex v. We then add an edge $e = uv$. The combined operation is called an *edge-insertion* operation, which we denote by $G \odot e$. If we insert edges $e_1, \ldots, e_k$ successively in G, then we denote the resulting graph by $G \odot (e_1, \ldots, e_k)$, or in the case where S is a subset of edges to be inserted, we let $G \odot S$ denote the resulting graph.

We define a corresponding insertion operation for vertices, whereby we subdivide edges 3 times in succession, inserting vertices u_1, u_2, and u_3. We then add a vertex v and join it to u_1, u_2, and u_3 by edges. The operation is called a *vertex-insertion* operation, and we denote the resulting graph by $G \odot v$.

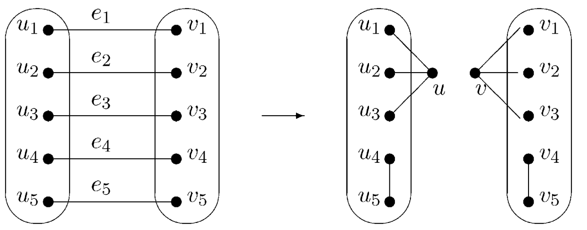

Fig. 3. 5-edge reduction.

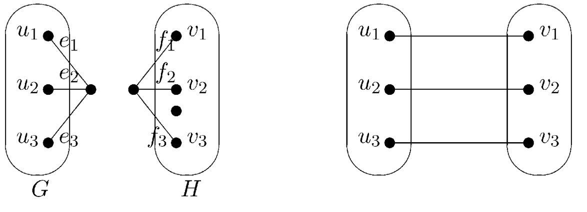

Fig. 4. Vertex expansion at u.

For each of the reduction operations defined above, we can define the reverse operation, namely, an *expansion* operation. Suppose G is a cubic graph and let $e = u_1u_2 \in E(G)$. Let H be a cubic graph and let $f = v_1v_2 \in E(H)$. Given that the endvertices of e and f are ordered as u_1, u_2 and v_1, v_2, respectively, we define $(G; u_1; e) \otimes (H; v_1; f) = (G\backslash\{e\}) \cup (H\backslash\{f\}) \cup \{u_1v_1, u_2v_2\}$ and the corresponding operation we call a 2-*edge expansion*.

Suppose $u \in V(G)$. Let e_1, e_2, e_3 be an ordering of the edges incident to u where $e_i = u_iu,\ i = 1, 2, 3$. Let H be a cubic graph and let $v \in V(H)$. We suppose f_1, f_2, f_3 is an ordering of the edges incident to v where $f_i = v_iv,\ i = 1, 2, 3$. We define an operation called a *vertex expansion* at u whereby we delete u, and add the graph $H\backslash v$ together with the edges $u_iv_i,\ i = 1, 2, 3$. (see Fig. 4). We denote the resulting graph by $(G; u; e_1, e_2, e_3) \otimes (H; v; f_1, f_2, f_3)$ and denote the corresponding operation by $u \to H$. Note that a vertex expansion may yield the same graph, for example when H is a multiple 3-edge. In the case where H is 3-edge colourable, we refer to the vertex expansion as being 3-*chromatic*.

If we perform an expansion at each vertex, then we say that the resulting graph is an *expansion* of G. If in addition each vertex expansion is 3-chromatic, then we say that the expansion is 3-chromatic.

We may define the reverse operations to 4- and 5-edge reductions as well. Let G and H be cubic graphs. Pick 2 non-incident edges of G which we order as e_1, e_2 where we assume $e_1 = u_1u_2$, and $e_2 = u_3u_4$. Here we order the endvertices of e_1 and e_2 as u_1, u_2 and u_3, u_4, respectively. Similarly, we pick 2 edges f_1, f_2 in H where $f = v_1v_2$ and $f_2 = v_3v_4$. We order the endvertices of f_1 and f_2 as v_1, v_2 and v_3, v_4, respectively. We define a 4-*edge expansion* whereby we delete e_1, e_2, f_1, f_2 from $G \cup H$ and then add the edges $u_iv_i,\ i = 1, 2, 3, 4$. (see Fig. 5) We denote this graph by $(G; u_1, u_3; e_1, e_2) \otimes (H; v_1, v_3; f_1, f_2)$.

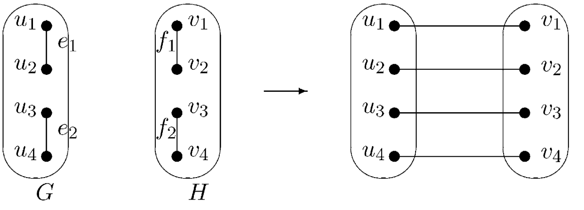

Fig. 5. 4-edge expansion.

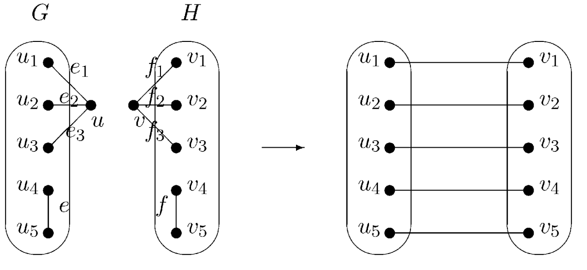

Fig. 6. 5-edge expansion.

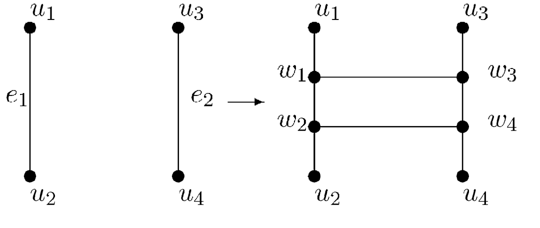

Fig. 7. Expanding a square.

Suppose we are given cubic graphs G and H as before. We let u be a vertex of G and let $e \in E(G)$ be an edge non-incident with u. We order the edges of E_u as e_1, e_2, e_3 where $e_i = uu_i,\ i = 1, 2, 3$. We let $e = u_4u_5$ where the vertices are ordered as u_4, u_5. In a similar way, let $v \in V(H)$ and let $f \in E(H),\ f \notin E_v$. We suppose f_1, f_2, f_3 is an ordering of the edges at v where $f_i = vv_i,\ i = 1, 2, 3$. We let $f = v_4v_5$ where the endvertices are ordered as v_4, v_5. We define a 5-*edge expansion* by deleting u, v, e, f from $G \cup H$ and adding the edges $uv_i,\ i = 1, \ldots, 5$ (see Fig. 6). We denote the resulting graph by $(G; u, u_4; e_1, e_2, e_3, e) \otimes (H; v, v_4; f_1, f_2, f_3, f)$.

We can define a $\square$-*expansion* in the following way: let G be a cubic graph and let $e_1 = u_1u_2 \in E(G)$ and $e_2 = u_3u_4 \in E(G)$ where the endvertices of e_1 and e_2 are ordered as u_1, u_2 and u_3, u_4, respectively. We subdivide e_1 by 2 vertices w_1 and w_2, so that the vertices lie in order u_1, w_1, w_2, u_2. Next, we subdivided e_2 by the vertices w_3 and w_4 so that the vertices lie in order u_3, w_3, w_4, u_4.. We then add the edges w_1w_3 and w_2w_4. It is permissable that $e_1 = e_2$ but w_1w_2 and w_3w_4 must be edges in the subdivided graph. We denote the resulting graph by $G \otimes_\square (e_1, e_2; u_1, u_2; u_3, u_4)$. See Fig. 7.

Some simple observations are given in the proposition below whose proof is straightforward and left as an exercise for the reader.

Proposition 2.3. *Let G and H be cubic graphs.*

(a) *If* $u \in V(G)$ *and* G' *is the result of a vertex expansion* $u \to H$, *then* $\chi'(G') = \max\{\chi'(G), \chi'(H)\}$.
(b) *Suppose* $G' = (G; u_1, u_3; e_1, e_2) \otimes (H; v_1, v_3; f_1, f_2)$. *If* $\chi'(G) = \chi'(H) = 3$, *and there exist proper* 3*-edge colourings* $c_1, c_2 : E(G) \to \{1, 2, 3\}$ *such that* $c_1(e_1) = c_1(e_2)$ *and* $c_2(e_1) \neq c_2(e_2)$, *then* $\chi'(G') = 3$.
(c) *Suppose* $G' = (G; u, u_4; e_1, e_2, e_3, e) \otimes (H; v, v_4; f_1, f_2, f_3, f)$. *If* $\chi'(G) = \chi'(H) = 3$ *and there exist proper* 3*-edge colourings* $c_1, c_2, c_3 : E(G) \to \{1, 2, 3\}$ *such that* $c_i(e) = c_i(e_i),\ i = 1, 2, 3$, *then* $\chi'(G') = 3$.
(d) *Suppose* $G' = G \otimes_{\square} (e_1, e_2; u_1, u_2; u_3, u_4)$. *If* $\chi'(G) = 3$, *then* $\chi'(G') = 3$.

Example 2.4. Suppose $e = uu' \in E(P_{10})$ where $N(u) = \{u', u_1, u_2\}$ and $N(u') = \{u, u_3, u_4\}$. Let $P_8 = P_{10} \oplus e$, and let $f_1 = u_1u_2 \in E(P_8)$ and $f_2 = u_3u_4 \in E(P_8)$. We have that $\chi'(P_8) = 3$ and moreover, P_8 has 2 proper 3-edge colourings $c_1, c_2 : E(P_8) \to \{1, 2, 3\}$ where $c_1(f_1) = c_1(f_2)$ and $c_2(f_1) \neq c_2(f_2)$.

Suppose now that $v_1a_1v_2a_2v_3$ is a path of length 2 in P_{10}. Let $E_{v_1} = \{a_1, b_1, b_2\}$, $E_{v_2} = \{a_1, a_2, b_3\}$, and $E_{v_3} = \{a_2, b_4, b_5\}$ where $b_1 = u_1v_1, b_2 = u_2v_1, b_3 = u_3v_2, b_4 = u_4v_3, b_5 = u_5v_3$. The graph $G' = P_{10} \otimes \{b_1, b_2, b_3, b_4, b_5\}$ has 2 components G'_1 and G'_2, where G'_1 is P_8. The graph G'_1 is obtained from P_{10} by deleting the vertices v_1, v_2, v_3 and adding a vertex u together with the edges uu_1, uu_2, uu_3 and u_4u_5. There exist proper 3-edge colourings $c_1, c_2, c_3 : E(G) \to \{1, 2, 3\}$ such that $c_i(uu_i) = c_i(u_4u_5), i = 1, 2, 3$ (see Fig. 8).

A sequence of $\square$-expansions is said to be *disjoint* if each expansion preserves the 4-cycles created in the previous $\square$-expansions. Given that we perform any number of disjoint $\square$-expansions on P_{10} the resulting graph is either 3-edge colourable or is a 3-chromatic expansion of P_{10}. We have something slightly more general:

Theorem 2.5. *Let* $Q = P_{10} \odot (e_1, \ldots, e_k)$. *Then either* $\chi'(P_{10} \odot S) = 3$ *for some ordered subset* $S \subseteq \{e_1, \ldots, e_k\}$ *or Q is an expansion of* P_{10}.

The above theorem follows from results in Section 7. It implies the following result:

Theorem 2.6. *Let* P'_{10} *be a* 3*-chromatic expansion of* P_{10} *where* $v(P'_{10}) \leqslant 16$. *Let Q be a cubic graph obtained from* P'_{10} *via a sequence of disjoint* $\square$*-expansions. Then either* $\chi'(Q) = 3$ *or Q is a* 3*-chromatic expansion of* P_{10}.

Proof. To minimize the burden of details, we shall only prove the case where $P'_{10} = P_{10}$, the proof for the general scenario being the same in spirit. The graph Q is also obtained by inserting edges into P_{10}. Now by Theorem 2.5, we have that either we obtain a graph Q' with $\chi'(Q') = 3$ via a subsequence of edge insertions (in which case $\chi'(Q) = 3$), or Q is

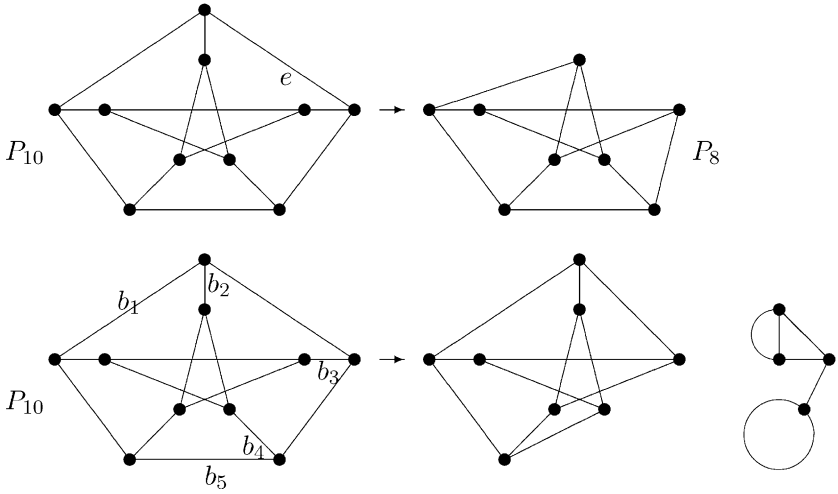

Fig. 8. Edge-reduction on P_{10} and 5-edge reduction on P_{10}.

an expansion $v \to A_v,\ v \in V(P_{10})$ of P_{10}. In the former case, we could obtain a 3-edge colourable graph via a subsequence of $\square$-expansions, which would imply $\chi'(Q) = 3$. In the latter case, each A_v would be obtained by performing disjoint $\square$-expansions on a multiple 3-edge, and thus $\chi'(A_v) = 3$. This shows that such an expansion would be 3-chromatic. This completes the proof. $\square$

Given that P_{10} is the only snark with 16 or fewer vertices, if G is a graph with 18 vertices which is not a snark, then either $\chi'(G) = 3$ or G is a 3-chromatic expansion of P_{10}.

Proposition 2.7. *Let G be a* 2*-connected cubic graph with* $v(G) \leqslant 16$. *Then either* $\chi'(G) = 3$ *or G is a* 3*-chromatic expansion of* P_{10}. *Moreover, if* $v(G) = 18$, *and G is not a snark, then the above conclusion is still valid.*

Let H_1 be a cubic graph and let $u \in V(H_1)$. Let e_1, e_2, e_3 be an ordering of the edges incident to u where $e_i = u_i u,\ i = 1, 2, 3$. Let H_2 be a cubic graph and let $v \in V(H_2)$. We suppose f_1, f_2, f_3 is an ordering of the edges incident to v where $f_i = v_i v,\ i = 1, 2, 3$. We suppose $\mathcal{C}^1$ and $\mathcal{C}^2$ are collections of cycles in H_1 and H_2, respectively, where each e_i(resp., f_i) is covered twice by cycles in $\mathcal{C}^1$(resp., $\mathcal{C}^2$). We define a splicing operation where the cycles of $\mathcal{C}^1$ and $\mathcal{C}^2$ are "spliced" together to form a collection of cycles $\mathcal{C}$ of $H = (H_1; u; e_1, e_2, e_3) \otimes (H_2; v; f_1, f_2, f_3)$. Let C_1^1, C_2^1, C_3^1 be the cycles of $\mathcal{C}^1$ which contain the pairs of edges $\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}$, respectively, and let C_2^2, C_2^2, C_3^2 be the cycles of $\mathcal{C}^2$ which contain the pairs of edges $\{f_1, f_2\}, \{f_1, f_3\}, \{f_2, f_3\}$, respectively. Let h_i be the edge $u_i v_i \in E(H),\ i = 1, 2, 3.$ and let

$$C_1 = (C_1^1 \backslash u) \cup (C_1^2 \backslash v) \cup \{h_1, h_2\},$$

$$C_2 = (C_2^1 \backslash u) \cup (C_2^2 \backslash v) \cup \{h_1, h_3\},$$

$$C_3 = (C_3^1 \backslash u) \cup (C_3^2 \backslash v) \cup \{h_2, h_3\}.$$

Let $\mathcal{C} = (\mathcal{C}^1 \backslash \{C_1^1, C_2^1, C_3^1\}) \cup (\mathcal{C}^2 \backslash \{C_1^2, C_2^2, C_3^2\}) \cup \{C_1, C_2, C_3\}$. We call $\mathcal{C}$ a collection of cycles obtained by *splicing* together $\mathcal{C}^1$ and $\mathcal{C}^2$.

3. 3-colourable subgraphs

A circuit which is a vertex-disjoint collection of cycles which partitions the vertices of the graph is called a 2-*factor*. It is well-known that every bridgeless cubic graph contains a perfect matching and hence also a 2-factor (see [2, p. 79]).

Suppose G is a 2-connected, cubic, 3-edge colourable graph, and let C be a circuit of G. Given G has a 3-edge colouring with colours 1, 2, 3, we let C_{ij} be the 2-factor induced by the edges having colours i or j where $i, j = 1, 2, 3$. Let $C'_{ij} = C_{ij} \bigtriangledown C,\ i, j = 1, 2, 3$, where '$\bigtriangledown$' denotes symmetric difference. Now $C'_{ij},\ i, j = 1, 2, 3$ are 3 circuits which cover all the edges of G twice, except for the edges of C which are covered once. To summarize:

Lemma 3.1. *Let G be a cubic 3-edge colourable graph and let C be a circuit of G. Then there are 3 circuits which cover the edges of C once, and the edges of $E(G)\backslash E(C)$ twice.*

We also have a specific variation of this lemma which we will need:

Lemma 3.2. *Let P'_{10} be a 3-chromatic expansion of P_{10} given by $v \to A_v$, $v \in V(P_{10})$. Let $\mathcal{C}'$ be a disjoint collection of cycles of P'_{10} where, with the exception of possibly one cycle, each cycle of $\mathcal{C}'$ is contained in some A_v. Then P'_{10} contains a collection of cycles $\mathcal{D}'$ which cover the edges of $\bigcup_{C' \in \mathcal{C}'} E(C')$ once and the other edges of P'_{10} twice.*

Proof. For each $v \in V(P_{10})$ let A'_v be the subgraph of P'_{10} induced by the edges in P_{10} corresponding to those in A_v. We shall assume that $\mathcal{C}'$ contains one cycle K' which is not contained in any $A_v,\ v \in V(P_{10})$. In the case where no such cycle exists, the proof is similar. We first observe that given any cycle C in P_{10}, there is a collection of cycles in P_{10} covering C once, and the other edges of P_{10} twice. Let K be the cycle of P_{10} corresponding to the cycle K'. Let $\mathcal{D}$ be a collection of cycles of P_{10} which cover K once and the other edges of P_{10} twice. For any cycle $C' \in \mathcal{C}'$, if C' intersects A'_v, then the intersection corresponds to a cycle in A_v. Moreover, the intersection of the cycles of $\mathcal{C}'$ with A'_v corresponds to a disjoint collection of cycles in A_v which we denote by $\mathcal{C}_v$. Since A_v is 3-edge colourable, Lemma 3.1 implies that there is a collection of cycles $\mathcal{D}_v$ in A_v covering the cycles of $\mathcal{C}_v$ once and the other edges of A_v twice. One can now splice together the collections $\mathcal{D}_v,\ v \in V(P_{10})$ with $\mathcal{D}$ to obtain the desired collection of cycles $\mathcal{D}'$ of P'_{10}. $\square$

Example 3.3. Let G be the cubic graph consisting of t independent vertices joined to a cycle C of length $3t$. If $v(G) \leqslant 16$ (that is, $t \leqslant 4$), then according to Proposition 2.7 we have that either $\chi'(G) = 3$ or G is a 3-chromatic expansion of P_{10}. It follows by Lemmas

3.1 and 3.2 that there is a collection of cycles in G covering C once and the other edges twice.

Suppose that G is a cubic graph with a pseudo 2-factor $(X, \mathcal{C})$ and suppose that there are two bridgeless subgraphs H_1 and H_2 where $G = H_1 \cup H_2$, $E(H_1) \cap E(H_2) = \bigcup_{C \in \mathcal{C}} E(C)$, and each H_i $i = 1, 2$ has a collection of cycles $\mathcal{D}_i$ which cover all the edges of H_i twice except the edges of $\mathcal{C}$ which are covered once. The collection $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a cycle double cover of G. Our strategy for the proof of the main theorem is, when possible, to find two such subgraphs H_1 and H_2. We note that if $\chi'_h(H_1) = \chi'_h(H_2) = 3$, then Lemma 3.1 implies that the desired cycle collections $\mathcal{D}_1$ and $\mathcal{D}_2$ exist.

Lemma 3.4. *Let $(X, \mathcal{C})$ be a pseudo 2-factor of a cubic 2-connected graph G. Suppose there is a degree-compatible subgraph $H_{\mathcal{C}}$ of $G_{\mathcal{C}}$ such that $h(H)$ is $\mathcal{C}$-compatible, and $\chi'_h(H) = 3$. Then G has a cycle double cover comprised of cycles from 5 circuits.*

Proof. Suppose $H_{\mathcal{C}}$ is a subgraph as specified in the Lemma. By Lemma 3.1 there is a collection of cycles $\mathcal{C}_H$ belonging to 3 circuits which cover the edges of $\mathcal{C}$ once, and the edges of $H \backslash \bigcup_{C \in \mathcal{C}} E(C)$ twice. Let $H' = (G \backslash E(H) \cup X) \ \cup \ \bigcup_{v \in V(H_{\mathcal{C}}) \backslash X} C(v)$. In H' there is a 2-factor $\mathcal{C}'$ corresponding to $\mathcal{C}$. Each cycle $C' \in \mathcal{C}'$ is such that $h(C')$ is an even cycle. Consequently, $\chi'_h(H') = 3$. Thus, we can find 2 perfect matchings P_1 and P_2 in $h(H')$ where $P_1 \cup P_2 = \bigcup_{C' \in \mathcal{C}'} E(h(C'))$. For $i = 1, 2$ $h(H') \backslash P_i$ is a disjoint union of cycles. Let $\mathcal{C}^i_{H'}$ be the corresponding collection of cycles in H'. Then $\mathcal{C}_{H'} = \mathcal{C}^1_{H'} \cup \mathcal{C}^2_{H'}$ is a collection of cycles belonging to 2 circuits which cover the edges of $\mathcal{C}'$ once and the other edges of H' twice. It follows that $\mathcal{C}_H \cup \mathcal{C}_{H'}$ is the desired cycle double cover of G. $\square$

Let G be a 2-connected cubic graph and let $(X, \mathcal{C})$ be a pseudo 2-factor. We suppose that, apart from loops, $G_{\mathcal{C}}$ is a 2-connected graph and has 4 odd vertices v_1, v_2, v_3, v_4. We wish to show that there exists a subgraph containing v_1, v_2, v_3, v_4 which is one of the subgraphs illustrated in Fig. 9. In $H^1_{\mathcal{C}}$ there is a cycle containing all 4 vertices. In $H^2_{\mathcal{C}}$ and $H^3_{\mathcal{C}}$ there is a cycle containing exactly 3 of the vertices $v_1, \ldots, v_4$ which are denoted v_{i_1}, v_{i_2}, and v_{i_3}. There are 2 internally disjoint paths from the fourth vertex v_{i_4} to the cycle. In $H^4_{\mathcal{C}}$, we have 2 disjoint cycles each containing exactly 2 of the vertices $v_1, \ldots, v_4$. In $H^5_{\mathcal{C}}$ there are exactly 2 cycles meeting at one vertex, each cycle containing 2 of the vertices $v_1, \ldots, v_4$. In $H^6_{\mathcal{C}}$, there are 2 cycles meeting at 2 vertices (labelled v_{13} and v_{24}) where each cycle contains 2 vertices of $v_1, \ldots, v_4$.

Lemma 3.5. *Let G be a 2-connected loopless multigraph and let v_1, v_2, v_3, v_4 be 4 vertices of G. The graph G has a subgraph H containing v_1, v_2, v_3, v_4 where H is one of the graphs specified in Fig. 9. In* (f)*, the vertices v_{13} and v_{24} form a 2-separating set in G which separates each pair of vertices v_i and v_j, $i, j = 1, 2, 3, 4$.*

Proof. If there is a cycle in G containing v_1, v_2, v_3, v_4, then we have the subgraph $H^1_{\mathcal{C}}$ in Fig. 9. We may therefore assume that no such cycle exists. Suppose instead that there is a cycle C containing exactly 3 of the vertices, say $v_{i_1}, v_{i_2}, v_{i_3}$, where the remaining vertex

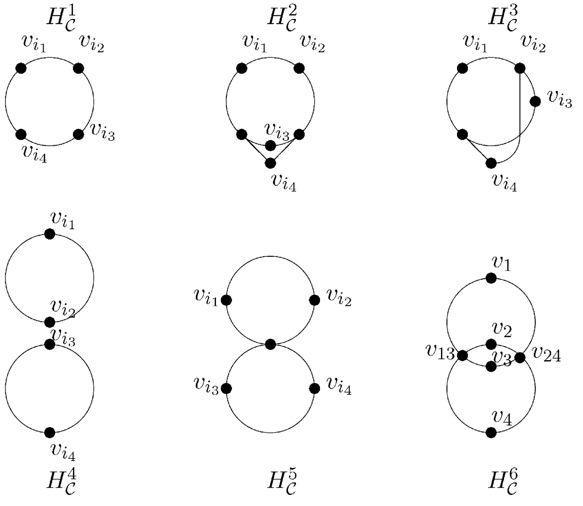

Fig. 9. Bridgeless subgraphs containing v_1, v_2, v_3, v_4.

v_{i_4} lies outside of C. By Mengers theorem [2, p. 46], there are 2 internally vertex-disjoint paths from v_{i_4} to C which meet only at v_{i_4}. In this case, we have the two possibilities $H^2_{\mathcal{C}}$ and $H^3_{\mathcal{C}}$ illustrated in Fig. 9. We suppose now that G has no cycle containing 3 or 4 of the vertices $v_i,\ i = 1, 2, 3, 4$. Since G is 2-connected, there is a cycle C containing v_1 and v_2 (but not v_3 or v_4). The cycle C is the union of 2 paths, say P_1 and P_2 from v_1 to v_2. Since G is 2-connected, there are 2 internally disjoint paths P_3 and P_4 from v_3 to C which meet only at v_3. Since it is assumed that G has no cycle containing 3 or more of the vertices $v_i,\ i = 1, 2, 3, 4$, we may assume that P_3 meets C along P_1 at a vertex $v_{13} \neq v_1, v_2$. Similarly, P_4 meets C along P_2 at a vertex $v_{24} \neq v_1, v_2$. Let $H = C \cup P_3 \cup P_4$. We have that $v_4 \notin V(H)$, for otherwise there would be a cycle containing v_1, v_3, and v_4 (given that $v_3 \notin V(C)$). Again, by the 2-connectedness of G, there are 2 internally disjoint paths P_5 and P_6 from v_4 to H which meet only at v_4. Depending on where P_5 and P_6 intersect H, the graph G must contain one of the subgraphs $H^4_{\mathcal{C}},\ H^5_{\mathcal{C}}$ or $H^6_{\mathcal{C}}$ as illustrated in Fig. 9. In the case that G contains neither $H^4_{\mathcal{C}}$ nor $H^5_{\mathcal{C}}$, it must be the case that the vertices v_{13} and v_{24} form a 2-separating set for each pair of vertices $v_i, v_j,\ i, j = 1, 2, 3, 4$. $\square$

Lemma 3.6. *Let G be a multigraph. There exists a forest $F \subseteq G$ such that $d_F(v) = d_G(v) \pmod 2,\ \forall v \in V(G)$.*

Proof. By induction on the number of edges. If $\varepsilon(G) = 0$, then the lemma holds trivially. Suppose the lemma holds for all multigraphs having fewer than m edges ($m > 0$), and suppose $\varepsilon(G) = m$. If G contains no cycles, then it is itself a forest and we can choose $F = G$. We suppose therefore that G contains a cycle C. Let $G' = G \backslash E(C)$. By assumption, there is a forest $F \subseteq G'$ such that $d_F(v) = d_{G'}(v) \pmod 2,\ \forall v \in V(G')$. This means that

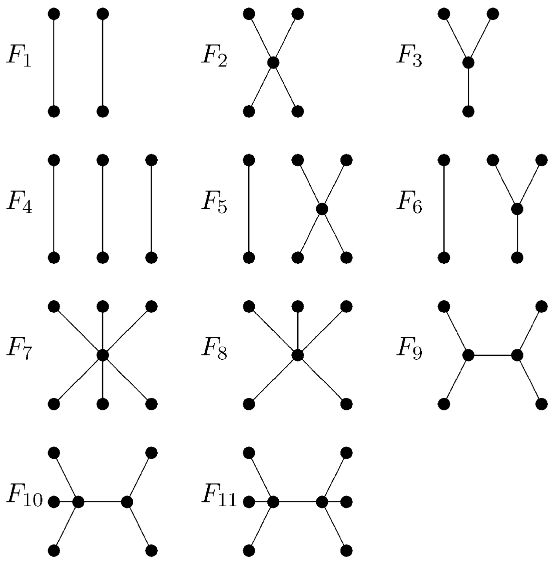

Fig. 10. Non-isomorphic, non-homeomorphic forests with 4 or 6 odd vertices.

$d_G(v) = d_F(v) \pmod 2) \forall v \in V(G)$. Thus the lemma holds for all graphs with m edges, and the proof follows by induction. $\square$

There are exactly 11 non-isomorphic, non-homeomorphic forests having 4 or 6 odd vertices. These are illustrated in Fig. 10.

Let G be a 2-connected cubic graph and let $(X, \mathcal{C})$ be a pseudo 2-factor of G. We suppose that $G_{\mathcal{C}}$ is 2-connected and has 4 odd vertices v_1, v_2, v_3, v_4. There is a bridgeless subgraph $H'_{\mathcal{C}} \subseteq G_{\mathcal{C}}$ as in Lemma 3.5. The graph $G'_{\mathcal{C}} = G_{\mathcal{C}} \backslash E(H'_{\mathcal{C}})$ has 4 or 6 odd vertices (depending on $H'_{\mathcal{C}}$) and hence by Lemma 3.6 there is a forest $F'_{\mathcal{C}} \subseteq G'_{\mathcal{C}}$ homeomorphic to one of the forests given in Fig. 10 where $d_{F'_{\mathcal{C}}}(v) = d_{G'_{\mathcal{C}}}(v) \pmod 2\ \forall v \in V(G_{\mathcal{C}})$. Let $H_{\mathcal{C}} = H'_{\mathcal{C}} \cup F'_{\mathcal{C}}$. Then $H_{\mathcal{C}}$ is a degree-compatible subgraph of $G_{\mathcal{C}}$.

For a multigraph M, we have a list of 9 conditions:

(3.1.1) $v_3(M) = 4$ and $v_{\geqslant 5}(M) = 0$.
(3.1.2) $v_3(M) = 4$, $v_5(M) = 0$, $v_6(M) = 1$, and $v_{\geqslant 7}(M) = 0$.
(3.1.3) $v_3(M) = 3$, $v_5(M) = 1$, $v_{\geqslant 6}(M) = 0$.
(3.1.4) $v_3(M) = 3$, $v_5(M) = 1$, $v_6(M) = 1$, $v_{\geqslant 7}(M) = 0$.
(3.1.5) $v_3(M) = 3$, $v_5(M) = 0$, $v_6(M) = 0$, $v_7(M) = 1$, $v_{\geqslant 8}(M) = 0$.
(3.1.6) $v_3(M) = 2$, $v_5(M) = 2$, $v_{\geqslant 6}(M) = 0$.
(3.1.7) $v_3(M) = 4$, $v_5(M) = 0$, $v_6(M) = 2$, $v_{\geqslant 7}(M) = 0$.
(3.1.8) $v_3(M) = 4$, $v_5(M) = 0$, $v_6(M) = 0$, $v_7(M) = 0$, $v_8(M) = 1$, and $v_{\geqslant 9}(M) = 0$.
(3.1.9) $v_3(M) = 4$, $v_5(M) = 0$, $v_6(M) = 1$, $v_7(M) = 0$, $v_8(M) = 1$, and $v_{\geqslant 9}(M) = 0$.

By considering all the possible subgraphs, the subgraph $H_{\mathcal{C}}$ satisfies at least one of the 9 conditions listed above. The table below indicates for each combination of a forest from

Fig. 10 and a subgraph from Fig. 9 the subset of conditions which apply to $H_{\mathcal{C}}$. In each case, at least one of these conditions must hold.

	$H_{\mathcal{C}}^1$	$H_{\mathcal{C}}^2$	$H_{\mathcal{C}}^3$	$H_{\mathcal{C}}^4$	$H_{\mathcal{C}}^5$	$H_{\mathcal{C}}^6$
F_1	(3.1.1)		(3.1.1) (3.1.3)	(3.1.1)	(3.1.1) (3.1.2)	(3.1.1) (3.1.2) (3.1.7)
F_2	(3.1.1) (3.1.2)		(3.1.1) (3.1.2) (3.1.5)	(3.1.1) (3.1.2)	(3.1.1) (3.1.2) (3.1.8)	(3.1.1) (3.1.2) (3.1.9)
F_3	(3.1.1)		(3.1.2) (3.1.3) (3.1.4) (3.1.6)	(3.1.3)	(3.1.3) (3.1.4)	Not possible
F_4		(3.1.1)				
F_5		(3.1.1) (3.1.2)				
F_6		(3.1.2) (3.1.3)				
F_7		(3.1.2) (3.1.8)				
F_8		(3.1.5) (3.1.8)				
F_9		(3.1.4) (3.1.6) (3.1.7)				
F_{10}		(3.1.2) (3.1.3) (3.1.4) (3.1.7)				
F_{11}		(3.1.2) (3.1.7)				

4. Cycle covers

In this section, we prove some results on cycle coverings. Let G be a cubic graph and let $p : E(G) \to \{0, 1, 2\}$ be a non-negative edge weighting of G. Let $\mathcal{C}$ be a collection of cycles in G. For each edge $e \in E(G)$ we let $m_{\mathcal{C}}(e)$ be the number of cycles in $\mathcal{C}$ containing e. We say that $\mathcal{C}$ is a *cycle p-cover* for (G, p) if $m_{\mathcal{C}}(e) = p(e), \ \forall e \in E(G)$.

A weighting $p : E(G) \to \mathbf{Z}^+$ is *eulerian* if $\forall v \in V(G), \ \sum_{e \in E_v} p(e) = 0 \pmod 2$. For a weighted graph (G, p) with eulerian weighting p we define a *subdivision operation* where we subdivide an edge e_0 with a vertex u and give the subdivided edges weight $p(e_0)$. Suppose we are given a weighted graph (G, p) and we perform a subdivision operation twice in succession, where we subdivide with vertices u and v. We then add an edge e of weight 2 between u and v. The resulting graph is $G \odot (e)$, and we denote the resulting (eulerian) weighting by $p_{\odot(e)}$.

Suppose we perform a subdivision operation 3 times, where we subdivide with vertices u_1, u_2, and u_3. We add a vertex v and join it to u_1, u_2, and u_3 with edges of weight 2. The resulting graph is $G \odot (v)$ and we denote the corresponding weighting by $p_{\odot(v)}$.We say that $(G \odot (e), p_{\odot(e)})$(resp., $(G \odot (v), p_{\odot(v)})$) *preserves* cycle coverings if, given (G, p) has a cycle p-cover, then $(G \odot (e), p_{\odot(e)})$ has a cycle $p_{\odot(e)}$-cover (resp., $(G \odot (v), p_{\odot(v)})$ has a cycle $p_{\odot(v)}$-cover. Similarly, we say that an insertion operation preserves 3-edge colourings if, given G is 3-edge colourable, the graph resulting from G after the insertion operation is also 3-edge colourable.

We define the *distance* between two edges e_0 and e_1 in a connected graph G to be the number of edges in the shortest path containing e_o and e_1 minus 1. This distance we denote by $dist_G(e_0, e_1)$.

Theorem 4.1. *Let (G, p) be a weighted cubic graph where $p : E(G) \to \{1, 2\}$ and p is eulerian.*

(i) *Let $G' = G \odot (e)$ and $p' = p_{\odot(e)}$ where e has endvertices in edges e_0 and e_1 in G. If $dist_G(e_0, e_1) \leqslant 2$, then (G', p') preserves cycle coverings, and G' preserves* 3*-edge colourings. Consequently, if e_0 and e_1 belong to a* 5*-cycle, then (G', p') preserves cycle coverings, and G' preserves* 3*-edge colourings.*
(ii) *Let $G' = G \odot (v)$ and $p' = p_{\odot(e)}$ where v has neighbours inserted in the edges e_0, e_1, and e_2 in G. If e_0, e_1, and e_2 belong to a cycle of length at most* 5 *in G, then (G', p') preserves cycle coverings, and G' preserves* 3*-edge colourings.*

Proof. (i) Let (G', p') and e_0, e_1 be as in (i). Suppose $dist_G(e_0, e_1) = 0$; that is, $e_0 = e_1$. Let $C \in \mathcal{C}$ be a cycle containing e_0. Then e is a chord of C in G' and we can replace C by 2 cycles $C_1, C_2 \subset C \cup \{e\}$ where C_1 and C_2 cover e twice and C once. It then follows that $\mathcal{C}' = (\mathcal{C}\backslash\{C\}) \cup \{C_1, C_2\}$ is a cycle p'-cover of (G', p'). We also see that G can be obtained from G' via an $\circ$-reduction. Thus $\chi'(G) = \chi'(G')$ and G' preserves 3-edge colourings.

Suppose that $dist_G(e_0, e_1) = 1$; that is, e_0 and e_1 are incident with a common vertex. Let $C_0, C_1 \in \mathcal{C}$ be cycles where C_0 contains e_0 and C_1 contains e_1. If $e_1 \in E(C_0)$, then e is a chord of C_0 and we may adopt the previous argument. So we may assume $e_1 \notin E(C_0)$ and likewise, $e_1 \notin E(C_1)$. Let $H = h(C_0 \cup C_1)$. We have that $\chi'(H) = 3$, as $C_0 \bigtriangledown C_1$ corresponds to a 2-factor with even cycles in H. Moreover, we see that e is a chord of some cycle in $C_0 \bigtriangledown C_1$, and consequently $H' = h(C_0 \cup C_1 \cup \{e\})$ is also 3-edge colourable. By Lemma 3.1 there is a collection of cycles $\mathcal{C}_{H'}$ in H' which covers $C_0 \bigtriangledown C_1$ once, and the other edges of H' twice. Let $\mathcal{D}$ be the collection of cycles of $C_0 \cup C_1 \cup \{e\}$ corresponding to $\mathcal{C}_{H'}$. Then $\mathcal{C}' = (\mathcal{C}\backslash\{C_0, C_1\}) \cup \mathcal{D}$ is seen to be a cycle p'-cover for (G', p'). We note that G can be obtained from G' via a $\triangle$- reduction and consequently $\chi'(G) = \chi'(G')$. Thus G' preserves 3-edge colourings.

We suppose now that $dist_G(e_0, e_1) = 2$. Let $C_0, C_1 \in \mathcal{C}$ where $e_0 \in E(C_0)$ and $e_1 \in E(C_1)$. We may assume that $e_0 \notin E(C_1)$, $e_1 \notin E(C_0)$, and there is an edge $e_{01} \in E(G)$ lying on a path of length 3 between e_0 and e_1. We will consider 2 cases:

Case 1: $E(C_0) \cap E(C_1) = \emptyset$.

Let $C_{01} \in \mathcal{C}$ be a cycle containing e_{01}. We may assume $e_0, e_1 \notin E(C_{01})$, for otherwise we can jump ahead to the second case. Let $H = h(C_0 \cup C_1 \cup C_{01})$. We have that $\chi'(H) = 3$

and moreover, e is a chord of a cycle in the 2-factor of H $C_0 \bigtriangledown C_1 \bigtriangledown C_{01}$. We can now apply the previous argument to obtain a cycle p'-cover for G'.

Case 2: $E(C_0) \cap E(C_1) \neq \emptyset$.

Consider $H = h(C_0 \cup C_1)$. Suppose $e = xy$. We shall assume that $\mathcal{C}$ is a cycle cover having a maximum number of cycles. Let C_0^h and C_1^h be the cycles of H corresponding to C_0 and C_1, respectively. We have that $\chi'(H) = 3$ and $C_0^h \bigtriangledown C_1^h$ is a 2-factor (with even cycles). If e is a chord of some cycle in $C_0 \bigtriangledown C_1$, then we proceed as before. So we may assume that e lies between 2 cycles of $C_0^h \bigtriangledown C_1^h$. Colour the edges of the cycles of $C_0^h \bigtriangledown C_1^h$ alternatively with colours green and blue in such a way that the edges containing e_0 and e_1 are given the same colour, say green. Colour the remaining edges of H red. Let D_{gr}^h and D_{br}^h be the circuits induced by the green–red and blue–red edges, respectively. Let $\mathcal{D}_{gr}^h$ and $\mathcal{D}_{br}^h$ be the set of cycles in D_{gr}^h and D_{br}^h, respectively. We let $\mathcal{D}_{gr}$ and $\mathcal{D}_{br}$ be the sets of cycles in G corresponding to D_{gr}^h and D_{br}^h, respectively. Similarly, we let $\mathcal{D}'_{gr}$ and D'_{br} be the sets of cycles in G' corresponding to $\mathcal{D}_{gr}$ and $\mathcal{D}_{br}$, and we let D'_{gr} and D'_{br} be the circuits of G' corresponding to D_{gr} and D_{br}. If $|\mathcal{D}_{gr}^h| > 1$, or $|\mathcal{D}_{br}^h| > 1$, then $(\mathcal{C} \backslash \{C_0, C_1\}) \cup \mathcal{D}_{gr} \cup \mathcal{D}_{br}$ would be a cycle p-cover of (G, p) with more cycles than $\mathcal{C}$, contradicting the maximality of $\mathcal{C}$. Thus both D_{gr} and D_{br} are cycles. This means that e is a chord of D'_{gr} in G', and we can split $D'_{gr} \cup \{e\}$ into 2 cycles D_{gr}^1 and D_{gr}^2 where $\{D_{gr}^1, D_{gr}^2\}$ cover D'_{gr} once and e twice. Let $\mathcal{C}' = (\mathcal{C} \backslash \{C_0, C_1\}) \cup \{D_{gr}^1, D_{gr}^2, D'_{br}\}$. Then $\mathcal{C}'$ is a cycle p'-cover for (G', p').

To show that G' preserves 3-edge colourings, we first note that a cubic graph is 3-edge colourable iff it has three 2-factors which form a double cycle cover. Suppose $\chi'(G) = 3$, and let $\mathcal{C}$ be a double cover consisting of cycles from three 2-factors. We may assume C_0 and C_1 are disjoint (as in case 1) or C_0 and C_1 are the same cycle. Let D be the 4-cycle in G' containing e and e_{01}. Let $\mathcal{C}' = (\mathcal{C} \backslash \{C_0, C_1, C_{01}\}) \cup \{(C_0 \cup C_1) \bigtriangledown D,\ C_{01} \bigtriangledown D\}$. Now $\mathcal{C}'$ is seen to be a cycle p'-cover of G', and $\mathcal{C}'$ is a union of three 2-factors. Thus $\chi'(G') = 3$. This shows that G' preserves 3-edge colourings. This proves (i).

To prove (ii) we note that one can obtain (G', p') by performing an edge insertion operation twice, each time inserting an edge which is a chord of a 5-cycle. The proof then follows by (i). $\square$

Lemma 4.2. *Let (G, p) be a weighted cubic graph where $p : E(G) \rightarrow \{1, 2\}$ is an eulerian weighting. Let C be a chordless cycle of G where $p(e) = 1,\ e \in E(C)$. Suppose that G is the union of subgraphs $H_i,\ i = 1, \ldots, t\ \ t \leqslant 4$ which intersect along C. For $i = 1, \ldots, t$ let p_i be the weighting p restricted to H_i.*

(a) *If $d_{H_i}(C) = 3\ \forall i$ and each (H_i, p_i) has a cycle p_i-cover, then (G, p) has a cycle p-cover.*

(b) *Suppose $t = 2$ and $d_{H_1}(C) \leqslant 5$, and $d_{H_2}(C) \leqslant 3$. If for $i = 1, 2$ each (H_i, p_i) has a cycle p_i-cover, then (G, p) has a cycle p-cover. Moreover, if $\chi'_h(H_1) = \chi'_h(H_2) = 3$, then $\chi'(G) = 3$.*

Proof. We shall first prove (a). For $i = 1, \ldots, t$ let H'_i be the graph obtained from H_i by contracting C into a single vertex, and we let p'_i be a weighting of H'_i where p'_i is the same as p_i restricted to $H_i \backslash E(C)$. Given each (H_i, p_i) has a cycle p_i-cover, we have that each

(H_i', p_i') has a cycle p_i'-cover, say $\mathcal{D}_i'$. We form a cubic graph H' from G by contracting each graph $H_i \backslash C$ into a single vertex, so that H' consists of C together with t independent vertices joined to C. Let q' be a weighting of H' where $q'(e) = 1,\ \forall e \in E(C)$ and $q'(e) = 2,\ \forall e \notin E(C)$. According to Example 2, (H', q') has a cycle q'-cover, say $\mathcal{D}'$. We can now splice together the cycle collections $\mathcal{D}_i',\ i = 1, \ldots, t$ with $\mathcal{D}'$ to obtain a cycle collection $\mathcal{D}$ which is a cycle p-cover for G.

To prove (b) let H_1' be the cubic graph obtained from G by contracting $H_2 \backslash C$ into single vertex u_1, and if necessary, performing a $\vee$-reduction on u_1 if it has degree 2. We define a weighting p_1' on H_1' where $p_1'(e) = p_1(e)\ \forall e \in E(H_1)$ and $p_1(e) = 2$ for all other edges. Let H_2' be the graph obtained from H_2 by contracting C into a single vertex u_2, performing a $\vee$-reduction on u_2 if it has degree 2. We define a weighting p_2' on H_2' where p_2' is the same p_2 restricted to $H_2 \backslash E(C)$. One obtains (H_1', p_1') from H_1 by either inserting a vertex of degree 3 in C, or inserting a chord in C. Assuming (H_1, p_1) has a cycle p_1-cover, we observe that $d_{H_1}(C) \leqslant 5$, and thus Theorem 4.1 (ii) implies that (H_1', p_1') has a cycle p_1'-cover, say $\mathcal{D}_1'$. Assuming (H_2, p_2) has a cycle p_2-cover, we have that (H_2', p_2') has a cycle p_2'-cover, say $\mathcal{D}_2'$. We now splice together $\mathcal{D}_1'$ and D_2' to obtain a cycle p-cover for (G, p).

If we assume that H_1 and H_2 are 3-edge colourable, then H_1' is 3-edge colourable (by Theorem 4.1 (ii)) and H_2' is 3-edge colourable. Since G is obtained from H_1 and H_2, either via a (3-chromatic) vertex expansion $u \to H_2'$ or via a 2-edge expansion, the graph G is 3-edge colourable. $\square$

5. K-joins

For a positive integer $k > 0$, we define a *k-join* of 2 graphs G and H where we join G and H by taking k vertices $g_1, g_2, \ldots, g_k$ in G and k vertices $h_1, h_2, \ldots, h_k$ in H and identify each pair of vertices $g_i, h_i,\ i = 1, 2, \ldots, k$ with single vertices. We denote the resulting graph by $(G; g_1, \ldots, g_k) \vee_k (H; h_1, \ldots, h_k)$. We define the 0-join of G and H to be the disjoint union of G and H, and denote this graph by $G \vee_0 H$. A k-join is said to be *odd* (resp., *even*) if $d_G(g_i)$ and $d_H(h_i)$ are odd (resp., even) for all i. Here, we use the symbol $\vee_k^o$ (resp., $\vee_k^e$) in place of $\vee_k$ to denote an odd (resp., even) k-join.

If $d_G(g_i)$ is even (resp., odd) for all i and $d_H(h_i)$ is odd (resp., even) for all i, then the k-join is said to be *even–odd* (resp., *odd–even*). We use the symbol $\vee_k^{eo}$ (resp., $\vee_k^{oe}$) in place of $\vee_k$ to denote an even–odd (resp., odd–even) k-join.

For two families of graphs $\mathcal{G}$ and $\mathcal{H}$ where each graph has at least k vertices, we define $\mathcal{G} \vee_k \mathcal{H}$ to be the set of k-joins of graphs in $\mathcal{G}$ with graphs in $\mathcal{H}$. We define $\mathcal{G} \vee_k^o \mathcal{H}$ (resp., $\mathcal{G} \vee_k^e \mathcal{H}$) to be the set of odd (resp., even) k-joins of graphs from $\mathcal{G}$ and $\mathcal{H}$. In a similar fashion, we define $\mathcal{G} \vee_k^{oe} \mathcal{H}$ and $\mathcal{G} \vee_k^{eo} \mathcal{H}$.

We define $(\mathcal{G})_k^1 = \mathcal{G}$, and for $i = 2, 3, \ldots$ we define $(\mathcal{G})_k^i = (\mathcal{G})_k^{i-1} \vee_k \mathcal{G}$. We let $(\mathcal{G})_k = \bigcup_{i \geqslant 1} (\mathcal{G})_k^i$, and define $(\mathcal{G})_k^{i,e}$ (resp., $(\mathcal{G})_k^{i,o}$) in a similar fashion, replacing the symbol $\vee_k$ with the symbol $\vee_k^e$ (resp., $\vee_k^o$) in the previous definition.

For collections of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_n$ we define a sequence of k-joins $\mathcal{G}_1 \vee_k \cdots \vee_k \mathcal{G}_n$ recursively by

$$\mathcal{G}_1 \vee_k \cdots \vee_k \mathcal{G}_n = (\mathcal{G}_1 \vee_k \cdots \vee_k \mathcal{G}_{n-1}) \vee_k \mathcal{G}_n.$$

We define $\mathcal{G}_1 \vee_k^o \cdots \vee_k^o \mathcal{G}_n$ and $\mathcal{G}_1 \vee_k^e \cdots \vee_k^e \mathcal{G}_n$ similarly.

Let $\mathcal{F}_2$ be the family of graphs consisting of graphs which are the edge-disjoint union of a cycle and a path, the path going between 2 vertices on the cycle. Each such graph has exactly 2 odd vertices (having degree 3). Let $\mathcal{F}_4$ be the family of bridgeless graphs with exactly 4 odd vertices v_1, v_2, v_3, v_4, being the union of a graph containing $v_1, \ldots, v_4$ as in Fig. 9, and a tree homeomorphic to one in Fig. 10. Let $\mathcal{F}_2^* = \bigcup_{i \geqslant 1} (\mathcal{F}_2)_1^{i,o}$. Each graph $F \in \mathcal{F}_2^*$ is a block chain whose blocks belong to $\mathcal{F}_2$. Moreover, each $F \in \mathcal{F}_2^*$ has exactly 2 odd vertices (having degree 3), one in each of its endblocks. Let

$$\begin{aligned}\mathcal{F}_4^* = {}& \mathcal{F}_4 \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^*) \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*) \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*)\\ & \cup (\mathcal{F}_4 \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^* \vee_1^o \mathcal{F}_2^*).\end{aligned}$$

Each member of $\mathcal{F}_4^*$ consists of a graph $G \in \mathcal{F}_4$ with block chains from $\mathcal{F}_2^*$ joined via an odd 1-join to some or none of the odd vertices of G.

Lemma 5.1. *Let G be a 2-edge connected graph having exactly 2 odd vertices v_1 and v_2. Then G contains a subgraph $H \in \mathcal{F}_2^*$ whose odd vertices are exactly v_1 and v_2.*

Proof. Suppose v_1 and v_2 belong to the same block B of G. Then there is a cycle C in B containing v_1 and v_2. Let $G' = G \backslash E(C)$. Then v_1 and v_2 are exactly the odd vertices of G'. They must belong to the same component in G', and consequently, there must be a path P in G' between them. Let $H = C \cup P$. Then $H \in \mathcal{F}_2$ (hence $H \in \mathcal{F}_2^*$) and moreover, v_1 and v_2 are exactly the odd vertices of H.

Suppose now that v_1 and v_2 belong to different blocks of G. Then there is a block chain $B_0 \cdots B_k$ where $v_1 \in V(B_0)$, $v_2 \in V(B_k)$, and $v_1, v_2 \notin V(B_i)$ for $0 < i < k$. Let $V(B_i) \cap V(B_{i+1}) = \{u_{i+1}\}$, $i = 0, \ldots, k-1$, and let $u_0 = v_1$, and $u_{k+1} = v_2$. Since $d_{B_0}(u_0)$ is odd, it follows that $d_{B_0}(u_1)$ is odd and thus $d_{B_{i-1}}(u_i)$, and $d_{B_i}(u_i)$ are odd for $i = 1, \ldots, k$. Since each B_i, $i = 0, \ldots, k$ is 2-connected (and is not a single edge), there are subgraphs $H_i \subseteq B_i$, $i = 0, \ldots, k$ where $H_i \in \mathcal{F}_2$ and u_i and u_{i+1} are exactly the odd vertices of H_i. Let $H = H_0 \cup \cdots \cup H_k$. Then $H \in \mathcal{F}_2^*$, and v_1 and v_2 are exactly the odd vertices of H.

Let G be a 2-connected cubic graph and let $(X, \mathcal{C})$ be a pseudo 2-factor of G.

Proposition 5.2. *Let $H_{\mathcal{C}} = (H_1)_{\mathcal{C}} \cup (H_2)_{\mathcal{C}}$ be a loopless subgraph of $G_{\mathcal{C}}$ where $(H_1)_{\mathcal{C}}$ intersects $(H_2)_{\mathcal{C}}$ at exactly one vertex v.*

(i) *If $h(H_1)$ and $h(H_2)$ are $\mathcal{C}$-compatible, $d_{(H_1)_{\mathcal{C}}}(v) \leqslant 5$, and $d_{(H_2)_{\mathcal{C}}}(v) \leqslant 3$, then $h(H)$ is $\mathcal{C}$-compatible. Moreover, if $\chi'_h(H_1) = \chi'_h(H_2) = 3$, then $\chi'_h(H) = 3$.*
(ii) *If $H_{\mathcal{C}} \in \mathcal{F}_2^*$, then $\chi'_h(H) = 3$.*

Proof. To prove (i) we first note that H_1 intersects H_2 along the cycle $C = C(v)$ which has no chords in $H_1 \cup H_2$ (since $H_{\mathcal{C}}$ is assumed to be loopless). We suppose that $h(H_1)$ and $h(H_2)$ are $\mathcal{C}$-compatible, $d_{(H_1)_{\mathcal{C}}}(v) \leqslant 5$, and $d_{(H_2)_{\mathcal{C}}}(v) \leqslant 3$. We have that $(h(H_i), p_{h(H_i)})$ has a cycle $p_{h(H_i)}$-cover for $i = 1, 2$, and $d_{H_1}(C) \leqslant 5$, and $d_{H_2}(C) \leqslant 3$. Now Lemma 4.2 b) implies that $(h(H), p_{h(H)})$ has a cycle $p_{h(H)}$-cover, and consequently $h(H)$ is $\mathcal{C}$-compatible. Moreover, if $\chi'_h(H_1) = \chi'_h(H_2) = 3$, then $\chi'_h(H) = 3$.

To prove (ii) suppose that $H_{\mathcal{C}} \in \mathcal{F}_2^*$. If $H_{\mathcal{C}} \in \mathcal{F}_2$, then we can reduce $h(H)$ to a multiple 3-edge via ∘-, △-, and ▫-reductions. In this case, $\chi'_h(H) = 3$. We suppose therefore that $H_{\mathcal{C}} \in \mathcal{F}_2^* \backslash \mathcal{F}_2$. Then $H_{\mathcal{C}} = (H_1)_{\mathcal{C}} \cup (H_2)_{\mathcal{C}}$ where $(H_1)_{\mathcal{C}}, (H_2)_{\mathcal{C}} \in \mathcal{F}_2^*$ and $(H_1)_{\mathcal{C}}$ intersects $(H_2)_{\mathcal{C}}$ at exactly one vertex, say v, where $d_{(H_i)_{\mathcal{C}}}(v) = 3,\ i = 1, 2$. We may assume that $\chi'(h(H_1)) = \chi'(h(H_2)) = 3$. It now follows from (i) that $\chi'_h(H) = 3$.

6. Proof of the main theorem

In this section, we give a proof of Theorem 1.2. Let G be a 2-connected cubic graph with $o(G) \leqslant 4$ and let $(X, \mathcal{C})$ be a pseudo 2-factor of G where $|X|$ plus the number of odd cycles in $\mathcal{C}$ is at most 4. Let $G_{\mathcal{C}}$ be the graph obtained from G by contracting the cycles of $\mathcal{C}$.

Theorem 6.1. *Either the graph $G_{\mathcal{C}}$ contains a degree-compatible subgraph $H_{\mathcal{C}}$ such that $h(H)$ is $\mathcal{C}$-compatible, or it contains a non-trivial 3-edge cut.*

Proof. If $o(G) = 0$, then $\chi'(G) = 3$ and result holds taking $H_{\mathcal{C}} = G_{\mathcal{C}}$. If $o(G) = 2$, then by Lemma 5.1 there is a degree compatible subgraph $H_{\mathcal{C}}$ of $G_{\mathcal{C}}$ belonging to $\mathcal{F}_2^*$. By Proposition 5.2 (ii), $\chi'_h(H) = 3$, and consequently $h(H)$ is $\mathcal{C}$-compatible. Thus we may assume that $o(G) = 4$, and v_1, v_2, v_3, v_4 are the odd vertices of $G_{\mathcal{C}}$.

Case 1: Suppose $G_{\mathcal{C}}$ has a block $B_{\mathcal{C}}$ containing all 4 of the odd vertices v_1, v_2, v_3, v_4.

The vertices $v_1, \ldots, v_4$ are easily seen to be the odd vertices of $B_{\mathcal{C}}$. According to Lemmas 3.5 and 3.6, there is a (loopless) subgraph $H_{\mathcal{C}} \subseteq B_{\mathcal{C}}$ where $H_{\mathcal{C}} \in \mathcal{F}_4$ and $v_1, \ldots, v_4$ are exactly the odd vertices of $H_{\mathcal{C}}$. For $v \in V(H_{\mathcal{C}}) \backslash X$ let $C'(v)$ be the cycle in $h(H)$ corresponding to $C(v)$ (ie. $C_{h(H)}(v)$).

For each 2- or 3-cycle $C'(v),\ v \in V(H_{\mathcal{C}}) \backslash X$, we perform ∘- and △-reductions, respectively. Next we perform ▫-reductions on all 4-cycles $C'(v)$, and this we do in such a way that the resulting (cubic) graph $h(H)'$ is bridgeless (this is possible by Lemma 1.7). Here, is an overview of the notation to be used in the ensuing proof.

$G_{\mathcal{C}}$: graph obtained from G by contracting cycles of $\mathcal{C}$.
$H_{\mathcal{C}}$: degree compatible subgraph of $B_{\mathcal{C}}$ belonging to $\mathcal{F}_4$.
H: subgraph in G corresponding to $H_{\mathcal{C}}$.
$h(H)$: cubic graph homeomorphic to H.
$h(H)'$: bridgeless graph obtained from $h(H)$ via ∘-, △-, and ▫-reductions.
$C(v)$: cycle in $\mathcal{C}$ corresponding to $v \in V(G_{\mathcal{C}}) \backslash X$.
$C'(v)$: cycle in $h(H)$ corresponding to $C(v)$.
$G'_{\mathcal{C}}$: the graph $G_{\mathcal{C}} \backslash E(H_{\mathcal{C}})$.
G': subgraph in G corresponding to $G'_{\mathcal{C}}$.

We know that exactly one of the conditions (3.1.1)–(3.1.9) holds for $H_{\mathcal{C}}$. We shall examine two subcases:

Case 1.1: Suppose $H_{\mathcal{C}}$ satisfies one of (3.1.1)–(3.1.8).

We have that $v(h(H)') \leqslant 16$. If $\chi'(h(H)') = 3$, then $\chi'(h(H)) = 3$ (by Lemma 1.7 and Corollary 2.2). It then follows from Lemma 3.1 that $h(H)$ is $\mathcal{C}$-compatible. Thus we may assume that $\chi'(h(H)') = \chi'(h(H)) = 4$. Since $v(h(H)') \leqslant 16$, Proposition 2.7 implies that $h(H)'$ is a 3-chromatic expansion of P_{10} and consequently $h(H)$ is a 3-chromatic expansion of P_{10}. Let $v \to A_v,\ v \in P_{10}$ be a representation of this expansion. For each $A_v,\ v \in P_{10}$ let A'_v be the subgraph of $h(H)$ induced by the edges corresponding to those in A_v. If $H_{\mathcal{C}}$ satisfies one of (3.1.1)–(3.1.3), (3.1.5), or (3.1.8), then all but at most one of the cycles $C'(v),\ v \in V(H_{\mathcal{C}})$ belongs to some $A'_v,\ v \in V(P_{10})$. In this case, Lemma 3.2 implies that there is a collection of cycles in $h(H)$ covering each of the cycles $C'(v),\ v \in V(H_{\mathcal{C}})$ once, and the other edges of $h(H)$ twice. This means that $h(H)$ is $\mathcal{C}$-compatible.

We suppose that $H_{\mathcal{C}}$ satisfies exactly one of (3.1.4), (3.1.6), or (3.1.7) and exactly 2 of the cycles $C'(v),\ v \in V(H_{\mathcal{C}})$ say $C'(u_1)$ and $C'(u_2)$, do not belong to any A'_v. We may assume that $C'(u_1)$ intersects exactly 5 of the subgraphs A'_v and $C'(u_2)$ intersects the other 5 A'_v's ; that is, they correspond to 2 vertex-disjoint 5-cycles of P_{10}. Thus $h(H)\backslash E(C'(u_1) \cup C'(u_2))$ has at least 5 components. However, since we are given that $H_{\mathcal{C}}$ is the union of 2 graphs, one from each of Figs. 9 and 10, and $H_{\mathcal{C}}$ satisfies one of (3.1.4), (3.1.6), or (3.1.7), one sees that $h(H)\backslash(C'(u_1) \cup C'(u_2))$ can have at most 4 components. This yields a contradiction, and this concludes the proof for case 1.1.

Case 1.2: Suppose $B_{\mathcal{C}}$ contains no degree-compatible subgraph in $\mathcal{F}_4$ which satisfies one of (3.1.1)–(3.1.8).

By Theorem A.1 in the Appendix A, either $B_{\mathcal{C}}$ contains a degree-compatible subgraph $H_{\mathcal{C}}$ which is $\mathcal{C}$-compatible, or G has a non-trivial 3-edge cut which separates a vertex of X or odd cycle of $\mathcal{C}$ in B from the other vertices of X or odd cycles in $\mathcal{C}$ in B. In this case, the theorem is seen to hold.

Case 2: Suppose no block of $G_{\mathcal{C}}$ contains v_1, v_2, v_3, v_4.

We shall divide this case into 2 subcases:

Case 2.1: There is a block $B_{\mathcal{C}} \subseteq G_{\mathcal{C}}$ having 4 odd vertices.

We may assume that $B_{\mathcal{C}}$ has odd vertices u_1, u_2, u_3, u_4. For each u_i which is odd in $G_{\mathcal{C}}$ we may assume $u_i = v_i$. If u_i is not odd in $G_{\mathcal{C}}$, we may assume there is a block chain $(B_i)_{\mathcal{C}} = (B_{i0})_{\mathcal{C}} \cup \ \ldots\ \cup (B_{ir_i})_{\mathcal{C}}$ where $u_i \in V((B_{i0})_{\mathcal{C}})$, $v_i \in V((B_{ir_i})_{\mathcal{C}})$ and u_i and v_i are exactly the odd vertices of the chain. Let $G^1_{\mathcal{C}}$ be the subgraph obtained from $G_{\mathcal{C}}$ where for each $i = 1, \ldots, 4$ we delete all the vertices of $(B_i)_{\mathcal{C}}$ except u_i. Now $u_1, \ldots, u_4$ are seen to be the odd vertices of $G^1_{\mathcal{C}}$ which belong to the block $B_{\mathcal{C}}$. If G^1 has a non-trivial 3-edge cut which separates a vertex or odd cycle corresponding to one of the vertices $u_1, \ldots, u_4$, then such a cut will also be a non-trivial 3-edge cut of G. So we may assume that no such cuts exist in G^1. Now according to Theorem A.1, there is a degree-compatible subgraph $J_{\mathcal{C}}$ for which $h(J)$ is $\mathcal{C}$-compatible and one of two things hold: either $J_{\mathcal{C}} \in \mathcal{F}_4$ and one of (3.1.1)–(3.1.8) holds, or every odd degree vertex of $J_{\mathcal{C}}$ has degree three.

According to Lemma 5.1 the chain $(B_i)_{\mathcal{C}}$ contains a subgraph $(H_i)_{\mathcal{C}} \in \mathcal{F}^*_2$ whose odd vertices are exactly u_i and v_i. If $u_i = v_i$, then we let $(H_i)_{\mathcal{C}} = u_i = v_i$. Let $H_{\mathcal{C}} = J_{\mathcal{C}} \cup \bigcup_i (H_i)_{\mathcal{C}}$. We have that $H_{\mathcal{C}} \in \mathcal{F}^*_4$.

By assumption, $u_i,\ i = 1, \ldots, 4$ cannot all be odd in $G_{\mathcal{C}}$. We may therefore assume that at least one of the u_i's, say u_1, is not odd in $G_{\mathcal{C}}$. Suppose first that $d_{J_{\mathcal{C}}}(u_i) \leqslant 5$, for $i = 1, \ldots, 4$. For each i where $u_i \neq v_i$ we have $h(B_i)$ is $\mathcal{C}$-compatible since $\chi'_h(B_i) = 3$

by Proposition 5.2 (ii). It follows from repeated application of Proposition 5.2(i) that $h(H)$ is $\mathcal{C}$-compatible. As such we can assume that $J_{\mathcal{C}}$ has odd vertices of degree at least 7. This means that $J_{\mathcal{C}}$ must satisfy one of (3.1.1)–(3.1.8), and in particular, it must satisfy (3.1.5). Thus $J_{\mathcal{C}}$ has one vertex of degree 7, and 3 vertices of degree 3. If for some $i,\ u_i = v_i$, and $d_{J_{\mathcal{C}}}(u_i) = 7$, then $h(H)$ is $\mathcal{C}$-compatible by Proposition 5.2(i). Thus we may assume that $d_{J_{\mathcal{C}}}(u_i) = 7$ for some $u_i \neq v_i$, and this we can assume this holds for u_1 (and $d_{J_{\mathcal{C}}}(u_i) = 3,\ i = 2, 3, 4$).

Let J_1 be the graph obtained from $J \cup H_1$ where we contract $H_1 \backslash C(u_1)$ into a single vertex w_1. We can reduce each 2-, 3-, and 4-cycle $C_{h(J)}(v) \subseteq h(J_1)$ via $\circ$-, $\triangle$-, or $\square$-reductions so that the resulting cubic graph, which we denote by $h(J_1)'$ is 2-connected. We see that $h(J_1)'$ has 14 vertices, and thus according to Proposition 2.7 either $\chi'(h(J_1)') = 3$ or $h(J_1)'$ is a 3-chromatic expansion of P_{10}. Now Theorem 2.6 implies that either $\chi'_h(J_1) = 3$ or $h(J_1)$ is a 3-chromatic expansion of P_{10}. Since $h(J \cup H_1)$ is a 3-chromatic expansion of $h(J_1)$, it follows that $\chi'_h(J \cup H_1) = 3$ or $h(J \cup H_1)$ is a 3-chromatic expansion of P_{10}. Since $\chi'_h(H_i) = 3$ if $u_i \neq v_i$, it follows that $\chi'_h(H) = 3$ or $h(H)$ is a 3-chromatic expansion of P_{10} given by $v \to A_v,\ v \in V(P_{10})$ where we may assume that all cycles $C'(v),\ v \in V(H_{\mathcal{C}}) \backslash X$ belong to some A_v, except for possibly $C'(u_1)$. Thus, Lemma 3.2 implies that $h(H)$ is $\mathcal{C}$-compatible and this completes the proof of Case 2.1.

Case 2.2: Suppose each block of $G_{\mathcal{C}}$ has at most 2 odd vertices.

If each block of $G_{\mathcal{C}}$ has at most 2 odd vertices, then it is seen that $G_{\mathcal{C}}$ contains 2 disjoint block chains $(B_0)_{\mathcal{C}}$ and $(B_1)_{\mathcal{C}}$ (not having any common blocks) where the endblocks of the block chains each contain exactly one odd vertex of $G_{\mathcal{C}}$. We may assume v_1, v_2 and v_3, v_4 belong to the endblocks of $(B_0)_{\mathcal{C}}$ and $(B_1)_{\mathcal{C}}$, respectively. By Lemma 5.1, there exists subgraphs $(H_0)_{\mathcal{C}} \subseteq (B_0)_{\mathcal{C}}$ and $(H_1)_{\mathcal{C}} \subseteq (B_1)_{\mathcal{C}}$ where $(H_0)_{\mathcal{C}},\ (H_1)_{\mathcal{C}} \in \mathcal{F}_2^*$ and moreover, v_1, v_2 and v_3, v_4 are exactly the odd vertices of $(H_0)_{\mathcal{C}}$ and $(H_1)_{\mathcal{C}}$, respectively. Let $H_{\mathcal{C}} = (H_0)_{\mathcal{C}} \cup (H_1)_{\mathcal{C}}$. The graph $H_{\mathcal{C}}$ belongs to either $\mathcal{F}_2^* \vee_0 \mathcal{F}_2^*,\ \mathcal{F}_2^* \vee_1^e \mathcal{F}_2^*$, or $\mathcal{F}_2^* \vee_1^{oe} \mathcal{F}_2^*$. If $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_0 \mathcal{F}_2^*$, then $\chi'(h(H)) = 3$ (according to Proposition 5.2 (ii)) In this case $h(H)$ is $\mathcal{C}$-compatible. Thus we may assume that either $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_1^e \mathcal{F}_2^*$ or $H_{\mathcal{C}} \in \mathcal{F}_2^* \vee_1^{oe} \mathcal{F}_2^*$, and $(H_0)_{\mathcal{C}}$ and $(H_1)_{\mathcal{C}}$ intersect at a vertex u.

Let $(H_u)_{\mathcal{C}}$ be the subgraph of $H_{\mathcal{C}}$ which is the union of the blocks of $H_{\mathcal{C}}$ which contain u. Let $h(H_u)'$ be the graph obtained from $h(H_u)$ by reducing all 2-, 3-, or 4-cycles $C_{h(H_u)}(v) \subseteq h(H_u)$ via $\circ$-, $\triangle$-, or $\square$-reductions (where as usual, bridgelessness is preserved). The resulting graph has at most 16 vertices, and according to Proposition 2.7 either $\chi'(h(H_u)') = 3$ or $h(H_u)'$ is a 3-chromatic expansion of P_{10}. It then follows from Theorem 2.6 that either $\chi'_h(H_u) = 3$ or $h(H_u)$ is a 3-chromatic expansion of P_{10}. If $\chi'_h(H_u) = 3$, then by Lemma 4.2(b) we have $\chi'_h(H) = 3$. In this case, $h(H)$ is $\mathcal{C}$-compatible. On the other hand, if $h(H_u)$ is a 3-chromatic expansion of P_{10}, then $h(H)$ is a 3-chromatic expansion of P_{10} where we may assume that the expansion has a representation $v \to A_v,\ v \in V(P_{10})$ such that all cycles $C_{h(H)}(v),\ v \in V(H_{\mathcal{C}})$ belong to some A_v except for possibly $C_{h(H)}(u)$. Lemma 3.2 implies that $h(H)$ is $\mathcal{C}$-compatible. This completes the proof of case 2.2. $\square$

Proof of Theorem 1.2. We suppose again that G is a 2-connected, cubic graph and let $(X, \mathcal{C})$ be a pseudo 2-factor of G where $|X|$ plus the number of odd cycles in $\mathcal{C}$ is at most 4. We may assume that the theorem holds for any graph with fewer vertices than G.

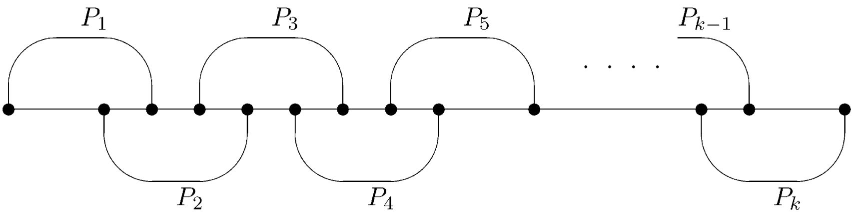

Fig. 11. A vine $P_1, \ldots, P_k$.

Suppose that G has a non-trivial 3-edge cut. Then G can be expressed as a vertex expansion $G = (G_1; u; e_1, e_2, e_3) \otimes (G_2; v; f_1, f_2, f_3)$ where $v(G_i) < v(G),\ i = 1, 2$. For $i = 1, 2$ let $(X_i, \mathcal{C}_i)$ be the pseudo 2-factor of G_i obtained from G in the natural way. Then $|X_i|$ plus the number of odd cycles in $\mathcal{C}_i$ is at most 4. Thus $o(G_i) \leqslant o(G)$, and hence by assumption, G_1 and G_2 each admit double cycle covers $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Now one can construct a cycle double cover $\mathcal{D}$ via splicing $\mathcal{D}_1$ and $\mathcal{D}_2$ together.

If we now assume that G has no non-trivial 3-edge cuts, then Theorem 6.1 implies that $G_{\mathcal{C}}$ has a bridgeless degree compatible subgraph $H_{\mathcal{C}}$ for which $h(H)$ is $\mathcal{C}$-compatible. By Lemma 3.4, we can construct a cycle double cover for G. This completes the proof of the theorem. $\square$

7. Vines

Let P be a path $v_0 v_1 \cdots v_n$ and let $P_1, \ldots, P_k$ be a collection of paths which intersect P at exactly their terminal vertices, where for each i, P_i has terminal vertices $v_{t(i)}$ and $v_{h(i)}$ and $t(i) < h(i)$. If the paths $P_i,\ i = 1, \ldots, k$ are internally vertex-disjoint and satisfy,

(i) $t(1) = 0,\ h(k) = n.$
(ii) $t(i) < t(i+1) \leqslant h(i) < h(i+1),\ i = 1, \ldots, k-1.$
(iii) $h(i) < t(i+2),\ i = 1, \ldots, k-2$

then we say that $P_1, \ldots, P_k$ form a *vine* along P. Note that a vine may consist of just one path. We say that vertices u and v are *joined by a vine* if there exists a path P from u to v and a vine along P (see Fig. 11).

Let G be a graph and let H be a subgraph. Let P be a path from u to v in H and let $P_1, \ldots, P_k$ be a vine along P where each P_i intersects H only at its terminal vertices. Then we say that $P_1, \ldots, P_k$ is an *H-vine*. In this case, we say that there is an H-vine from u to v in H.

Let $P_1, \ldots, P_k$ be a vine along $P = v_0 v_1 \cdots v_n$ as above. We shall now define what we call the *circuit of the vine* $C_{P_1,\ldots,P_k}$. If $k = 1$, let $C_{P_1,\ldots,P_k} = P \cup P_1$. If $k = 2l+1,\ l \geqslant 1$, then let

$$\begin{aligned} C_{P_1,\ldots,P_k} = {} & P_1 \cup P[v_{h(1)}, v_{t(3)}] \cup \cdots \cup P_{2i-1} \cup P[v_{h(2i-1)}, v_{t(2i+1)}] \cdots \\ & \cup\ P_{2l+1} \cup P[v_0, v_{t(2)}] \cup P_2 \cup \cdots \cup P[v_{h(2i)}, v_{t(2i+2)}] \\ & \cup\ P_{2i+2} \cup \cdots \cup P[v_{h(2l)}, v_n]. \end{aligned}$$

If $k = 2l,\ l \geqslant 1$, then let

$$\begin{aligned} C_{P_1,\dots,P_k} = {} & P_1 \cup P[v_{h(1)}, v_{t(3)}] \cup \cdots \cup P_{2i-1} \cup P[v_{h(2i-1)}, v_{t(2i+1)}] \cdots \cup P_{2l-1} \\ & \cup\ P[v_{h(2l-1)}, v_{2l}] \cup P[v_0, v_{t(2)}] \cup P_2 \cup \cdots \cup P[v_{h(2i)}, v_{t(2i+2)}] \\ & \cup\ P_{2i+2} \cup \cdots \cup P[v_{h(2l-2)}, v_{2l}] \cup P_{2l}. \end{aligned}$$

Let G be a connected cubic graph containing a subgraph H which is homeomorphic to a cubic graph $\tilde{H}$. For each edge $\tilde{e} \in E(\tilde{H})$, let $[\tilde{e}]_H$ be the corresponding path in H, and for any subgraph $\tilde{I} \subseteq \tilde{H}$, we let $[I]_H$ be the corresponding subgraph in H. We leave the verification of the following theorem to the reader.

Theorem 7.1. *Suppose for any two edges $\tilde{e}, \tilde{f} \in E(\tilde{H})$ it holds that if there is an H-vine from a vertex of $[\tilde{e}]_H$ to a vertex of $[\tilde{f}]_H$, then $\tilde{e}$ and $\tilde{f}$ are incident in $\tilde{H}$. Then the graph G is an expansion of H.*

Theorem 7.2. *Let G be a connected, cubic graph and let H be a subgraph homeomorphic to $\tilde{H} \simeq P_{10}$. Suppose $\tilde{e}, \tilde{f} \in E(\tilde{H})$ are two non-incident edges. If there is an H-vine from a vertex $v_0 \in [\tilde{e}]_H$ to a vertex $v_n \in [\tilde{f}]_H$. Then for some such vine $P_1, \dots, P_k$ it holds that $\chi'_h(H \cup P_1 \cup \cdots \cup P_k) = 3$.*

Proof. Suppose that there are non-incident edges $\tilde{e}, \tilde{f} \in E(\tilde{H})$ for which there is an H-vine from a vertex of $[\tilde{e}]_H$ to a vertex of $[\tilde{f}]_H$. Pick such a vine having a fewest number of paths, say $P_1, \dots, P_k$, and assume that it is an H-vine along a path $P \subseteq H$ from a vertex $v_0 \in [\tilde{e}]_H$ to a vertex $v_n \in [\tilde{f}]_H$. Given that $\tilde{H} \simeq P_{10}$, $\tilde{H}$ has a 2-factor $\tilde{C}_1$ and $\tilde{C}_2$ being two 5-cycles where $\tilde{e} \in E(\tilde{C}_1)$ and $\tilde{f} \in \tilde{C}_2$. If the vine consists of only one path P_1, then $h(H \cup P_1)$ has a 2-factor consisting of two 6-cycles, $[\tilde{C}_i]_H,\ i = 1, 2$. In this case, $\chi'(H \cup P_1) = 3$. We suppose therefore that the vine has more than one path. Since we chose $P_1, \dots, P_k$ to have as few paths as possible, we have that the distance between $\tilde{e}$ and $\tilde{f}$ in $\tilde{H}$ equals 2, and moreover, there is a path $u\tilde{e}v\tilde{g}w\tilde{f}z$ in $\tilde{H}$ such that $P \subseteq [u\tilde{e}v\tilde{g}w\tilde{f}z]_H$.

Let $C_i = [\tilde{C}_i]_H,\ i = 1, 2$. Now $(C_1 \cup C_2) \bigtriangledown C_{P_1 \cdots P_k}$ is a cycle in G, which is also a hamilton cycle in $h(H \cup P_1 \cup \cdots \cup P_k)$. thus we have that $\chi'_h(H \cup P_1 \cup \cdots \cup P_k) = 3$. $\square$

From the above, we obtain the following corollary.

Corollary 7.3. *Let G be a connected cubic graph and let H be a subgraph homeomorphic to P_{10}. Then either G is an expansion of P_{10}, or there is an H-vine $P_1, \dots, P_k$ such that $\chi'_h(H \cup P_1 \cup \cdots \cup P_k) = 3$.*

We also see that Theorem 2.5 is a consequence of the above result.

Appendix A.

Let G be a 2-connected cubic graph with $o(G) = 4$ and let $(X, \mathcal{C})$ be a pseudo 2-factor of G where $|X|$ plus the number of odd cycles in $\mathcal{C}$ equals 4. Let $G_{\mathcal{C}}$ be the graph obtained from G by contracting the cycles of $\mathcal{C}$.

Theorem A.1. *Suppose that the odd vertices of $G_{\mathcal{C}}$ are contained in a block $B_{\mathcal{C}}$. Then one of the three statements holds*:

(i) *$G_{\mathcal{C}}$ contains a degree-compatible subgraph $H_{\mathcal{C}}$ satisfying one of* (3.1.1)–(3.1.8) *for which $h(H)$ is $\mathcal{C}$-compatible.*

(ii) *$G_{\mathcal{C}}$ contains a degree-compatible subgraph $H_{\mathcal{C}}$ where each odd degree vertex has degree three and for which $h(H)$ is $\mathcal{C}$-compatible.*

(iii) *G contains a non-trivial* 3*-edge cut which separates a vertex of X or an odd cycle of $\mathcal{C}$ from the other vertices of X and odd cycles of $\mathcal{C}$.*

Proof. The odd vertices of $G_{\mathcal{C}}$ are easily seen to be exactly the odd vertices of $B_{\mathcal{C}}$. Since $B_{\mathcal{C}}$ is a block with more than one edge, Lemmas 3.5 and 3.6 imply that it contains a degree compatible subgraph $H_{\mathcal{C}} \in \mathcal{F}_4$ (which is also degree-compatible in $G_{\mathcal{C}}$) satisfying one of (3.1.1)–(3.1.9). If $H_{\mathcal{C}}$ satisfies one of (3.1.1)–(3.1.8), then following the proof of Theorem 6.1, case 1.1, the graph $h(H)$ would be $\mathcal{C}$-compatible. In this case, (i) holds. So we may assume that $H_{\mathcal{C}}$ satisfies (3.1.9), and moreover, $G_{\mathcal{C}}$ contains no degree compatible subgraph in $\mathcal{F}_4$ which satisfies one of (3.1.1)–(3.1.8). We shall assume that $v_1, \ldots, v_4$ are exactly the odd vertices of $G_{\mathcal{C}}$. We shall let $C'(v)$, $h(H)'$, $G'_{\mathcal{C}}$, and G' be as defined in the proof of Theorem 6.1.

We shall first show that $h(H)$ is a 3-chromatic expansion of P_{10}. We have that $v(h(H)') = 18$. If $\chi'(h(H)') = 3$, then $\chi'_h(H) = 3$ (by Lemma 1.7 and Corollary 2.2). It would then follow from Lemma 3.1 that $h(H)$ is $\mathcal{C}$-compatible. This being the case, we may assume that $\chi'(h(H)') = \chi'_h(H) = 4$. According to [3], there are only 3 different cubic graphs of order 18 having girth at least 5 and chromatic index $\chi' = 4$. Two such graphs are obtained by performing a 4-edge expansion with the graphs P_8 and P_{10}. The third graph is obtained by performing a vertex expansion at one vertex u of P_{10}, where $u \to P_{10}$.

Since $H_{\mathcal{C}} \in \mathcal{F}_4$, we have that $H_{\mathcal{C}} = H'_{\mathcal{C}} \cup F'_{\mathcal{C}}$ where $H'_{\mathcal{C}}$ is homeomorphic to one of the graphs in Fig. 9 and $F'_{\mathcal{C}}$ is homeomorphic to a forest in Fig. 10. From the table in Section 3, we see that there is only one possibility for $H'_{\mathcal{C}}$ and $F'_{\mathcal{C}}$; the graph $F'_{\mathcal{C}}$ is homeomorphic to F_2, and $H'_{\mathcal{C}}$ is homeomorphic to $H^6_{\mathcal{C}}$. Given that we are assuming that $G_{\mathcal{C}}$ has no degree-compatible subgraphs in $\mathcal{F}_4$ satisfying one of (3.1.1)–(3.1.8), we have that the vertices v_{13} and v_{24} (as specified by Lemma 3.5) form a 2-separating set for $G_{\mathcal{C}}$ which separates each pair of vertices v_i and v_j, $i \neq j$. Let $u_1 = v_{13}$ and $u_2 = v_{24}$. The graph $H'_{\mathcal{C}}$ consists of 4 internally vertex-disjoint paths $P^1_{\mathcal{C}}, P^2_{\mathcal{C}}, P^3_{\mathcal{C}}, P^4_{\mathcal{C}}$ between u_1 and u_2, where $v_i \in V(P^i_{\mathcal{C}}),\ i = 1, \ldots, 4$. The graph $F'_{\mathcal{C}}$ is homeomorphic to F_2 and consists of 4 internally vertex-disjoint paths originating at u_1 and terminating at v_i. One of these paths contains u_2, and we may assume that this path terminates at v_4. For $i = 1, 2, 3$, we denote the path terminating at v_i by $Q^i_{\mathcal{C}}$, and we denote the path terminating at v_4 by $P^5_{\mathcal{C}} \cup Q^4_{\mathcal{C}}$ where $P^5_{\mathcal{C}}$ is the portion of the path between u_1 and u_2, and $Q^4_{\mathcal{C}}$ is the portion of the path between u_2 and v_4 (see Fig. 12).

The cycles $C'(u_1)$ and $C'(u_2)$ are vertex-disjoint cycles of $h(B)'$ having lengths 8 and 6, respectively. Suppose $h(H)'$ is a 4-edge expansion of P_8 with P_{10}. Let $A = \{f_1, f_2, f_3, f_4\}$ be the 4-edge cut formed via this expansion. Then $h(B)' \backslash \{f_1, f_2, f_3, f_4\}$ has exactly 2 components K_1 and K_2 having 10 and 8 vertices, respectively. Suppose first that neither

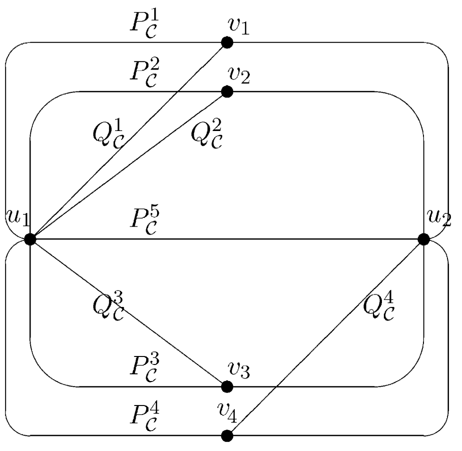

Fig. 12. The graph $H_{\mathcal{C}}$.

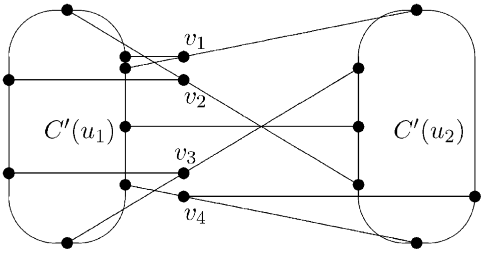

Fig. 13. The graph $h(H)'$.

$C'(u_1)$ nor $C'(u_2)$ contain edges of A. Then either both cycles belong to one component, or they belong to separate components. The former is impossible considering that each component has at most 10 vertices. The latter is also seen to be impossible since there are 5 edge-disjoint paths between u_1 and u_2 in $H_{\mathcal{C}}$, and hence no 4-edge cut in $h(H)'$ can separate $C'(u_1)$ and $C'(u_2)$. We conclude that at least one of the cycles contains edges of A (see Fig. 13).

Suppose $C'(u_1)$ contains no edges of A, but $C'(u_2)$ does. Then $C'(u_1) \subseteq K_1$; for otherwise, if $C'(u_1) \subseteq K_2$, then it would follow that $C'(u_2) \subseteq K_1$. Given that $C'(u_2)$ contains at least 2 edges of A, one sees upon examination of $H_{\mathcal{C}}$ that K_1 would contain at least 3 of the vertices $v_i,\ i = 1, \ldots, 4$ and hence $v(K_1) \geqslant 8 + 3 = 11$ vertices. This yields a contradiction. We may therefore assume that $C'(u_1)$ contains edges of A, and hence it must have at least 2 such edges.

Suppose $C'(u_2)$ contains no edges of A. Given $C'(u_1)$ contains at least 2 edges of A, one sees by inspecting $H_{\mathcal{C}}$ that for at least 2 of the vertices $v_i,\ i = 1, \ldots, 4$, no edge of A is incident with v_i. Thus the component(K_1 or K_2) containing $C'(u_2)$ would have at least $6 + 5 = 11$ vertices. This yields a contradiction. We may therefore assume that $C'(u_1)$ and $C'(u_2)$ both contain edges of A, and hence they contain 2 edges apiece.

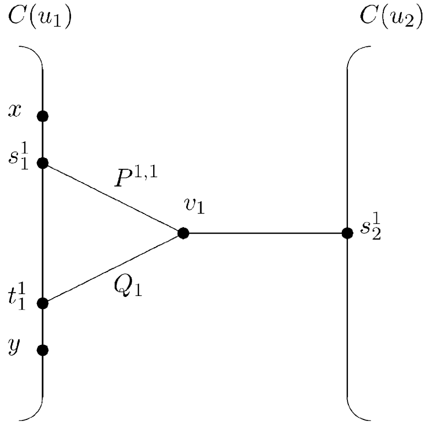

Fig. 14.

We now have that no edge of A is incident with the vertices v_i, $i = 1, \ldots, 4$ and hence the neighbours of v_i in $h(H)$ belong to the same component (K_1 or K_2) as v_i. Thus, neither K_1 nor K_2 can contain 3 or more or the vertices v_i, and each component contains 2 vertices apiece. Suppose v_i and v_j belong to K_2. Given that K_2 is in the P_8 part of the 4-edge expansion and no edge of A is incident with v_i or v_j, it follows that $dist_{K_2}(v_i, v_j) \leqslant 2$. However, upon inspection of $H_{\mathcal{C}}$, one sees that for $i \neq j$, $dist_{h(H)'}(v_i, v_j) \geqslant 3$. Here we reach a final contradiction. We conclude that $h(H)'$ cannot be a 4-edge expansion of P_8 with P_{10}. Similar arguments also demonstrate that $h(H)'$ is not a vertex-expansion of P_{10} where for a vertex $u \in V(P_{10})$ we expand the vertex via $u \to P_{10}$.

From the above, Proposition 2.7 implies that $h(H)'$ must be a 3-chromatic expansion of P_{10}. Hence $h(H)$ is also a 3-chromatic expansion of P_{10}, and we let $v \to A_v$, $v \in V(P_{10})$ be a representation of this expansion. For each v, let A'_v be the subgraph of $h(H)$ induced by those edges of $h(H)$ coinciding with those in A_v. If one of the cycles $C(u_i)$, $i = 1, 2$ belongs to some A'_v, then all but one of the cycles $C(v)$, $v \in V(B_{\mathcal{C}})\backslash X$ belong to A'_v's, and as was demonstrated before, $h(H)$ is $\mathcal{C}$-compatible in this case. Thus, we may assume that neither $C(u_1)$ nor $C(u_2)$ are contained in any A'_v. Thus each cycle intersects exactly 5 of the subgraphs A'_v.

Suppose $P_{\mathcal{C}}$ is a path in $G_{\mathcal{C}}$ and u is one of its endvertices. We define a *stem-vertex* of P in the following way: if $u \in X$, then it is a stem vertex. Otherwise, we define a vertex of $C(u)$ to be a stem-vertex if it is a separating vertex of P.

For $i = 1, 2$ and $j = 1, \ldots, 5$ let s_i^j denote the stem-vertex of P^j on $C(u_i)$. For $j = 1, 2, 3$ let t_1^j denote the stem-vertex of Q^j on $C(u_1)$ and let t_2^4 denote the stem-vertex of Q^4 lying on $C(u_2)$. Let $P^{1,1}$ denote the portion of P^1 lying between $C(v_1)$ and v_1. Let x and y denote the stem-vertices lying on either side of s_1^1 and t_1^1, and let $C_1[x, y]$ denote the portion of C_1 between x and y which contains s_1^1 and t_1^1 (see Fig. 14). Let $J = C_1[s_1^1, t_1^1] \cup P^{1,1} \cup Q^1$. One can show that $\{x, y, s_2^1\}$ is a 3-separating set of H which separates H into two subgraphs H_1 and H_2 (so that $H_1 \cap H_2 = \{x, y, s_2^1\}$) where $C_1[x, y] \cup P^1 \cup Q^1 \subseteq H_1$. If there is a H-vine in G from a vertex in J to a vertex in H_2, then we could modify $H_{\mathcal{C}}$ to obtain a degree-compatible graph of $G_{\mathcal{C}}$ for which $h(H)$ is 3-edge colourable and hence $\mathcal{C}$-compatible (we leave the verification of this to the reader). In addition, each odd degree vertex in such a subgraph has degree three, in which case (ii) holds. We may thus assume that no such H-vine exists. Let x' be the vertex of $C_1[x, y]$ closest to x, where x' is joined to a vertex in J by an H-vine in G. We define y' analogously for y. Let $s_2^{1'}$ be the vertex of P^1 closest to s_2^1 which is joined by an H-vine to a vertex in J. Now $\{x', y', {s_1}^{1'}\}$ is a 3-separating set in G. Consequently, G contains a non-trivial 3-edge cut which separates the odd cycle or vertex in G corresponding to v_1 from the odd cycles or vertices in G corresponding to v_2, v_3, and v_4. In this case, (iii) is seen to hold. This concludes the proof of the theorem. □

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