Squaring a Tournament: A Proof of Dean's Conjecture

David C. Fisher
UNIVERSITY OF COLORADO AT DENVER
DENVER, CO 80217-3364
e-mail: dfisher@orphan.cudenver.edu

ABSTRACT

Let the square of a tournament be the digraph on the same nodes with arcs where the directed distance in the tournament is at most two. This paper verifies Dean's conjecture: any tournament has a node whose outdegree is at least doubled in its square. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

In a "round robin tournament," each team plays every other team exactly once. Assuming no ties, for each pair of teams i and j, either i beats j, or j beats i, but not both (see the table below). Team i "sort-of-beats" team j if either i beats j, or i beats some team which beat j. Instead of looking at the teams they beat, a team with a modest record might prefer to look at the usually larger set they sort-of-beam.

The results can be recorded with a "tournament". A *digraph* D is a set of *nodes* $V(D) = \{v_1, v_2, \ldots, v_n\}$ and a set of ordered pair of nodes $R(D)$ called *arcs*. If $(v_i, v_j) \in R(D)$, we will say v_i *beats* v_j in D. Let the *outset* $O_D(v_i)$ be the nodes that v_i beats and let the *inset* $I_D(v_i)$ be the nodes that beat v_i. Let $d_D(v_i) = |O_D(v_i)|$ be the *outdegree* of v_i. A *tournament* T is a digraph where either $(v_i, v_j) \in T$ or $(v_j, v_i) \in T$, but not both. Figure 1(a) shows the tournament T which records the results from the table. If team i beats team j in the table, then v_i beats v_j in T depicted with an arrow pointing from v_i to v_j.

The relation "sort-of-beats" is shown with the "square" of a digraph. The *square* of a digraph D (notated D^2) is a digraph with $V(D^2) = V(D)$ and with $(v_i, v_j) \in R(D^2)$ if either $(v_i, v_j) \in R(D)$ or there is a k with $(v_i, v_k) \in R(D)$ and $(v_k, v_j) \in R(D)$. Figure 1(b) shows the square of the tournament in Figure 1(a). It has an arc from v_i to v_j if the directed distance from v_i to v_j in the tournament is at most two.

Journal of Graph Theory Vol. 23, No. 1, 43–48 (1996)

CCC 0364-9024/96/010043-06

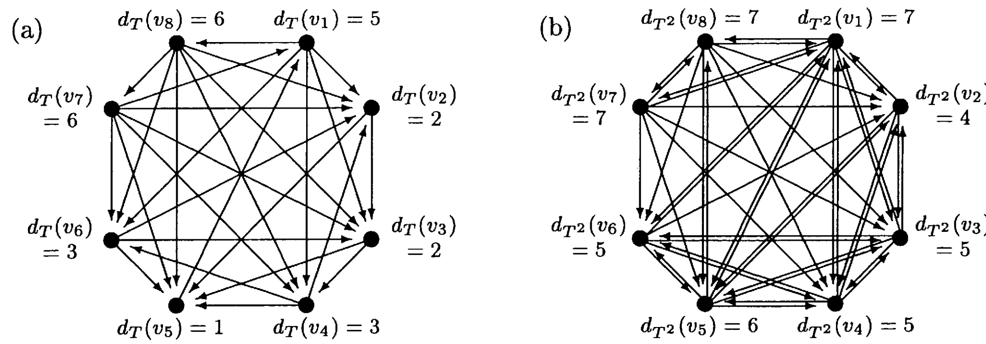

FIGURE 1. A tournament and its square. The labels show the outdegree of each node. Dean conjectured that any tournament has a node whose outdegree at least doubles in its square. Here there are three such nodes: v_2, v_3, and v_5.

Dean's conjecture is that any tournament T has a node v_i with $d_{T^2}(v_i) \geq 2d_T(v_i)$. It is clearly true if $d_T(v_i) = 0$ for some node v_i, because then $d_{T^2}(v_i) = 0$. If the minimum outdegree of T is one, some node v_i with $d_T(v_i) = 1$ beats v_j which must beat another node v_k. Thus v_i beats both v_j and v_k in T^2 giving $d_{T^2}(v_i) \geq 2$. Dean and Latka [1] continued this to verify the conjecture for tournaments whose minimum outdegree is five or less. They also verified the conjecture for regular and almost regular tournaments (where outdegrees differ by at most one).

The problem with an approach based on the minimum outdegree of a tournament is that the outdegree of nodes with minimum outdegree may not double in its square. Figure 2 gives an example of this. Instead this paper uses a "losing" probability density to verify Dean's conjecture. Section 2 defines a losing density and shows it exists for every digraph. Section 3 shows that when a node is picked from a losing density on a tournament T, its expected outdegree in T^2 is at least twice what it is in T.

Dean's conjecture is a special case of another conjecture. An *oriented graph* is a loopless digraph with at most one arc between each pair of nodes. Seymour (quoted from [1]) conjectured that an oriented graph D has a node v_i with $d_{D^2}(v_i) \geq 2d_D(v_i)$. The more general conjecture remains unresolved. To highlight why the approach here does not extend to oriented graphs, results are proved to the greatest extent possible for digraphs.

TABLE I. A fictitious round-robin tournament. A team "sort-of-beats" another if it either beats that team, or beats a team which beat that team. For example, team 6 sort-of-beats teams 1, 2, 3, 4 and 5, because it beats 2, 3 and 5, it beats 3 which beats 4, and it beats 5 which beats 1.

Team	Teams they beat	Teams they "sort-of-beat"
1	2, 3, 4, 6, 8	2, 3, 4, 5, 6, 7, 8
2	3, 5	1, 3, 4, 5
3	4, 5	1, 2, 4, 5, 6
4	2, 5, 6	1, 2, 3, 5, 6
5	1	1, 2, 3, 4, 6, 8
6	2, 3, 5	1, 2, 3, 4, 5
7	1, 2, 3, 4, 5, 6	1, 2, 3, 4, 5, 6, 8
8	2, 3, 4, 5, 6, 7	1, 2, 3, 4, 5, 6, 7

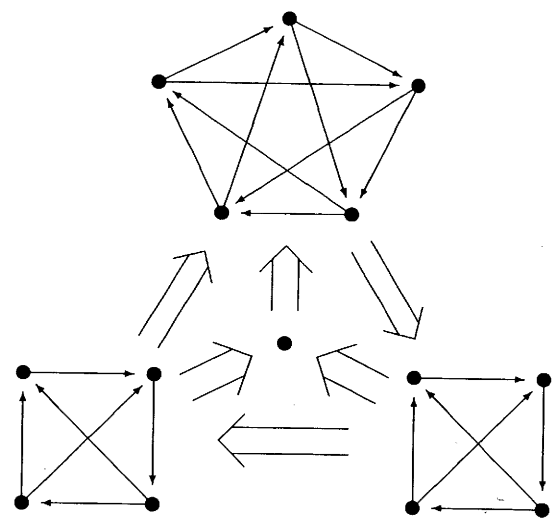

FIGURE 2. An Example. The center node in this tournament has outdegree five, while all other nodes have outdegree at least six. However the center node beats only nine nodes in T^2. Thus, it not always true that a node of minimum outdegree beats at least twice as many nodes in T^2 as it beats in T (adapted from [1]).

2. WINNING AND LOSING DENSITIES

If a node of a digraph never loses, it is a *winner. What if there are no winners?* Fisher and Ryan [2] and Laslier, Laffond, and Le Breton [3] independently developed the idea of a "winning density" of a tournament. Here this idea is extended to digraphs.

A (probability) *density* f on a digraph D gives each node a nonnegative value with $f(V(D)) = 1$ (let the probability of a set be the sum of the probabilities of its members). A density w is *winning* if $w(I_D(v_i)) \geq w(O_D(v_i))$ for all nodes v_i. In other words for any node v_i, a random node picked from a winning density is at least as likely to beat v_i as it is to lose to v_i (see Figure 3(a)).

If a digraph has a winner, picking that node with probability 1 (and any other node with probability 0) gives a winning density. So a winner can be thought of as a winning density, but digraphs without winners can have winning densities as seen in Figure 3. Thus winning densities are a generalization of winners.

Do all digraphs have winning densities? Fisher and Ryan [2] and Laslier, Laffond, and Le Breton [3] showed the answer is yes for tournaments (they also showed that a tournament has only one winning density, and it gives positive probabilities to an odd number of nodes). Theorem 1 uses Farkas's Lemma (see for example, Solow [4, p. 279]) to extend this. Let **0** and **1** be vectors of all zeros and ones, respectively. For vectors x and y, let $x \geq y$ mean that $x_i \geq y_i$ for all i.

Farkas's Lemma. Given a matrix M and a vector $\mathbf{b}$, exactly one of these systems has a solution: (a) $Mx = \mathbf{b}$ with $x \geq \mathbf{0}$; or (b) $M^T y \geq 0$ with $\mathbf{b}^T y < 0$.

Let the *adjacency matrix* $A(D)$ of a digraph D with n nodes be the $n \times n$ matrix with $a_{ij} = 1$ if v_i beats v_j and $a_{ij} = 0$ otherwise. Let $K(D) = A(D) - A(D)^T$. For a density w,

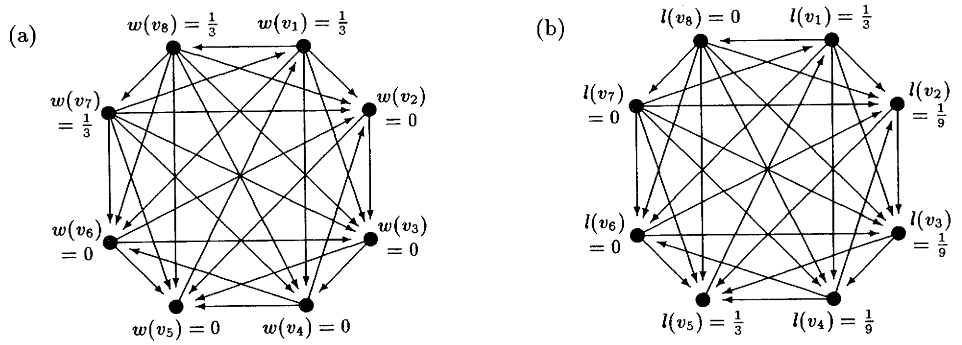

FIGURE 3. A Winning and Losing Density. This shows two densities on the tournament in Figure 1. Density w is winning because the inset of any node is at least as likely as its outset. For example, a node picked from this density will beat v_5 with probability 2/3 and will lose with probability 1/3. Density l is losing because the outset of any node is at least as likely as its inset.

let the associated vector be $\mathbf{w} = (w(v_1), w(v_2), \ldots, w(v_n))^T$. Then $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{1}^T\mathbf{w} = 1$. Further $(K(D)w)_i = w(O_T(v_i)) - w(I_T(v_i))$. So w is winning if $K(D)\mathbf{w} \leq \mathbf{0}$.

Theorem 1. Any digraph D has a winning density. Further for a winning density w, if $w(v_i) > 0$, then $w(O_D(v_i)) = w(I_D(v_i))$.

Proof. Suppose D has no winning density. Then this system has no solutions (I is the identity matrix):

$$\begin{bmatrix} K(D) & I \\ \mathbf{1}^T & \mathbf{0}^T \end{bmatrix} \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $K(D)^T = -K(D)$, Farkas's Lemma shows this system has a solution:

$$\begin{bmatrix} -K(D) & \mathbf{1} \\ I & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ v \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ with } (\mathbf{0}^T \quad 1) \begin{pmatrix} \mathbf{u} \\ v \end{pmatrix} < 0.$$

Thus $\mathbf{u} \geq \mathbf{0}$ and $K(D)\mathbf{u} \leq v\mathbf{1}$ with $v < 0$. So $K(D)\mathbf{u} < \mathbf{0}$ and hence $(\mathbf{1}^T\mathbf{u})^{-1}\mathbf{u}$ is the associated vector of a winning density, a contradiction. Therefore D has a winning density.

Now let w be a winning density on D with associated vector $\mathbf{w}$. Then $K(D)\mathbf{w} \leq \mathbf{0}$ and $\mathbf{w} \geq \mathbf{0}$ and hence $(\mathbf{w})_i(K(D)\mathbf{w})_i \leq 0$ for all i. Since $K(D)$ is skew-symmetric, $\mathbf{w}^T K(D)\mathbf{w} = 0$. Thus $(\mathbf{w})_i(K(D)\mathbf{w})_i = 0$ for all i. Therefore if $(\mathbf{w})_i = w(v_i) > 0$, then

$$0 = (K(D)\mathbf{w})_i = w(O_T(v_i)) - w(I_T(v_i)).$$

So if $w(v_i) > 0$, then $w(O_D(v_i)) = w(I_D(v_i))$. ∎

A density l on a digraph D is *losing* if $l(I_D(v_i)) \leq l(O_D(v_i))$ for all nodes v_i (see Figure 3(b)). Since a losing density is a winning density on the digraph formed by reversing its arcs, Theorem 1 has a counterpart for losing densities.

Corollary. Any digraph D has a losing density. Further for a losing density l, if $l(v_i) > 0$, then $l(O_D(v_i)) = l(I_D(v_i))$.

3. AN ANALYTICAL PROOF

Given a real value function r on the nodes of a digraph D, the expected value of r on a random node picked from density f is

$$E_f(r) \equiv \sum_{i=1}^{n} f(v_i)r(v_i).$$

Lemma 1 gives an indirect way to find the expected outdegree. For the density in Figure 3(b), the expected outdegree is $\frac{1}{3}\cdot 5 + \frac{1}{9}\cdot 2 + \frac{1}{9}\cdot 2 + \frac{1}{9}\cdot 3 + \frac{1}{3}\cdot 1 = 2\frac{7}{9}$. Lemma 1 shows this equals the sum of the probabilities of the insets: $\frac{1}{3} + \frac{4}{9} + \frac{4}{9} + \frac{4}{9} + \frac{1}{3} + \frac{4}{9} + 0 + \frac{1}{3} = 2\frac{7}{9}$.

Lemma 1. Let D be a digraph with density f. Then $E_f(d_D) = \sum_{j=1}^{n} f(I_D(v_j))$.

Proof. Since $v_j \in O_D(v_i)$ if and only if $v_i \in I_D(v_j)$, we have

$$E_f(d_D) = \sum_{i=1}^{n} f(v_i)d_D(v_i) = \sum_{i=1}^{n} \sum_{v_j \in O(v_i)} f(v_i) = \sum_{j=1}^{n} \sum_{v_i \in I_D(v_j)} f(v_i) = \sum_{j=1}^{n} f(I_D(v_j)).$$

∎

Lemma 2 only applies to a losing density l on a tournament T. It shows that given a node v_i, the probability that a node picked from l will beat v_i is at least twice as much in T^2 as in T. For example, the probability a node picked from the density in Figure 3(b) will beat v_6 is $\frac{4}{9}$ in T and $\frac{8}{9}$ in T^2.

Lemma 2 does not generalize to oriented graphs. For example, let D be a directed 4-cycle. Placing $\frac{1}{2}, 0, \frac{1}{2}$, and 0 on the consecutively labeled nodes v_1, v_2, v_3, and v_4, respectively, gives a losing density on D. But the probability a node picked from this density will beat v_4 is $\frac{1}{2}$ in both D and D^2.

Lemma 2. Let l be a losing density on a tournament T. Then $l(I_{T^2}(v_i)) \geq 2l(I_T(v_i))$ for all nodes v_i.

Proof. Since l is a losing density, $l(I_T(v_i)) \leq \frac{1}{2}$. If $l(I_{T^2}(v_i)) = 1$, we are done. Otherwise let Q be the subtournament $V(T) - I_{T^2}(v_i)$. Within Q, we have

$$\begin{aligned} \sum_{v_j \in V(Q)} l(v_j)l(I_Q(v_j)) &= \sum_{v_j \in V(Q)} \sum_{v_k \in I_Q(v_j)} l(v_j)l(v_k) \\ &= \sum_{v_k \in V(Q)} \sum_{v_j \in O_Q(v_k)} l(v_j)l(v_k) = \sum_{v_k \in V(Q)} l(v_k)l(O_Q(v_k)). \end{aligned}$$

Since $l(I_{T^2}(v_i)) < 1$ and hence $l(V(Q)) > 0$, we have $l(I_Q(v_h)) \geq l(O_Q(v_h))$ for some $v_h \in V(Q)$ with $l(v_h) > 0$.

Since $v_h \notin I_{T^2}(v_i)$, no node of $I_T(v_i)$ can lose to v_h. Then each node of $I_T(v_i)$ beats v_h because T is a tournament. So $I_T(v_i) \subseteq I_T(v_h)$. Since $I_T(v_i) \subseteq I_{T^2}(v_i)$, we then get $I_T(v_i) \subseteq I_T(v_h) \cap I_{T^2}(v_i)$. Since Q is the subtournament $V(T) - I_{T^2}(v_i)$, we have

$$l(I_T(v_h)) = l(I_Q(v_h)) + l(I_T(v_h) \cap I_{T^2}(v_i)) \geq l(I_Q(v_h)) + l(I_T(v_i)).$$

Similarly, $O_T(v_h) \subseteq O_T(v_i)$ and hence $O_T(v_h) \cap I_{T^2}(v_i) \subseteq O_T(v_i) \cap I_{T^2}(v_i) = I_{T^2}(v_i) - I_T(v_i)$. Therefore

$$l(O_T(v_h)) = l(O_Q(v_h)) + l(O_T(v_h) \cap I_{T^2}(v_i)) \leq l(O_Q(v_h)) + l(I_{T^2}(v_i) - I_T(v_i)).$$

Since $l(v_h) > 0$, the corollary in Section 2 gives $l(O_T(v_h)) = l(I_T(v_h))$. Thus

$$l(O_Q(v_h)) + l(I_{T^2}(v_i) - I_T(v_i)) \geq l(I_Q(v_h)) + l(I_T(v_i)).$$

Since $l(O_Q(v_h)) \geq l(I_Q(v_h))$, we then have $l(I_{T^2}(v_i) - I_T(v_i)) \geq l(I_T(v_i))$. The result then follows because $I_T(v_h) \subseteq I_{T^2}(v_h)$. ∎

Theorem 2 shows that for a node picked from a losing density on a tournament T, its expected outdegree in T^2 is at least twice what it is in T. So for the density in Figure 3(b), the expected outdegree in T^2 is $\frac{1}{3}\cdot 7 + \frac{1}{9}\cdot 4 + \frac{1}{9}\cdot 5 + \frac{1}{9}\cdot 5 + \frac{1}{3}\cdot 6 = 5\frac{8}{9}$ which is more than double the expected outdegree in T (calculated above to be $2\frac{7}{9}$).

Theorem 2. Let l be a losing density on a tournament T. Then $E_l(d_{T^2}) \geq 2E_l(d_T)$.

Proof. Lemma 1 (applied to both T and T^2) and Lemma 2 give

$$E_l(d_{T^2}) = \sum_{j=1}^{n} l(I_{T^2}(v_j)) \geq \sum_{j=1}^{n} 2l(I_T(v_j)) = 2E_l(d_T).$$

∎

Corollary. In any tournament T, there is a node v_i with $d_{T^2}(v_i) \geq 2d_T(v_i)$.

References

[1] N. Dean and B. Latka, Squaring a tournament—an open problem, *Congre. Numer.*, **109** (1995), 73–80.

[2] D. C. Fisher and J. Ryan, Optimal strategies for a generalized "Scissors, Paper and Stone" game, *Amer. Math. Monthly* **99** (1992), 935–942.

[3] G. Laffond, J. F. Laslier, and M. Le Breton, The bipartisan set of a tournament game, *Games and Economic Behavior* **5** (1993), 182–201.

[4] D. Solow, *Linear programming: An introduction to finite improvement algorithms.* North-Holland, Amsterdam (1984).

Received June 8, 1995

Bipartite Graphs and Their Endomorphism Monoids

Suohai Fan
DEPARTMENT OF MATHEMATICS
JINAN UNIVERSITY
GUANGZHOU, 510632
PEOPLES REPUBLIC OF CHINA
e-mail: tfsh@jnu.edu.cn

ABSTRACT

It is shown that any connected bipartite graph is determined by its endomorphism monoid up to isomorphism. © 1996 John Wiley & Sons, Inc.

1. INTRODUCTION

Throughout this paper, all graphs are finite, simple, and undirected. Let X be a graph. We denote its vertex set and edge set by $V(X)$ and $E(X)$ respectively. Write $\{x_1, x_2\} \in E(X)$ if vertices x_1 and x_2 are adjacent in X. Suppose X and Y are graphs. A homomorphism of X to Y is a mapping from $V(X)$ to $V(Y)$ such that $\{f(x_1), f(x_2)\} \in E(Y)$ whenever $\{x_1, x_2\} \in E(X)$. A homomorphism f of X to Y is an isomorphism if f is bijective and f^{-1} is also a homomorphism of Y to X. An endomorphism of X is a homomorphism of X to itself, and an automorphism is an isomorphism of X to itself. Denote by End X the set of endomorphisms of X and Aut X the set of automorphisms of X. Then End X forms a monoid under composition (the operations are written from right to left), and is called the endomorphism monoid of the graph X. Similarly, we have the automorphism group Aut X of X. Moreover, End $X \supseteq$ Aut X.

For references in Graph Theory and Semigroup Theory, see [3] and [4] respectively.

Since the endomorphism monoid End X captures more information about the structure of the graph X than the automorphism group Aut X does, it is sure that the study of graphs and their endomorphism monoids will be of much interest and yield fruitful results.

In this paper, we investigate the question of when End X determines X. An early result on determination by endomorphism monoids is for partially ordered sets (see L. M. Gluskin [2]). It asserts that two partially ordered sets S and T whose endomorphism monoids are isomorphic must themselves be either isomorphic or anti-isomorphic. Analogous results

Journal of Graph Theory Vol. 23, No. 1, 49–52 (1996)

CCC 0364-9024/96/010049-04