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Journal of Combinatorial Theory, Series B 87 (2003) 244–253
Journal of Combinatorial Theory Series B
http://www.elsevier.com/locate/jctb

Packing cycles in graphs, II

Guoli Ding,[a,*,1] Zhenzhen Xu,[b] and Wenan Zang[b,2]

[a] Mathematics Department, Louisiana State University, Baton Rouge, LA 70803-4918, USA
[b] Department of Mathematics, The University of Hong Kong, Hong Kong, China

Received 21 August 2001

Abstract

A graph G *packs* if for every induced subgraph H of G, the maximum number of vertex-disjoint cycles in H is equal to the minimum number of vertices whose deletion from H results in a forest. The purpose of this paper is to characterize all graphs that pack.

1. Introduction

This is a follow-up of a paper by Ding and Zang [3]. Like before, all graphs considered are finite, simple, and undirected. We first present the main result of [3].

Let $G = (V, E)$ be a graph with a nonnegative integral weight $w(v)$ on each $v \in V$. A collection $\mathscr{C}$ of cycles (repetition is allowed) of G is called a *cycle w-packing* if each vertex v of G is used at most $w(v)$ times by members of $\mathscr{C}$; a set X of vertices in G is called a *feedback set* if $G \backslash X$ is a forest. Let $\nu_w(G)$ denote the maximum size of a cycle w-packing and let $\tau_w(G)$ denote the minimum total weight of a feedback set. It is well known, and it is also easy to see, that $\nu_w(G) \leqslant \tau_w(G)$, while the equality does not have to hold in general. If G is a graph for which the equality $\nu_w(G) = \tau_w(G)$ holds for all nonnegative integral w, then G is called *cycle Mengerian* (CM). The main result of [3] is a characterization of CM graphs in terms of forbidden structures, which we define now. A *wheel* is a graph obtained from a cycle by adding a new vertex and making it adjacent to all vertices of the cycle. An *odd ring* is a graph obtained from an odd cycle, called the *base cycle*, by replacing each edge $e = uv$ with *either* a triangle

*Corresponding author.
E-mail address: gding1@lsu.edu (G. Ding).
[1] Research partially supported by NSF Grant DMS-9970329.
[2] Supported by the Research Grants Council of Hong Kong (Project No. HKU 7109/01P).

PII: S0095-8956(02)00007-2

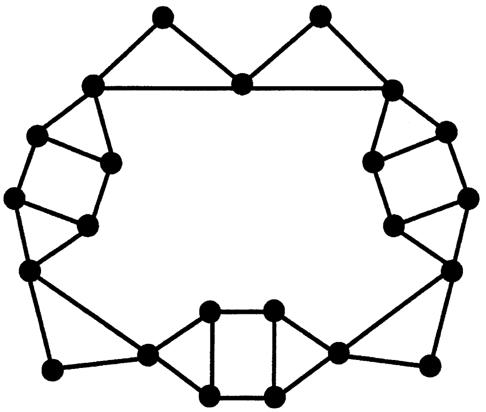

Fig. 1. An odd ring with base cycle C_7.

containing *e or* two triangles *uab, vcd* together with two additional edges *ac* and *bd* (see Fig. 1).

The following is the main theorem of [3].

(1.1) A graph is CM if and only if none of its induced subgraphs is isomorphic to a subdivision of $K_{2,3}$, a wheel, or an odd ring.

The present paper is concerned with graphs enjoying a similar min–max property. Let G be a graph. We will call a collection of vertex-disjoint cycles of G a *cycle packing* (instead of cycle 1-packing) of G. Then, let $v(G)$ denote the maximum size of a cycle packing in G and let $\tau(G)$ denote the minimum size of a feedback set in G. We say that G *packs* if $v(H) = \tau(H)$ for all induced subgraphs H of G. It is easy to see that G packs if and only if the equality $v_w(G) = \tau_w(G)$ holds for all $\{0,1\}$-valued w. Intuitively speaking, CM graphs are graphs that hold the desired min–max relation in the *weighted* case, while graphs that pack are the counterparts of CM graphs in the *unweighted* case.

The purpose of this paper is to prove a theorem similar to (1.1) that characterizes all graphs that pack. First, it is worth pointing out that when proving G, a subdivision of a wheel or an odd ring, is not CM, it was actually proved (cf. [3, Proof of Lemma 4.2]) that $v_w(G) < \tau_w(G)$, for $w \equiv 1$. Therefore, subdivisions of wheels and odd rings do not pack and thus should be excluded, as induced subgraphs. In addition, it is not difficult to see that, if G is a subdivision of $K_{3,3}$, then $v(G) = 1 < 2 = \tau(G)$. It follows that subdivisions of $K_{3,3}$ do not pack and thus should also be excluded. The next, our main result of this paper, states that these are the only graphs we need to exclude in order to characterize graphs that pack.

Theorem 1. *A graph packs if and only if none of its induced subgraphs is isomorphic to a subdivision of* $K_{3,3}$, *a wheel, or an odd ring.*

The rest of this paper is a proof of Theorem 1, using (1.1). Our proof is constructive and it yields a polynomial-time algorithm for finding, in graphs that pack, a maximum cycle packing as well as a minimum feedback set. Since converting our proof to an algorithm is quite standard, we will not discuss the algorithmic aspect any further, except for pointing out that, for CM graphs, both τ_w and v_w can be computed in polynomial time [3] while for general graphs computing τ and v are already NP-hard [4].

2. A Proof of Theorem 1

We begin by proving two lemmas.

Lemma 1. *Let x and y be two distinct vertices in a graph G, and let $\bar{G}$ be the graph obtained from G by introducing a new vertex z, then adding the edges zx, zy, and finally adding xy if x and y are nonadjacent in G. Suppose both G and $\bar{G}$ pack. If $v(G\backslash x) = v(G\backslash y) = v(G)$, then $v(G\backslash\{x,y\}) = v(G)$.*

Proof. Let $\mathscr{C}_x = \{C_1, C_2, \ldots, C_{v(G)}\}$ be a cycle packing in $G\backslash x$, let $\mathscr{C}_y = \{D_1, D_2, \ldots, D_{v(G)}\}$ be a cycle packing in $G\backslash y$, and let S be a minimum feedback set in $\bar{G}$. Let T be the triangle xyz and let $\mathscr{C} = \{T, C_1, \ldots, C_{v(G)}, D_1, \ldots, D_{v(G)}\}$ (notice that C_i and D_j should be viewed as two different members of $\mathscr{C}$ even though they may correspond to the same cycle in G). Then, as all members of $\mathscr{C}$ are cycles of $\bar{G}$, we observe that every member of $\mathscr{C}$ must intersect S. On the other hand, from the definitions of $\mathscr{C}_x$, $\mathscr{C}_y$, and $\bar{G}$, we also observe that each vertex in S is contained in at most two members of $\mathscr{C}$. Based on these two observations we deduce that $2|S| \geqslant |\mathscr{C}| = 2v(G) + 1$ and hence $|S| > v(G)$. Since $\bar{G}$ packs, $v(\bar{G}) = \tau(\bar{G}) = |S|$, so $v(\bar{G}) > v(G)$. Let $\mathscr{D}$ be a maximum cycle packing $\bar{G}$. Then the last inequality implies that some cycle C of $\mathscr{D}$ uses an edges in $E(\bar{G})\backslash E(G)$. Since $(\mathscr{D} - \{C\}) \cup \{T\}$ is a cycle packing of $\bar{G}$, we may thus assume that $T \in \mathscr{D}$. Consequently, $\mathscr{D}\backslash\{T\}$ is a cycle packing in $G\backslash\{x,y\}$ of size at least $v(G)$, which implies $v(G\backslash\{x,y\}) \geqslant v(G)$ and thus $v(G\backslash\{x,y\}) = v(G)$. □

For convenience, subdivisions of $K_{2,3}$, $K_{3,3}$, wheels, and odd rings will be called *Θ-graphs*, *K-graphs*, *W-graphs*, and *R-graphs*, respectively. We also simply say that a graph G *has* a graph H if H is isomorphic to an *induced* subgraph of G. The following is Lemma 3.1 in [3].

Lemma 2. *Let H be a subdivision of K_4 and let x and y be two of the four degree-three vertices. Let G be obtained from H by adding edges such that all these edges are incident with either x or y. Then G has a W-graph.*

Lemma 3. *Let G be a graph having neither K-graphs nor W-graphs. If G has a Θ-graph Σ which consists of three paths, P_1, P_2, P_3, linking distinct vertices x and y, then $P_i\backslash\{x,y\}$, $i = 1, 2, 3$, are contained in three different components of $G\backslash\{x,y\}$.*

Proof. For $i = 1, 2, 3$, let $P'_i = P_i \backslash \{x, y\}$. We remark that P'_i may consist of a single vertex. Suppose, on the contrary, that some distinct P'_i and P'_j are contained in the same component of $G \backslash \{x, y\}$. For convenience, let us choose the pair $\{i, j\}$ such that the shortest path Q linking P'_i and P'_j in $G \backslash \{x, y\}$ is as short as possible. Rename the subscripts if necessary, we may assume that $i = 1$ and $j = 2$. Note that Q is an induced path. Let $z_0, z_1, \dots, z_p, z_{p+1}$ be the vertices of Q such that $z_0 \in V(P'_1)$, $z_{p+1} \in V(P'_2)$, and they are ordered as in Q. By the minimality of Q, no z_i $(i > 1)$ has a neighbor in P'_1 and no z_i $(i < p)$ has a neighbor in P'_2.

Suppose some z_i is adjacent to a vertex z on P'_3. Then $1 \leqslant i \leqslant p$ as Σ is an induced subgraph of G. Since $z_0 z_1 \dots z_i z$ is a path linking P'_1 and P'_3 of length $i + 1$, and $z z_i z_{i+1} \dots z_{p+1}$ is a path linking P'_2 and P'_3 of length $p + 2 - i$, in view of the minimality of Q (which has length $p + 1$), we have $p + 1 \leqslant \min\{i + 1, p + 2 - i\}$. Thus $p = i = 1$, in other words, Q is of length two and z_1 is adjacent to z. If z_1 is adjacent to at least three vertices in $V(P_i) \cup V(P_j)$ for some $i \neq j$, then $V(P_i) \cup F(P_j) \cup \{z_1\}$ induces a W-graph in G; else, z_1 is adjacent to no vertices in $V(P_1) \cup V(P_2) \cup F(P_3)$, except for z_0, z_2 and z. Thus $V(P_1) \cup V(P_2) \cup V(P_3) \cup \{z_1\}$ induces a K-graph in G. So we reach a contradiction in either case, and hence we may assume hereafter that no vertex on Q is adjacent to any vertex on P'_3. Let us now distinguish among three cases.

Case 1: z_1 has three or more neighbors in P_1, or z_p has three or more neighbors in P_2. In this case, $V(P_1) \cup V(P_3) \cup \{z_1\}$ or $V(P_2) \cup V(P_3) \cup \{z_p\}$ induces a W-graph, a contradiction.

Case 2: z_1 has precisely one neighbor in P'_1 and z_p has precisely one neighbor in P'_2. In this case, let H denote the K_4 subdivision consisting of Σ and Q. Then it is easy to see that, in the subgraph induced by $V(H)$, edges not in $E(H)$ must have one end in $\{x, y\}$ and the other in $V(Q) \backslash \{z_0, z_{p+1}\}$. By Lemma 2, we deduce that G has a W-graph.

Case 3: If neither of the previous cases occurs, then, by symmetry, we may assume that z_1 has precisely two neighbors on P_1 and neither of them is x or y. It follows that $p \geqslant 2$, for otherwise $V(P_1) \cup V(P'_2) \cup \{z_1\}$ would induce a W-graph in G. Next, observe that some z_i, with $2 \leqslant i \leqslant p$, is adjacent to a vertex in $V(P_2) \backslash \{z_{p+1}\}$, since otherwise $V(P_1) \cup V(P_2) \cup V(Q)$ would also induce a W-graph in G. Let R be a shortest path between z_1 and $\{x, y\}$, which only use vertices in $(V(Q) \backslash \{z_0\}) \cup V(P_2)$. If j is the largest subscript such that $z_j \in V(R)$, then z_j is adjacent to some vertex on P_2 and so $j \geqslant 2$. It is easy to see that a W-graph in G is induced by $V(P_1) \cup \{z_1, z_2, \dots, z_j\}$ if z_j is adjacent to both x and y, and by $V(P_1) \cup V(P_3) \cup V(R)$ if z_j is nonadjacent to x or y; this contradiction completes the proof of Lemma 3. □

We are now ready to establish the main result. An induced subgraph is called an *obstruction* if it is a K-graph, a W-graph, or an R-graph.

Proof of Theorem 1. The "only if" part has been justified before stating the theorem. Here we prove the "if" part. Let $G = (V, E)$ be a graph having no obstructions. To show G packs, we apply induction on $|V|$. The statement clearly holds when $|V| = 1$. So we proceed to the induction step. In view of the induction hypothesis, it suffices to verify that $\nu(G) = \tau(G)$.

If G has no Θ-graphs, then the desired statement follows directly from (1.1). So we assume that G has a Θ-graph Σ. Let x, y be the two branch vertices of Σ and let P_1, P_2, and P_3 be the three paths linking x and y in Σ. By Lemma 3, $P_1 \backslash \{x, y\}$, $P_2 \backslash \{x, y\}$, and $P_3 \backslash \{x, y\}$ are contained in three different components of $G \backslash \{x, y\}$, say C_1, C_2, and C_3, respectively. Let H_i be the subgraph of G induced by $V(C_i) \cup \{x, y\}$. Rename the subscripts if necessary, we may assume that

(2.1) $|V(H_1)| \leqslant |V(H_2)| \leqslant |V(H_3)|$.

In addition, we may also assume that

(2.2) $|V(H_1) \backslash \{x, y\}| \geqslant 2$.

If $H_1 \backslash \{x, y\}$ has only one vertex, say z, then P_1 is the path xzy. Let F denote the graph obtained from G be replacing the path P_1 with the edge xy. Clearly, G is a subdivision of F. It follows that F has no obstructions, and it also follows that $\nu(F) = \nu(G)$ and $\tau(F) = \tau(G)$. Since $|V(F)| < |V(G)|$, we conclude from the induction hypothesis that F packs. Therefore, $\nu(F) = \tau(F)$, which implies $\nu(G) = \tau(G)$, and thus we may assume (2.2) holds.

(2.3) If $\nu(H_1 \backslash x) = \nu(H_1 \backslash y) = \nu(H_1)$ then $\nu(H_1 \backslash \{x, y\}) = \nu(H_1)$.

To justify it, let $\bar{H}_1$ be the graph obtained from H_1 by introducing a new vertex z and then adding the edges zx, zy and xy. Clearly, $F = H_1 \cup P_2 \cup P_3$ is a subdivision of $\bar{H}_1$. It follows that $\bar{H}_1$ has no obstructions. Since $|V(H_1)| \leqslant |V(\bar{H}_1)| < |V(F)| \leqslant |V(G)|$, by induction hypothesis both H_1 and $\bar{H}_1$ pack. Thus (2.3) follows from Lemma 1.

Let us now make two copies of H_1, denoted by H_{11} and H_{12}. For $i = 1, 2$, let x_i and y_i be the two vertices in H_{1i} that correspond to x and y, respectively. Let H be the graph obtained from the vertex-disjoint union of H_{11} and H_{12} by identifying x_1 and y_2 (with a as the new vertex), and y_1 and x_2 (with b as the new vertex). In the following, we prove some properties of H.

(2.4) *H packs.*

By (2.1) and our induction hypothesis, we only need show that H has no obstructions. Suppose the contrary: H has an obstruction Q. We aim to show that G has an obstruction. For $i = 1, 2$, let Q_i be the induced subgraph of Q in H_{1i}. Then $V(Q_i) \backslash \{a, b\} \neq \emptyset$, for otherwise Q is entirely contained in H_{11} or H_{12}, thus Q is an induced subgraph of G, contradicting the assumption on G. It follows that $\{a, b\} \subseteq V(Q_i)$ as Q is 2-connected. If one of Q_1 and Q_2, say Q_1, is a path, let $\bar{Q}$ denote the graph obtained from Q by replacing Q_1 with P_2, then $\bar{Q}$ is an induced subgraph of G and it is also an obstruction, a contradiction. So we may assume that neither of Q_1 and Q_2 is a path. From the definitions of a K-graph, a W-graph, and an R-graph, it follows instantly that Q must be an R-graph, and $\{a, b\}$ separates the base cycle of this R-graph into even and odd two paths (recall the definition of an odd ring). Without loss of generality, let us assume that Q_1 corresponds to the odd path. Let $\bar{Q}$ be the graph obtained from Q by replacing Q_1 with the cycle $P_2 \cup P_3$. Then $\bar{Q}$ is an R-graph of G; this contradiction completes the proof of (2.4).

(2.5) $\nu(H) \leqslant 2\nu(H_1) + 1$, and equality holds only if H_1 has a maximum cycle packing $\mathscr{D}$ and a path P connecting x and y such that $\mathscr{D}$ is contained in $H_1 \backslash V(P)$.

Let $\mathscr{C}$ be a maximum cycle packing in H, and let $\mathscr{C}_i$ be the collection of all cycles in $\mathscr{C}$ that are entirely contained in H_{1i}. If $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)$ has a cycle, say C, then C must pass through both a and b and thus C is the only cycle in $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)$ as cycles in $\mathscr{C}$ are pairwise vertex disjoint. Hence $\nu(H) = |\mathscr{C}| \leqslant |\mathscr{C}_1| + |\mathscr{C}_2| + 1 \leqslant 2\nu(H_1) + 1$. If the equality holds, then $|\mathscr{C}_1| = |\mathscr{C}_2| = \nu(H_1)$ and $|\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)| = 1$. Thus $\mathscr{C}_1$ corresponds to a maximum cycle packing $\mathscr{D}$ in H_1, and one portion of the unique cycle in $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)$ corresponds to a path P linking x and y in H_1 such that $\mathscr{D}$ is contained in $H_1 \backslash V(P)$. So (2.5) is established.

(2.6) $\tau(H) \geqslant 2\tau(H_1) - 2$, and equality holds only if H_1 has a minimum feedback set T which contains both x and y.

Let S be a minimum feedback set in H and let $S_i = S \cap V(H_{1i})$. Then S_i is a feedback set in H_{1i} and $S_1 \cap S_2 \subseteq \{a, b\}$. Note that $|S_1| \geqslant \tau(H_{11})$, $|S_2| \geqslant \tau(H_{12})$, and $|S_1 \cap S_2| \leqslant 2$. So $\tau(H) = |S| = |S_1| + |S_2| - |S_1 \cap S_2| \geqslant 2\tau(H_1) - 2$. If the equality holds then $|S_1| = |S_2| = \tau(H_1)$ and $|S_1 \cap S_2| = 2$. Thus S_1 corresponds to a minimum feedback set T of H_1 which contains both x and y. So (2.6) is proved.

(2.7) If $\tau(H) = 2\tau(H_1) - 1$, then H_1 has two minimum feedback sets T_1 and T_2 such that $x \in T_1 \backslash T_2$ and $y \in T_2 \backslash T_1$. Moreover, no minimum feedback set in H_1 contains both x and y.

Let S, S_1, and S_2 be as in the proof of (2.6). We claim that S contains at most one vertex from $\{a, b\}$, for otherwise, we have $\{a, b\} \subseteq S_1 \cap S_2$. Since $|S| = 2\tau(H_1) - 1$, it follows from the pigeonhole principle that $|S_1 \backslash \{a, b\}| \leqslant \tau(H_1) - 2$, or $|S_2 \backslash \{a, b\}| \leqslant \tau(H_1) - 2$, say the former. Then $|S_1| \leqslant \tau(H_1)$ and thus equality must hold, which implies that S_1 is a minimum feedback set in H_{11} that contains both a and b. Now let $\bar{S}_2$ be the set of all the vertices in H_{12} that correspond to those in S_1 (recall that both H_{11} and H_{12} are isomorphic to H_1). Then $S_1 \cup \bar{S}_2$ is a feedback set in H of size $2\tau(H_1) - 2$, contradicting the hypothesis $\tau(H) = 2\tau(H_1) - 1$, and so the claim is proved. (Similarly, we can prove that no minimum feedback set in H_1 contains both x and y.) From this claim we conclude that $|S_1 \cap S_2| \leqslant 1$. Once again using $|S| = |S_1| + |S_2| - |S_1 \cap S_2|$ and $|S| = \tau(H) = 2\tau(H_1) - 1$, we obtain $|S_1| = |S_2| = \tau(H_1)$ and $|S_1 \cap S_2| = 1$. By symmetry we may assume that $S_1 \cap S_2 = \{a\}$. Then $b \notin S_1 \cup S_2$ according to the above claim. Thus S_1 and S_2 correspond to two minimum feedback sets T_1 and T_2, respectively, in H_1 such that $x \in T_1 \backslash T_2$ and $y \in T_2 \backslash T_1$. So (2.7) is justified.

(2.8) If $\tau(H) = 2\tau(H_1)$ and H has a minimum feedback set S with $S \cap \{a, b\} \neq \emptyset$, then $\nu(H_1 \backslash \alpha) < \nu(H_1)$ holds for precisely one $\alpha \in \{x, y\}$.

We first show that the inequality $\nu(H_1 \backslash \alpha) < \nu(H_1)$ holds for at least one $\alpha \in \{x, y\}$. Suppose the contrary that $\nu(H_1 \backslash x) = \nu(H_1 \backslash y) = \nu(H_1)$. Then, by (2.3), we have $\nu(H_1 \backslash \{x, y\}) = \nu(H_1)$. Let $S_i = S \cap V(H_{1i})$, for $i = 1, 2$. Since $|S| = 2\tau(H_1)$ and $S \cap \backslash \{a, b\} \neq \emptyset$, we must have $|S_1 \backslash \{a, b\}| < \tau(H_1)$ or $|S_2 \backslash \{a, b\}| < \tau(H_1)$, say the former. Now we have a contradiction from $\nu(H_1 \backslash \{x, y\}) = \nu(H_1) = \tau(H_1) > |S_1 \backslash \{a, b\}| \geqslant \tau(H_1 \backslash \{x, y\})$.

Without loss of generality, assume $v(H_1 \backslash x) < v(H_1)$. Next, we show that $v(H_1 \backslash y) = v(H_1)$. Let $\mathscr{C}$ be a maximum cycle packing in H. By (2.4) and the assumption in (2.8), $|\mathscr{C}| = v(H) = \tau(H) = 2\tau(H_1) = 2v(H_1)$. For $i = 1, 2$, let $\mathscr{C}_i$ be the set of cycles of $\mathscr{C}$ that are contained in H_{1i}. Since each cycle in $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)$ passes through both a and b, $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2)$ contains at most one cycle as cycles in $\mathscr{C}$ are pairwise vertex-disjoint. Thus $|\mathscr{C}_1| + |\mathscr{C}_2| \geqslant |\mathscr{C}| - 1 = 2v(H_1) - 1$, implying $|\mathscr{C}_1| - v(H_1)$ or $|\mathscr{C}_2| = v(H_1)$, say the former. From $v(H_1 \backslash x) < v(H_1)$ it follows that $\mathscr{C}_1$ has a cycle that passes through a. Therefore, $\mathscr{C} \backslash (\mathscr{C}_1 \cup \mathscr{C}_2) = \emptyset$, which, in turn, implies that $|\mathscr{C}_2| = v(H_1)$, and hence, by $v(H_1 \backslash x) < v(H_1)$ again, $\mathscr{C}_2$ contains a cycle which passes through b. Consequently, b is not contained in any cycle of $\mathscr{C}_1$ and thus $v(H_1 \backslash y) = |\mathscr{C}_1| = v(H_1)$, which finishes the proof of (2.8).

(2.9) If $\tau(H) = 2\tau(H_1)$ and $S \cap \{a, b\} = \emptyset$ for all minimum feedback sets S of H, then $\tau(H_1 \backslash \{x, y\}) = \tau(H_1)$, and x and y are contained in different components of $H_1 \backslash T$ for some minimum feedback set T of H_1 with $T \cap \{x, y\} = \emptyset$.

Let S' be a minimum feedback set in $H_1 \backslash \{x, y\}$ and, for $i = 1, 2$, let S_i be the copy of S' in H_{1i}. Then $S = S_1 \cup S_2 \cup \{a, b\}$ is a feedback set in H. Clearly, S is not minimum since S meets $\{a, b\}$. It follows that $2\tau(H_1) = \tau(H) < |S| = 2\tau(H_1 \backslash \{x, y\}) + 2$ and so $\tau(H_1 \backslash \{x, y\}) = \tau(H_1)$.

Next, let S be a minimum feedback set in H, and, for $i = 1, 2$, let $S_i = S \cap V(H_{1i})$. Then $S_i \cap \{a, b\} = \emptyset$. Since $|S| = 2\tau(H_1)$ and $|S_i| \geqslant \tau(H_1)$, we have $|S_1| = |S_2| = \tau(H_1)$. Hence S_i is a minimum feedback set in H_{1i}. If each of $H_{11} \backslash S_1$ and $H_{12} \backslash S_2$ contains a path between a and b. Then the union of these two paths yields a cycle in $H \backslash S$, contradicting the hypothesis that S is a feedback set in H. Hence one of $H_{11} \backslash S_1$ and $H_{12} \backslash S_2$ contains no path between a and b, say the former. Then S_1 corresponds to a minimum feedback set T of H_1 with the desired property. The proof of (2.9) is now complete.

Recall that our goal is to prove $v(G) = \tau(G)$. Since $v(G) \leqslant \tau(G)$ is always true, we only need to prove in the following that $\tau(G) \leqslant v(G)$. To this end, let us apply reduction methods. By (2.4), (2.5), and (2.6), $2\tau(H_1) - 2 \leqslant \tau(H) \leqslant 2\tau(H_1) + 1$. Depending on the relationship between $\tau(H)$ and $\tau(H_1)$, we distinguish among the following four cases.

Case 1: $\tau(H) = 2\tau(H_1) + 1$. Let F be the graph obtained from $G \backslash V(H_1 \backslash \{x, y\})$ by adding the edge $e = xy$. Then $(F \backslash e) \cup P_1$ is an induced subgraph of G and it is also a subdivision of F. It follows that F has no obstructions. Since $|V(F)| < |V(G)|$, by induction, F packs. To settle Case 1, we prove the following claim.

(2.10) $\tau(G) \leqslant \tau(F) + \tau(H_1)$ and $v(F) + v(H_1) \leqslant v(G)$.

To prove the first inequality, let S and T be minimum feedback sets in F and H_1, respectively. For any cycle C in G, if C is entirely contained in $F \backslash e$ or in H_1, then C is covered by S or T; if C is not entirely contained in $F \backslash e$ nor in H_1, then C passes through both x and y. Denote by $\bar{C}$ the cycle obtained from C by replacing its portion in H_1 with the edge e. Then $\bar{C}$ is a cycle in F which is covered by S, so C intersects S. Thus, we can conclude that $S \cup T$ is a feedback vertex of G, and hence $\tau(G) \leqslant |S| + |T| = \tau(F) + \tau(H_1)$.

Next, since we are in Case 1, we deduce from (2.4) that $v(H) = 2v(H_1) + 1$. It follows that we can choose $\mathscr{D}$ and P as in (2.5). Let $\mathscr{C}$ be a maximum cycle packing in F. Then we define a cycle packing $\mathscr{B}$ in $(F\backslash e) \cup P$ as follows: put $\mathscr{B} = \mathscr{C}$ if no cycle in $\mathscr{C}$ contains e; else, let C be the cycle in $\mathscr{C}$ containing e, and let $\bar{C}$ be the cycle obtained from C by replacing e with P. Set $\mathscr{B}$ to be the cycle packing obtained from $\mathscr{C}$ by replacing C with $\bar{C}$. Then $\mathscr{B} \cup \mathscr{D}$ is a cycle packing in G. Hence $v(F) + v(H_1) = |\mathscr{B}| + |\mathscr{D}| \leqslant v(G)$, and so the proof of (2.10) is complete.

Since both F and H_1 pack, $\tau(F) = v(F)$ and $\tau(H_1) = v(H_1)$. By (2.10), we thus have the desired inequality $\tau(G) \leqslant v(G)$ in Case 1.

Case 2: $\tau(H) = 2\tau(H_1) - 2$. Let $F = G\backslash V(H_1)$. Then F packs for it is a proper induced subgraph of G. Let $\mathscr{C}$ and $\mathscr{D}$ be maximum cycle packings in F and H_1, respectively. Let S be a minimum feedback set in F. In addition, by (2.6), we can choose a minimum feedback set T of H_1 with $\{x, y\} \subseteq T$. Clearly, $S \cup T$ is a feedback set in G and $\mathscr{C} \cup \mathscr{D}$ is a cycle packing in G. Hence $\tau(G) \leqslant |S| + |T| = \tau(F) + \tau(H_1)$ and $v(F) + v(H_1) = |\mathscr{C}| + |\mathscr{D}| \leqslant v(G)$. Now, similar to the proof in Case 1, we immediately have $\tau(G) \leqslant v(G)$, which settles Case 2.

Case 3: $\tau(H) = 2\tau(H_1) - 1$. Let F be the graph obtained from the graph $G\backslash V(H_1\backslash\{x, y\})$ by introducing a new vertex z and then adding the edges zx, zy and xy. Let us show that

(2.11) *F packs*.

By (2.2), $|V(F)| < |V(G)|$. Hence, by the induction hypothesis, we only need to show that F has no obstructions. Suppose the contrary that F has an obstruction Q. We aim to show that G also has an obstruction. If $z \notin V(Q)$ and $xy \notin E(Q)$, then Q is an obstruction of G; if $z \notin V(Q)$ and $xy \in E(Q)$, denote by $\bar{Q}$ the graph obtained from Q by replacing edge xy with the path P_1, then $\bar{Q}$ is an obstruction of G, we thus reach a contradiction in either case. So Q must contain z. Since z has only two neighbors, x and y, in F and since $\{x, y, z\}$ induces a triangle, from the structures of the obstructions it can be seen that Q is an R-graph and it contains the triangle xyz. Consequently, $Q' = Q\backslash\{x, y, z\}$ is a connected subgraph of $G\backslash\{x, y\}$. Recall that $P_1\backslash\{x, y\}$, $P_2\backslash\{x, y\}$, and $P_3\backslash\{x, y\}$ are contained, respectively, in three different components, C_1, C_2, and C_3 of $G\backslash\{x, y\}$. It follows that $V(Q')$ is disjoint from $V(C_i)$ for $i = 2$ or 3. Thus $\bar{Q}$, the graph obtained from Q by replacing the triangle xyz with the cycle $P_1 \cup P_i$, is an R-graph in G; this contradiction completes the proof of (2.11).

Similar to the proofs in the last two cases, we prove the following, which implies $\tau(G) \leqslant v(G)$.

(2.12) $\tau(G) \leqslant \tau(F) + \tau(H_1) - 1$ and $v(F) + v(H_1) - 1 \leqslant v(G)$.

By (2.7), H_1 has two minimum feedback sets T_1 and T_2 with $x \in T_1\backslash T_2$ and $y \in T_2\backslash T_1$. Let S be a minimum feedback set in F such that $|S \cap \{x, y\}|$ is maximized. Then at least one of x and y is in S, for otherwise $z \in S$ as the triangle xyz is covered by S. Now it is easy to see that $\bar{S} = (S\backslash\{z\}) \cup \{x\}$ is also a minimum feedback set in F, but $|\bar{S} \cap \{x, y\}| > |S \cap \{x, y\}|$, contradicting the assumption on S. By symmetry, let $x \in S$. Then it is not difficult to see that $S \cup T_1$ is a feedback set in G, and so $\tau(G) \leqslant |S \cup T_1| \leqslant \tau(F) + \tau(H_1) - 1$.

Now let $\mathscr{C}$ be a maximum cycle packing in F. We may assume that if the vertex z or the edge xy is contained in a cycle C in $\mathscr{C}$, then C is the triangle xyz, for otherwise we can replace C by the triangle xyz to get a new maximum cycle packing in F with the desired property. By (2.7), no minimum feedback set in H_1 contains both x and y, in other words, $\tau(H_1 \backslash \{x,y\}) \geqslant \tau(H_1) - 1$. Since both $H_1 \backslash \{x,y\}$ and H_1 pack, we have $\nu(H_1 \backslash \{x,y\}) \geqslant \nu(H_1) - 1$, which guarantees the existence of a cycle packing $\mathscr{B}$ in $H_1 \backslash \{x,y\}$ with $|\mathscr{B}| - \nu(H_1) - 1$. Once again let $\mathscr{D}$ be a maximum cycle packing in H_1. Observe that if triangle $xyz \in \mathscr{C}$, then $(\mathscr{C} \backslash \{xyz\}) \cup \mathscr{D}$ is a cycle packing in G with size $\nu(F) + \nu(H_1) - 1$; if $xyz \notin \mathscr{C}$, then the assumption on $\mathscr{C}$ implies that $\mathscr{C}$ is a cycle packing in G, and so $\mathscr{B} \cup \mathscr{C}$ is a cycle packing in G with size $\nu(F) + \nu(H_1) - 1$. Thus we always have $\nu(F) + \nu(H_1) - 1 \leqslant \nu(G)$, which proves (2.12) and completes the proof for Case 3.

Case 4: $\tau(H) = 2\tau(H_1)$.

We consider two subcases.

Case 4.1: *H has a minimum feedback set that intersects* $\{a,b\}$. By (2.8), we may assume that $\nu(H_1 \backslash x) < \nu(H_1) = \nu(H_1 \backslash y)$ and so $\tau(H_1 \backslash x) < \tau(H_1)$, as H_1 packs. Let $\mathscr{D}$ be a maximum cycle packing in $H_1 \backslash y$ and let T be a minimum feedback set in H_1. Then $|\mathscr{D}| = \nu(H_1)$ and $x \in T$. Clearly, $F = G \backslash V(H_1 \backslash y)$ packs since it is a proper induced subgraph of G. Let $\mathscr{C}$ and S be a maximum cycle packing and a minimum feedback set in F, respectively. Then it is routine to check that $\mathscr{C} \cup \mathscr{D}$ is a cycle packing in G, and $S \cup T$ is a feedback set in G. Thus $\tau(G) \leqslant |S| + |T| = \tau(F) + \tau(H_1) = \nu(F) + \nu(H_1) = |\mathscr{C}| + |\mathscr{D}| \leqslant \nu(G)$, which settles case 4.1.

Case 4.2: *All minimum feedback sets of H are disjoint from* $\{a,b\}$. Let $\mathscr{D}$ be a maximum cycle packing in $H_1 \backslash \{x,y\}$ and let T be a minimum feedback set in H_1 as chosen in (2.9). Let $F = G \backslash V(H_1 \backslash \{x,y\})$. Then F packs since it is a proper induced subgraph of G. Now let $\mathscr{C}$ and S be a maximum cycle packing and a minimum feedback vertex set in F, respectively. It follows that $\mathscr{C} \cup \mathscr{D}$ is a cycle packing in G (by the definitions of $\mathscr{C}$ and $\mathscr{D}$), and $S \cup T$ is a feedback set in G (note that since there is no path linking x and y in $H_1 \backslash T$, every cycle of G which is not entirely contained in F nor in H_1 must intersect T). Similar to the proof in case 4.1, we have $\tau(G) \leqslant \nu(G)$.

The proof of Theorem 1 is complete. □

3. Concluding remarks

Graphs with the min–max relation on feedback sets and cycle packings in both weighted and unweighted cases are characterized in our two papers. The closely related problems of describing *digraphs* with the same min–max properties have also attracted much attention, see, for instances, [1,2,5–8]. However, the general problems remain open, and certainly they deserve more research efforts.

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