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NOTE

DENSE MINORS IN GRAPHS OF LARGE GIRTH

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We show that a graph of girth greater than $6\log k+3$ and minimum degree at least 3 has a minor of minimum degree greater than k. This is best possible up to a factor of at most 9/4. As a corollary, every graph of girth at least $6\log r+3\log\log r+c$ and minimum degree at least 3 has a K_r minor.

1. Introduction

Thomassen [9] proved that, in graphs of minimum degree at least 3, sufficiently high girth forces a minor of any given minimum degree:

Theorem (Thomassen 1983). *For any integer k, every graph G of girth $g(G)\geq 4k-3$ and $\delta(G)\geq 3$ has a minor H with $\delta(H)\geq k$.*

Our aim in this note is to reduce the upper bound for the required girth to the correct order of magnitude:

Theorem 1. *For any integer k, every graph G of girth $g(G)>6\log k+3$ and $\delta(G)\geq 3$ has a minor H with $\delta(H)>k$.*

The best lower bound implied by known examples is $\frac{8}{3}\log k-c$, but we note that existing conjectures about cubic graphs of large girth would raise this to about $4\log k$.

Since an average degree of at least $cr\sqrt{\log r}$ forces a K_r minor [5,10], Theorem 1 has the following consequence:

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Corollary 2. *There exists a constant $c \in \mathbb{R}$ such that every graph G of girth $g(G) \geq 6\log r + 3\log\log r + c$ and $\delta(G) \geq 3$ has a K_r minor.* ∎

Asymptotically, Thomason [11] showed that a K_r minor is forced by an average degree of $(d + o(1))r\sqrt{\log r}$, where $d = 0.53131\ldots$ is an explicit constant that is best possible. This means that, for large enough r, Corollary 2 holds with $c = -2.4742$.

We adopt the notation of [4]. All our logarithms are binary, all graphs considered finite, and $0 \in \mathbb{N}$.

2. A lower bound

Minimum-order cubic graphs of girth at least some given integer g are called *g-cages* and have been studied in some detail (see [1] for an overview). Their exact order is known for $g \leq 12$. The best more general upper bound for the order of g-cages is due to Biggs & Hoare [2] and Weiss [12]:

Lemma 2.1. *There is a constant $c^* > 0$ such that for infinitely many integers g there exists a cubic graph of girth at least g and order at most $c^* 2^{3g/4}$.*

Now suppose that a graph G as in Lemma 2.1 has a minor of minimum degree k (say). Then G has at least $k+1$ branch sets, each of which sends out at least k edges and hence contains at least $k-2$ vertices (since G is cubic). Therefore

$$(k+1)(k-2) \leq |G| \leq c^* 2^{3g/4},$$

giving $g \geq \frac{8}{3}\log k - c$ for a suitable constant c. Choosing $k = k(g) \in \mathbb{N}$ maximal with this last inequality, we can thus deduce from Lemma 2.1 the following counterpart to Theorem 1:

Proposition 2.2. *There is a constant $c \in \mathbb{R}$ such that for infinitely many $k \in \mathbb{N}$ there exist cubic graphs of girth at least $\frac{8}{3}\log k - c$ that have no minor H with $\delta(H) > k$.* ∎

Any improvement on the bound in Lemma 2.1 will result in a corresponding improvement to Proposition 2.2. It has been conjectured (see [3] or [8]) that g-cages exist on as few as about $2^{g/2}$ vertices. This would increase our lower girth bound to $4\log k - c$.

3. The upper bound

In this section we prove Theorem 1. Following Mader [7], we start from the observation that in a graph G of girth $g(G)>2d+1$ and $\delta(G)\geq 3$ the d-ball around a vertex x is a tree T_x sending at least $|T_x|-2$ edges to the rest of G. Our main effort will go into proving that, depending on our lower bound for $g(G)$, not too many of these edges can go to the same tree T_y. Then partitioning $V(G)$ into such trees and contracting these will give us a minor of large minimum degree.

Given a tree T with root r and vertices $t,t'\in T$, we say that t' lies *above* t in T (and t *below* t') if $t\leq t'$ in the tree-order on $V(T)$ associated with r, ie. if t separates t' from r in T. Any neighbour of t above it is a *successor* of t in T, its unique neighbour below is its *predecessor*. For $i\in\mathbb{N}$ we write L^i_T for the set of *leaves* (maximal elements) of T at distance i from r.

Given a graph G, a vertex $x\in G$, and $d\in\mathbb{N}$, let us write $V^d_{G,x}$ for the set of vertices of G at distance exactly d from x. We need the following easy lemma:

Lemma 3.1. *Let T be a tree with root r in which no vertex has exactly one successor, and let $d\in\mathbb{N}$. Then $\sum_{i\geq d}2^{d-i}|L^i_T|\geq|V^d_{T,r}|$.* ∎

We are now ready to prove our main result, which we restate:

Theorem 1. *For any integer k, every graph G of girth $g(G)>6\log k+3$ and $\delta(G)\geq 3$ has a minor H with $\delta(H)>k$.*

Proof. Put $\lfloor\log k\rfloor =: d$. Let X be a maximal set of vertices such that $d(x,y)>2d$ for all distinct $x,y\in X$. Beginning with $T^0_x:=\{x\}$, let us define trees T^i_x rooted at x, for all $x\in X$ and $i=0,\dots,2d$. Assume that for some i the T^i_x have been defined and partition the set of vertices of G at distance at most i from X. We then add each vertex v at distance $i+1$ from X to one T^i_x to which it is adjacent, thereby obtaining a similar set of disjoint trees T^{i+1}_x. By the choice of X, the trees $T_x:=T^{2d}_x$ partition the entire vertex set of G, and

(1) T_x *contains all the vertices of G at distance at most d from x.*

As $g(G)>4d+1$, the T_x are induced subgraphs in G. Finally, we have

(2) $d(w,y)\leq d(v,x)+1$ *whenever* $vw\in E(G)$ *with* $v\in T_x$ *and* $w\in T_y$,

as otherwise w would have been added to T_x after v rather than to T_y.

Let us use Lemma 3.1 to estimate the number of edges leaving a tree T_x. For all $i \in \mathbb{N}$ let

$$E_x^i := \{vw \in E(G) \mid v \in T_x,\ w \in G - T_x,\ d(v,x) = i\}\,.$$

Let T'_x denote the subgraph of G induced by T_x and all its neighbours in G. As $g(G) > 4d+3$, T'_x is again a tree. Every vertex $v \in T_x$ has degree $d_G(v) \geq 3$ in T'_x, while all the vertices of $T'_x - T_x$ are leaves in T'_x. As $|E_x^i| = |L_{T'_x}^{i+1}|$ for all i, and $|L_{T'_x}^d| = 0$ by (1), Lemma 3.1 yields

$$\sum_{i \geq d} 2^{d-i-1} |E_x^i| = \sum_{i \geq d} 2^{d-i-1} |L_{T'_x}^{i+1}| = \sum_{i \geq d} 2^{d-i} |L_{T'_x}^i| \geq |V_{T'_x,x}^d| = |V_{G,x}^d|.$$

Multiplying by 2^{d+1} and setting $V_x^d := V_{G,x}^d$ we obtain

$$\sum_{i \geq d} 2^{2d-i} |E_x^i| \geq 2^{d+1} |V_x^d|.$$

Every edge in E_x^i joins T_x to a tree T_y distinct from T_x. This defines a partition of E_x^i into sets $A_{x,y}^i$ $(y \in X \setminus \{x\})$. Then the above inequality can be rewritten as

$$2^{d+1} |V_x^d| \leq \sum_y \sum_{i \geq d} 2^{2d-i} |A_{x,y}^i|\,, \tag{3}$$

where the first sum is taken over all $y \in X \setminus \{x\}$ such that G contains a T_x–T_y edge. We shall prove that, for each of these y,

$$\sum_{i \geq d} 2^{2d-i} |A_{x,y}^i| \leq |V_x^d|\,, \tag{4}$$

so that (3) can be satisfied only if there are at least 2^{d+1} distinct y, ie. if T_x sends edges to at least 2^{d+1} other trees T_y. Contracting all the trees T_x with $x \in X$ we then obtain a minor of G of minimum degree at least $2^{d+1} > k$, as desired.

For the proof of (4) let now x and y be fixed distinct vertices in X. Consider a T_x–T_y edge $e = vw$ of G, with $v \in T_x$ and $w \in T_y$ say. Then $i := d(v,x) \geq d$, by (1) and $w \notin T_x$. Let z_e be the vertex below v in T_x at distance d from v, ie. in V_x^{i-d}, and let B_e be the set of vertices in V_x^d that lie above z_e in T_x. These vertices have distance $2d - i$ from z_e, so

$$|B_e| \geq 2^{2d-i}. \tag{5}$$

Let us show that

$$B_e \cap B_{e'} = \emptyset \quad \text{for all distinct } T_x\text{–}T_y \text{ edges } \ e, e'. \tag{6}$$

Suppose not, ie. suppose that z_e and $z_{e'}$ are comparable in T_x, say $z_e \le z_{e'}$. Write $e =: vw$ and $e' =: v'w'$ with $v, v' \in T_x$ and $w, w' \in T_y$, and put $i := d(v, x)$. We show that the unique cycle C in $T_x \cup T_y + e + e'$ has length less than $g(G)$ (Fig. 1).

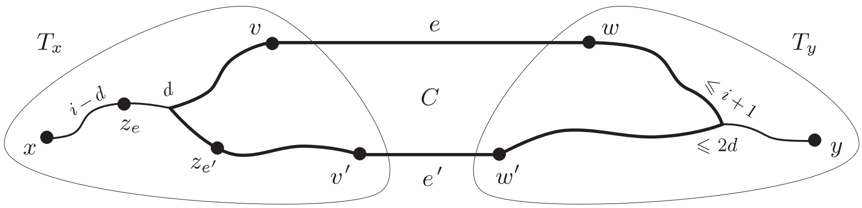

Fig. 1. The cycle C between T_x and T_y.

The portion of C in T_x is a subpath of the walk $v \dots z_e \dots v'$ in T_x, which has length at most $d + (2d - (i - d)) = 4d - i$. Its portion in T_y is a subpath of the walk $w \dots y \dots w'$ in T_y, which has length at most $(i+1) + 2d$ by (2). Thus $|C| \le 6d + 3 < g(G)$, as desired. This completes the proof of (6).

Now (5), (6) and the definition of the B_e imply (4):

$$\sum_{i \ge d} 2^{2d-i} |A^i_{x,y}| = \sum_{i \ge d} \sum_{e \in A^i_{x,y}} 2^{2d-i} \overset{(5)}{\le} \sum_{i \ge d} \sum_{e \in A^i_{x,y}} |B_e| \overset{(6)}{\le} |V^d_x| \,. \qquad \blacksquare$$

In order to improve the bound in Theorem 1 further, we have considered the question of whether the set X might be chosen more effectively. For the proof of (1) we need its points to be more than $2d$ apart. But if they were placed in G so that every other vertex v had distance $d(v, X) \le \alpha d$ from X for some $\alpha < 2$ (rather than just $d(v, X) \le 2d$, which we get simply by choosing X maximal), we would instantly shorten the cycle C in the proof of (6) to at most $(2 + 2\alpha)d + 3$, improving the girth bound in the theorem to $(2 + 2\alpha)\log k + 3$. Note that the theoretical optimum of $\alpha = 1$ would give us exactly (up to the additive constant) the conjectured lower bound from Section 2.

The problem of whether such a set X exists for given values of d and α has been shown to be NP-hard [8], and so we did not pursue this approach further. However, Kühn and Osthus [6] have recently shown that a random choice of X can indeed reduce the leading factor of 6 in Theorem 1 to the conjectured optimum of 4.

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