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Graphs with large maximum degree containing no odd cycles of a given length

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Abstract

Let us write $f(n, \Delta; C_{2k+1})$ for the maximal number of edges in a graph of order n and maximum degree Δ that contains no cycles of length $2k+1$. For $\frac{n}{2} \leqslant \Delta \leqslant n-k-1$ and n sufficiently large we show that $f(n, \Delta; C_{2k+1}) = \Delta(n-\Delta)$, with the unique extremal graph a complete bipartite graph.

There are numerous well-known results asserting that every graph on n vertices with sufficiently many edges, or satisfying some natural degree conditions, contains long cycles of certain lengths (see, for example, [5–7,13] or [2, Chapter 3]). More recently, there are many similar results giving stronger conclusions under (sometimes fairly specific) stronger assumptions (see, for example, [1,4,9–12]). Here we return to the basic extremal question, imposing only the at first sight rather weak additional condition that the graph has at least one vertex of large degree.

To be more specific, let G be a graph on n vertices, and let k be a fixed positive integer. It is well known that if G has more than $n^2/4$ edges and $2k+1 \leqslant \lfloor \frac{1}{2}(n+3) \rfloor$, then G contains a C_{2k+1}, a cycle of length $2k+1$ (see, for example, [2, p. 150]). Since $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ contains no odd cycles, the maximal number of edges in a graph containing no C_{2k+1} (the extremal number for C_{2k+1}) is $\lfloor n^2/4 \rfloor$ for sufficiently large n. The main aim of this paper is to prove a considerable strengthening of this result: we shall show that this extremal number for C_{2k+1} becomes significantly smaller if we specify that the maximum degree Δ of G takes a value somewhat larger than $n/2$. Let

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$f(n, \Delta; C_{2k+1})$ denote the maximal number of edges in a graph G of order n and maximum degree Δ containing no C_{2k+1}. We shall show that if n is sufficiently large and $n/2 \leqslant \Delta \leqslant n-k-1$, then $f(n, \Delta; C_{2k+1}) = \Delta(n-\Delta)$ with the unique extremal graph a complete bipartite graph with classes of size Δ and $n-\Delta$. Smaller values of Δ are less interesting: when $\Delta < n/2$ and n is even the trivial upper bound of $n\Delta/2$ is attained by any Δ-regular balanced bipartite graph. It is surprising that simply requiring one vertex in G of large degree can so affect the extremal number for odd cycles, lowering it from $\lfloor n^2/4 \rfloor$ to $\Delta(n-\Delta)$. As expected, even when $\Delta(G)$ is large the family of bipartite graphs provides the extremal examples.

Although our main result is to calculate $f(n, \Delta; C_{2k+1})$ when $n/2 \leqslant \Delta \leqslant n-k-1$, we first give a result for the case $\Delta \geqslant n-k$, for which we need a few lemmas. As in [3], throughout the paper we use the following standard notation: for a graph G we write $V(G)$ for its vertex set, $E(G)$ for its edge set, $|G|$ for the number of vertices, $e(G)$ for the number of edges, $\Delta(G)$ for the maximum degree, and $\delta(G)$ for the minimum degree. For a vertex v of G we write $\Gamma(v)$ for the set of neighbors of v in G, and for $U \subset V(G)$, $\Gamma_U(v)$ for $\Gamma(v) \cap U$. We write $d_G(v)$ for the degree of v in G, or just $d(v)$ when it is clear which graph is meant, and $G-v$ for the graph formed from G by deleting v. Finally, for $V \subset V(G)$ we write $G[V]$ for the subgraph of G induced by V.

Lemma 1. *Let G be a graph on n vertices that contains no C_{2k+1}, and let v be a vertex of G with $d_G(v) = \Delta(G) = n-1-m$. Let P be a maximal path in $G-v$ with endvertices x and y. Suppose that $2k > m+1$, and that P has $\ell \geqslant 2k$ vertices. If none of the neighbors of x lie closer to y on P than any neighbor of y then $\min\{d_G(x), d_G(y)\} \leqslant m$ except in the special case when $d_G(x) = d_G(y) = 3$, $m = 2$, $\ell = 2k+1$, and both endvertices of P are adjacent to v.*

Proof. Suppose for a contradiction that $d_G(x)$, $d_G(y) \geqslant m+1$. Let us write P as $x = v_1 \dots v_\ell = y$. Then by assumption there is a c such that if v_i is a neighbor of x then $i \leqslant c$, and if v_j is a neighbor of y then $j \geqslant c$. Let v_i be a neighbor of x and v_j a neighbor of y. If $vv_{i'}$ and $vv_{j'}$ are edges of G with $i' < i \leqslant j < j'$ then the cycle $vv_{i'} \dots v_1 v_i \dots v_j v_\ell \dots v_{j'}$ lies in G. Since this cannot be of length $2k+1$ we have

$$\ell - (i - i' - 1) - (j' - j - 1) \neq 2k.$$

Now $\ell \geqslant 2k$ by assumption. If we had $\ell \geqslant 2k+m$ then, as v has only m non-neighbors, for some $1 \leqslant i \leqslant m+1$ both vertices in $\{v_i, v_{i+2k-1}\}$ would be joined to v, forming a C_{2k+1} in G. (The pairs are disjoint as $2k > m+1$.) Thus $2k \leqslant \ell \leqslant 2k + m - 1$, and the value $f = \ell + 2 - 2k$ that the sum $(i - i') + (j' - j)$ cannot take satisfies

$$2 \leqslant f \leqslant m+1. \tag{1}$$

Let r be the number of non-neighbors v_t of v with $t < c$, and s the number with $t > c$. Let the neighbors of x on P be $v_{i_1}, \dots, v_{i_d}$ with $2 = i_1 < \dots < i_d$. Note that from the maximality of P we have $d = d_{G-v}(x)$, so $d \geqslant d_G(x) - 1 \geqslant m$. Suppose for a contradiction that $r = 0$. Since there are only m non-neighbors of v and at least $m+1$ neighbors of y, we have $v_h y, v_{h+1} v \in E(G)$ for some h. (If $yv \notin E(G)$ then y has at least

$m+1$ neighbors on P and the statement follows from the pigeonhole principle. If $yv \in E(G)$ then the statement holds with $h = \ell - 1$.) Also, $c \geqslant d+1 \geqslant m+1 > \ell - 2k+1$, so as v has no non-neighbors before v_c we have $vv_{\ell-2k+1} \in E(G)$. Thus the $(2k+1)$-cycle $vv_{\ell-2k+1} \ldots v_h y \ldots v_{h+1}$ lies in G. Hence we may assume $r \geqslant 1$. Similarly $s \geqslant 1$, so $r \leqslant m - s < m$ and $d \geqslant r+1$.

If $1 \leqslant q < i_{d-r}$ then one of the $r+1$ vertices $v_{i_t - q}$, $t = d-r, \ldots, d$ must be joined to v. (Recall that v has r non-neighbors on P to the left of v_c.) Hence we may choose $i - i' = q$ in the argument of the first paragraph. Moreover, at most one value of q with $i_{d-r} \leqslant q < i_{d-r+1}$ does not occur as a difference $i - i'$, since if $i - i' \neq q$ for any i and i' then the non-neighbors of v on the left of v_c must be precisely $v_{i_{d-r+1}-q}, \ldots, v_{i_d - q}$, and this can only happen for at most one value of q.

Hence we may choose i and i' so that $i - i'$ can take any value from 1 to $i_{d-r} - 1 \geqslant d - r$, and if $d-r+1$ is not possible then either $d-r+2$ is or $i_{d-r+1} = d-r+2$. In particular if $xv = v_1 v \in E(G)$ then either $d-r+1$ or $d-r+2$ is a possible difference $i - i'$. Now $d_G(x) = d$ if $xv \notin E(G)$ and $d_G(x) = d+1$ if $xv \in E(G)$. Thus $i - i'$ can take any value from 1 to $d_G(x) - r$, or any value from 1 to $d_G(x) - r + 1$ except $d_G(x) - r$. Note that $d_G(x) \geqslant m+1$ by assumption while $r \leqslant m - s \leqslant m - 1$, so $d_G(x) - r \geqslant 2$, and 1 is certainly a possible value of $i - i'$.

Arguing as above we can choose $j' - j$ to be any value from 1 to $d_G(y) - s \geqslant 2$ or any value from 1 to $d_G(y) - s + 1$ except $d_G(y) - s$. Now $S = d_G(x) + d_G(y) - r - s \geqslant 2(m+1) - m = m+2$. From the values taken by $i - i'$ and $j' - j$ we see that their sum can always take any value from 2 to $S - 2 \geqslant m$. As this sum cannot take the value f, from (1) we see that $S = m+2, f = m+1$, and $(i - i') + (j' - j)$ cannot take the value $S - 1$. From the last fact we must be in the case where $i - i'$ cannot take the value $d_G(x) - r$, nor $j' - j$ the value $d_G(y) - s$. Also, as $(d_G(x) - r - 2) + (d_G(y) - s + 1) = S - 1$, we must have $d_G(x) - r = 2$, and, similarly, $d_G(y) - s = 2$. But now

$$m \geqslant r + s = d_G(x) + d_G(y) - 4 \geqslant 2m - 2,$$

so $m \leqslant 2$, and as $m \geqslant r + s$, $m = 2$. Equality in the equation above together with $d_G(x), d_G(y) \geqslant m+1$ now gives $d_G(x) = d_G(y) = 3$, whence $\ell = 2k + f - 2 = 2k+1$. Finally, from the values not taken by $i - i'$ and $j' - j$ we have $xv, yv \in E(G)$, completing the proof. $\square$

Note that in the proof above we did not use the fact that the length of the cycle is odd, so the lemma applies with k a half-integer. We have stated it for C_{2k+1} for consistency with our notation in the rest of the paper.

The following is a variant of results of Ore and Pósa. We write $H[P]$ for $H[V(P)]$.

Lemma 2. *Let $P = v_1 v_2 \ldots v_\ell$ be a maximal path in a graph H, and let $m = d(v_1) + d(v_\ell)$. If there are indices $1 \leqslant i < j \leqslant \ell$ with v_j a neighbor of v_1 and v_i a neighbor of v_ℓ then $H[P]$ contains a cycle of length at least* $\min\{\ell, m\}$. *In particular, if in addition $v_1 v_h$, $v_h v_\ell \notin E(G)$ for $i < h < j$ then either $H[P]$ is Hamiltonian, or the cycle $C = v_1 v_2 \ldots v_i v_\ell v_{\ell-1} \ldots v_j$ has length at least m.*

Proof. Let us imitate the standard proof of Pósa's theorem, as in [3, p. 106]. Suppose that $H[P]$ is not Hamiltonian. We may assume that every vertex v_h with $i<h<j$ is not a neighbor of either v_1 or v_ℓ. Since P is maximal, the neighbors of v_1 and v_ℓ are vertices of P. Furthermore, there is no index h, $1\leqslant h\leqslant \ell-1$, such that v_1v_{h+1} and v_hv_ℓ are both edges since otherwise $v_1v_2\dots v_hv_\ell v_{\ell-1}\dots v_{h+1}$ is a Hamilton cycle in $H[P]$. Consequently, the sets $\Gamma(v_1)=\{v_h\colon v_1v_h\in E(G)\}$ and $\Gamma^+(v_\ell)=\{v_{h+1}\colon v_hv_\ell\in E(G)\}$ are disjoint subsets of $\{v_2,v_3,\dots,v_{i+1},v_j,v_{j+1},\dots,v_\ell\}$. Hence $m\leqslant \ell-(j-i)+1$. Since G contains the cycle $v_1v_2\dots v_iv_\ell v_{\ell-1}\dots v_j$ of length $\ell-(j-i)+1$, the proof is complete. $\square$

We shall also need an old result of Erdős and Gallai [8] proved using a method due to Dirac [7]: if G is a graph on n vertices whose longest path has ℓ edges, then $e(G)\leqslant \ell n/2$. (See, for example, [3].) The following lemma is a generalization of this result which, although not needed here, is simple enough that it seems likely to find applications elsewhere. For a vertex v of a graph G let us write $\ell_G(v)$ for the (edge) length of a longest path in G starting at v.

Lemma 3. *For every graph G we have*

$$e(G)\leqslant \frac{1}{2}\sum_{v\in V(G)}\ell_G(v), \tag{2}$$

with equality if and only if every component of G is a complete graph.

Proof. We use induction on $|G|$. We assume that G is connected, as otherwise we are done by induction.

Let $P=v_1\dots v_{\ell+1}$ be a longest path in G, so $\ell_G(v_1)=\ell_G(v_{\ell+1})=\ell$. We may assume that $\ell\geqslant 2$ as otherwise G is just a single edge or a single vertex and the result holds. Suppose first that G contains a cycle C of length $\ell+1$. As G is connected and any edge from a vertex on C to a vertex not on C would create a path longer than P, the cycle C is a Hamilton cycle, i.e., G has only $\ell+1$ vertices. Also, $\ell_G(v)=\ell$ for all $v\in V(G)$, so

$$e(G)\leqslant \binom{\ell+1}{2}=\frac{1}{2}\sum_{v\in V(G)}\ell_G(v),$$

with equality if and only if G is complete, as required.

From now on we assume as we may that G does not contain a cycle of length $\ell+1$. Again we imitate the standard proof of Pósa's Theorem. For $i=1,2,\dots,\ell$, v_1v_{i+1} and $v_iv_{\ell+1}$ cannot both be edges of G. As P is a longest path we have $\Gamma(v_1)\cup\Gamma(v_{\ell+1})\subset P$. Defining $\Gamma^+(v_{\ell+1})$ as in the proof of Lemma 2, we thus have

$$\Gamma(v_1)\cap\Gamma^+(v_{\ell+1})=\emptyset$$

and

$$\Gamma(v_1)\cup\Gamma^+(v_{\ell+1})\subset\{v_2,\dots,v_{\ell+1}\}.$$

Hence $d(v_1)+d(v_{\ell+1})=|\Gamma(v_1)|+|\Gamma^+(v_{\ell+1})|\leqslant\ell$.

Without loss of generality, we may assume that $d(v_1)\leqslant\ell/2$. Let $G'=G-v_1$. Then by induction

$$e(G')\leqslant\frac{1}{2}\sum_{v\in V(G')}\ell_{G'}(v)\leqslant\frac{1}{2}\sum_{v\in V(G')}\ell_G(v)=\frac{1}{2}\sum_{v\in V(G)}\ell_G(v)-\frac{\ell}{2},\tag{3}$$

as $\ell_G(v_1)=\ell$. Since

$$e(G)=e(G')+d(v_1)\leqslant e(G')+\ell/2,\tag{4}$$

this proves (2). To complete the proof suppose that equality holds in (2). Then we must have equality throughout (3) and (4). As G is connected and all the neighbors of v_1 lie on P, the graph G' is connected, and by induction G' is complete. Thus G consists of the vertex v_1 joined by $\ell/2$ edges to the complete graph G'. Since the longest path starting at v_1 has length ℓ, G' is K_ℓ. However, the other endvertex w of the path of length ℓ starting at v_1 has $\ell_G(w)=\ell>\ell_{G'}(w)=\ell-1$, so the second inequality in (3) is strict, a contradiction. □

Theorem 4. *Let G be a graph with n vertices containing no C_{2k+1} with maximum degree $\Delta=n-1-m$, $m<k$. Then $e(G)\leqslant\Delta+(k-1)(n-1)$.*

Proof. We use induction on n. Since $e(G)\leqslant\Delta+\binom{n-1}{2}=\Delta+(\frac{n}{2}-1)(n-1)$ we may assume that $k<\frac{n}{2}$. Hence $k<\frac{n}{2}<n-k\leqslant\Delta$. If $x\in V(G)$ has degree $d_G(x)\leqslant k-1$ then removing x gives a graph with no C_{2k+1} and maximum degree Δ', $\Delta-1\leqslant\Delta'\leqslant\Delta$. Thus $\Delta'=(n-1)-1-m'$, $m'<k$, and by induction this graph has at most $\Delta'+(k-1)(n-2)$ edges. Adding back x gives $e(G)\leqslant\Delta'+(k-1)(n-2)+d_G(x)\leqslant\Delta+(k-1)(n-1)$, as required. Similarly, if $d_G(x)=k$ and x is adjacent to all vertices of maximal degree, then $\Delta'<\Delta$ and $e(G)\leqslant\Delta'+(k-1)(n-2)+d_G(x)\leqslant\Delta+(k-1)(n-1)$. Hence we may assume that $\delta(G)\geqslant k$, and that if x is a vertex of degree k then there is some vertex of degree Δ that is not adjacent to x.

Select a vertex v of maximum degree and a (maximal) path P in $G-v$ so that the length of P is maximized (over all pairs (v,P)). Label the vertices of P as $x=v_1v_2\dots v_{\ell-1}v_\ell=y$. If P has at most $2k-1$ vertices then all paths in $G-v$ have (edge) length at most $2k-2$, so by the Erdős–Gallai result (or by Lemma 3) we have $e(G-v)\leqslant(2k-2)(n-1)/2$ and the result follows. We may thus assume that $\ell\geqslant 2k$.

We now consider two cases:

(i) Suppose first that none of the neighbors of x lie closer to y on P than any neighbor of y. Since $\delta(G)\geqslant k>m$, Lemma 1 implies that $m=2$, $d_G(x)=d_G(y)=3$, $k=3$, $\ell=2k+1=7$, and $v_1v,v_\ell v\in E(G)$. As v has only two non-neighbors and G contains no C_7, the non-neighbors of v are v_2 and v_6. Note that v_1 has no neighbors other than v off P. The only possible neighbors on P are v_2 and v_4 (otherwise $G[P\cup v]$ contains a C_7), so v_1 is joined to v_4. Similarly v_7 is joined to v_4.

None of the vertices outside $P \cup \{v\}$ can have degree Δ since otherwise the path vP would miss a vertex of degree Δ, contradicting the choice of P. Since G contains no C_7, the vertices $v_1, v_2, v_3, v_5, v_6, v_7$ are each non-adjacent to at least three vertices (for example v_2 is non-adjacent to v_5, v_6, v_7, and v). Hence the only vertex on P that can have degree Δ is v_4. Thus x is a vertex of degree $3 = k$ adjacent to all the degree Δ vertices, contradicting the assumption above.

(ii) Now suppose there is a neighbor of x that is closer on P to y than some neighbor of y. By Lemma 2, $G - v$ contains a cycle of length at least $r = \min\{\ell, d_{G-v}(x) + d_{G-v}(y)\}$. However, $G - v$ cannot contain a cycle C of length at least $2k$—otherwise, as v has fewer than k non-neighbors, v is joined to more than half the vertices of C, and $C \cup \{v\}$ contains a C_{2k+1}. As $\ell \geqslant 2k$ it follows that $d_{G-v}(x) + d_{G-v}(y) = r < 2k$. We can therefore assume without loss of generality that $d_{G-v}(x) \leqslant k-1$. Since $\delta(G) \geqslant k$, we have $d_{G-v}(x) = k-1$, $xv \in E(G)$, and $d_{G-v}(y) = k$ or $k-1$, so $r = 2k-1$ or $2k-2$. We claim also that y cannot have degree Δ; as $d_{G-v}(y) \leqslant k$ we have $d_G(y) \leqslant k+1$. Since $k < \frac{n}{2} < \Delta$, it follows that if $d_G(y) = \Delta$ then $yv \in E(G)$ and $n = \ell + 1 = 2k+1$, whence vP is a C_{2k+1}, giving a contradiction. We shall use this fact later: for any pair (v, P) as at the start of the proof, neither endvertex of P can have degree Δ. In particular, if v_h has degree Δ then xv_{h+1}, $v_{h-1}y \notin E(G)$; otherwise we can find a pair (v, P') where P' ends in the vertex v_h of degree Δ.

Our aim now is to show that contrary to our assumption there is a vertex z (one of x and y) of degree k adjacent to all vertices of degree Δ in G. Note that all vertices $w \neq v$ of degree Δ lie on P, as otherwise vP is a path longer than P avoiding the vertex w of maximal degree. Let us also note that $\Gamma_P(x) \neq \Gamma_P(y)$. (One of many ways to check this is as follows: suppose that $\Gamma_P(x) = \Gamma_P(y)$. Then $d_{G-v}(y) = d_{G-v}(x) = k-1$, so as $\delta(G) \geqslant k$ we have $yv \in E(G)$, and vP is a cycle. Thus $\ell \neq 2k$, so $\ell > 2k$. But $v_{\ell-1}$ is a neighbor of y and thus by assumption of x, so $G - v$ contains a cycle $xv_2 \dots v_{\ell-1}$ of length $\ell - 1 \geqslant 2k$. As noted above this gives a contradiction.)

Let i, j be a pair with $i < j$, xv_j, $v_i y \in E(G)$ and xv_h, $v_h y \notin E(G)$ for $i < h < j$. By Lemma 2, the cycle $C = v_1 \dots v_i v_\ell \dots v_j$ has length $|C| \geqslant 2k-2$.

Suppose first that there is a vertex $w = v_h$ with degree Δ where $i < h < j$. As noted above, xv_{h+1}, $v_{h-1}y \notin E(G)$, so $i+1 < h < j-1$. Now if v_h is adjacent to v_{r+1} for any neighbor v_r of x or y with $r \geqslant j$ we can find a cycle in $G[P]$ of length at least $|C| + 2 \geqslant 2k$, giving a contradiction as before. Similarly v_h cannot be adjacent to v_{r-1} for any neighbor v_r of x or y with $r \leqslant i$. Since $xy \notin E(G)$ and $\Gamma_P(x) \neq \Gamma_P(y)$, there are at least $d_{G-v}(x) + 1 = k$ non-neighbors of v_h on P. But v_h has degree Δ and thus only $m < k$ non-neighbors in the whole of G.

As $d_G(x) = k$, by assumption there is a vertex w of degree Δ not adjacent to x. As $w \neq v$ from the above we have $w = v_h$ with $h \leqslant i$ or $h \geqslant j$. Since v_h has degree Δ, as shown above $v_{h-1}y \notin E(G)$, so $v_h \notin \Gamma(x) \cup \Gamma^+(y)$. From the proof of Lemma 2, $\Gamma(x)$ and $\Gamma^+(y)$ are disjoint subsets of $S = \{v_2, v_3, \dots, v_{i+1}, v_j, v_{j+1}, \dots v_\ell\}$. As $|S| = |C| \leqslant 2k-1 \leqslant d_{G-v}(x) + d_{G-v}(y) + 1$, we see that v_h is the unique element of S missing from $\Gamma(x) \cup \Gamma^+(y)$, and that $|S| = |C| = 2k-1 = d_{G-v}(x) + d_{G-v}(y) + 1$, so $d_{G-v}(y) = k-1$. Thus $yv \in E(G)$. Now we can assume there is a vertex $w' = v_{h'}$ of degree Δ that is not adjacent to y. By a similar argument to the above,

$v_{h'+1} \notin \Gamma(x) \cup \Gamma^+(y)$. As there is only one missing vertex, $h' + 1 = h$. But then $v_{h-1} \dots v_1 v v_\ell \dots v_{h+1}$ is a path of length ℓ avoiding a vertex v_h of maximal degree but ending in a vertex $v_{h-1} = v_{h'}$ of maximal degree, giving a contradiction. $\square$

Lemma 5. *Suppose that $k \geqslant 1$ and $n \geqslant 2k^2$. If G is a graph with n vertices and maximum degree $\Delta = n - k - 1$ containing no C_{2k+1} then $e(G) \leqslant \Delta(n - \Delta)$, with equality if and only if G is a complete bipartite graph.*

Proof. Note first that the case $k = 1$ is trivial, as then G is in fact a subgraph of $K_{2,n-2}$. We assume from now on that $k \geqslant 2$. Pick a vertex v of degree Δ and let $V = \{v_1, \dots, v_k\}$ be the vertices that are not adjacent to v. If we remove one of the vertices v_i then by Theorem 4 we have $e(G - v_i) \leqslant \Delta + (k - 1)(n - 2)$. Thus if

$$d_G(v_i) < \Delta(n - \Delta) - (\Delta + (k - 1)(n - 2)) = n - k^2 + k - 2$$

then we are done. Hence we can assume that $d_G(v_i) \geqslant n - k^2 + k - 2$ for all $i = 1, \dots, k$. Let $U = V(G) \setminus (V \cup \{v\}) = \Gamma(v)$.

Suppose V contains an edge $v_1 v_2$, say. Construct a path starting with $v_1 v_2$ and then alternating between V and U through all the remaining v_i as follows: if we have a path $v_1 v_2 u_2 v_3 \dots v_i$, $i \leqslant k - 1$, then pick $u_i \in (\Gamma_U(v_i) \cap \Gamma_U(v_{i+1})) \setminus \{u_2, \dots u_{i-1}\}$ and extend the path by $u_i v_{i+1}$. (We are treating the labels u_i and v_i differently—the v_i are fixed vertices, only the u_i are being chosen.) Finally, pick $u_k \in \Gamma_U(v_k) \setminus \{u_2, \dots, u_{k-1}\}$ and $u_1 \in \Gamma_U(v_1) \setminus \{u_2, \dots, u_k\}$ to give a cycle $v u_1 v_1 v_2 u_2 \dots v_k u_k$ of length $2k + 1$.

This will be possible if $|\Gamma_U(v_i) \cap \Gamma_U(v_{i+1})| \geqslant k - 2$ and $|\Gamma_U(v_i)| \geqslant k$ for each i. However $|\Gamma_U(v_i)| \geqslant n - k^2 - 1 \geqslant k$ and $|\Gamma_U(v_i) \cap \Gamma_U(v_{i+1})| \geqslant n - 2k^2 + k - 1 \geqslant k - 2$.

Hence we can suppose that $V \cup \{v\}$ is an independent set.

Now suppose there is an edge $u_0 u_1$, where $u_0, u_1 \in U$ and $u_1 \in \Gamma(v_1)$. Recalling that now $\Gamma_U(v_i) = \Gamma(v_i)$, construct a $(2k + 1)$-cycle $v u_0 u_1 v_1 u_2 v_2 \dots v_{k-1} u_k$ by choosing $u_i \in (\Gamma(v_{i-1}) \cap \Gamma(v_i)) \setminus \{u_0, u_1, \dots, u_{i-1}\}$ for $i = 2, \dots, k - 1$ and $u_k \in \Gamma(v_{k-1}) \setminus \{u_0, \dots, u_{k-1}\}$.

This succeeds provided $|\Gamma(v_i) \cap \Gamma(v_{i+1})| \geqslant k$ and $|\Gamma(v_i)| \geqslant k + 1$. However $|\Gamma(v_i)| \geqslant |\Gamma(v_i) \cap \Gamma(v_{i+1})| \geqslant n - 2k^2 + 3k - 3 \geqslant k + 1$ for $k \geqslant 2$. Hence setting $U' = \bigcup_{i=1}^{k} \Gamma(v_i) \subseteq U$ we can now suppose that U' is an independent set and there are no edges between U' and $U \setminus U'$. Every vertex of $U \setminus U'$ is joined to v, so there are no paths of length $2k - 1$ in $G[U \setminus U']$. Hence by the Erdős–Gallai result (or by Lemma 3) we have $e(G[U \setminus U']) \leqslant (k - 1)|U \setminus U'|$. The total number of edges in G is at most $\Delta + (k - 1)|U \setminus U'| + |V||U'|$. This in turn is at most $\Delta + k|U| = \Delta(n - \Delta)$, with equality iff $U' = U$ and G is a complete bipartite graph with classes $V \cup \{v\}$ and U. $\square$

Theorem 6. *If G is a graph on n vertices with maximum degree Δ containing no C_{2k+1} and $2k^2 \leqslant \Delta \leqslant n - k - 1$, then $e(G) \leqslant \Delta(n - \Delta)$, with equality if and only if G is a complete bipartite graph.*

Proof. If $\Delta = n - k - 1$ this is just Lemma 5. For $\Delta < n - k - 1$ pick a vertex x not adjacent to some vertex v of maximal degree. Then $2k^2 \leqslant \Delta(G - x) = \Delta \leqslant (n - 1) -$

$k-1$. Hence by induction on n, $e(G-x) \leqslant \Delta(n-1-\Delta)$. Hence $e(G) \leqslant \Delta(n-\Delta)$ with equality iff $G-x$ is a complete bipartite graph with class sizes Δ and $n-1-\Delta$ and x is a vertex of degree Δ. We cannot have $xy \in E(G)$ for a vertex y in the class of size $n-1-\Delta$, as then y would have degree $\Delta+1$ in G, so $G = K_{\Delta,n-\Delta}$. $\square$

Although the condition $\Delta \geqslant n/2$ is not stated in the result above, it is only this case that is interesting, as otherwise the trivial bound $e(G) \leqslant n\Delta/2$ is better. When $n/2 \leqslant \Delta \leqslant n-k-1$ and n is large enough Theorem 6 shows that $f(n,\Delta; C_{2k+1}) = \Delta(n-\Delta)$. Returning to the perhaps more natural formulation of a lower bound on n, rather than Δ as in Theorem 6, for $k \geqslant 1$ let $n_0(k)$ be the minimal integer such that $f(n,\Delta; C_{2k+1}) = \Delta(n-\Delta)$ whenever $n \geqslant n_0(k)$ and $n/2 \leqslant \Delta \leqslant n-k-1$. Then by Theorem 6 we have $n_0(k) \leqslant 4k^2$. In the other direction we shall now show that $n_0(k) \geqslant (1+o(1))k^2$.

For any $r \geqslant 1$ and $k \geqslant 1$ consider the graph G formed by taking r copies of K_{2k} joined at a single vertex v and deleting k edges from v arbitrarily. Then G has $n = (2k-1)r+1$ vertices, contains no C_{2k+1} and has maximum degree $\Delta(G) = n-k-1$. Since $G-v$ is $(2k-2)$-regular, $e(G) = \Delta + (k-1)(n-1)$. Now $\Delta + (k-1)(n-1) - \Delta(n-\Delta) = k^2 - n + 1$, so if $n \leqslant k^2$ then G is an example showing that $n_0(k) > n$. In particular taking $r = \lfloor k/2 \rfloor$ shows that $n_0(k) \geqslant (1+o(1))k^2$ when $k \to \infty$.

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