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On immersions of uncountable graphs

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Abstract

In his paper on well-quasi-ordering infinite trees (Proc. Cambridge Philos. Soc. 61 (1965) 697), Nash-Williams proposed the conjecture that the class of all graphs (finite or infinite) is well-quasi-ordered by the immersion relation (which is denoted here by $\leqslant_1$). In addition, in a subsequent paper, Nash-Williams discussed a weaker version of his original conjecture to the effect that the class of graphs is well-quasi-ordered with respect to a relation $\leqslant_2$ which, roughly speaking, is obtained by redefining $H \leqslant_1 G$ so that distinct vertices of H can be mapped into the same vertex of G. It is the purpose of the present note to disprove Nash-Williams' two immersion conjectures.

Keywords: Immersion relation; Infinite graphs; Well-quasi-ordering; Antichains

1. Introduction

For graph theoretic terminology, we refer to the definitions and notational conventions collected at the end of the introduction or to the textbook of Diestel [1]. The graphs considered in this note do not contain loops or multiple edges. For graphs G, H, we write $H \leqslant G$ if G contains a subgraph which is isomorphic to a subdivision of H. A classical result in infinite graph theory states that the class of trees is well-quasi-ordered by the *subdivision relation* $\leqslant$:

Theorem A (Nash-Williams [2]). *If $T_1, T_2, \ldots$ is an infinite sequence of trees, then there exist positive integers i, j such that $i < j$ and $T_i \leqslant T_j$.*

The following well-known conjecture of E. Vázsonyi apparently dates back to the 1930s. For finite graphs, the truth of the conjecture is an immediate consequence of

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the graph minor theorem of Robertson and Seymour [4]; cf. also [1]. In the general case, however, the conjecture remains far from proved to this day, although partial results on the infinite case can be obtained as consequences of results of Thomas [6].

Conjecture (Vázsonyi). *If $G_1, G_2, \ldots$ is an infinite sequence of graphs in which every vertex has degree $\leqslant 3$, then there exist positive integers i, j such that $i<j$ and $G_i \leqslant G_j$.*

Theorem A and Vázsonyi's conjecture deal with restricted classes of graphs, namely, trees and graphs with maximum degree at most 3, respectively. This prompted Nash-Williams [2] to suggest another conjecture for the class of all graphs which, if true, would have both Theorem A and Vázsonyi's conjecture as corollaries. For a graph G, let $\mathscr{P}(G)$ denote the set of nontrivial paths of G. For graphs G, H, an *immersion* $\varphi : H \to G$ is a mapping $\varphi : V(H) \cup E(H) \to V(G) \cup \mathscr{P}(G)$ such that

(i) if $v \in V(H)$, then $\varphi(v) \in V(G)$;
(ii) if $v, v' \in V(H), v \neq v'$, then $\varphi(v) \neq \varphi(v')$;
(iii) if $e = vv' \in E(H)$, then $\varphi(e) \in \mathscr{P}(G)$ and the path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(v')$;
(iv) if $e, e' \in E(H)$ are distinct, then $\varphi(e)$ and $\varphi(e')$ are edge-disjoint and
(v) if $e = vv' \in E(H)$ and $v'' \in V(H)$, then $\varphi(v'') \notin V(\varphi(e)) \backslash \{\varphi(v), \varphi(v')\}$.

Writing $H \leqslant_1 G$ to indicate that there exists an immersion $\varphi : H \to G$, Nash-Williams' conjecture reads as follows.

Conjecture A (Nash-Williams [2]). *If $G_1, G_2, \ldots$ is an infinite sequence of graphs, then there exist positive integers i, j such that $i<j$ and $G_i \leqslant_1 G_j$.*

In his paper [3], Nash-Williams subsequently presented a weaker version of Conjecture A which is still strong enough to imply Theorem A and Vázsonyi's conjecture. Let $\mathscr{C}(G)$ denote the set of cycles of a graph G. For graphs G, H, an *immersion $\varphi : H \to G$ in the weak sense* is a mapping $\varphi : V(H) \cup E(H) \to V(G) \cup \mathscr{P}(G) \cup \mathscr{C}(G)$ such that (i), (ii′), (iii′), (iv), (v) hold, where (i), (iv) and (v) are as above and the statements (ii′), (iii′) are as follows.

(ii′) if $e = vv' \in E(H)$ and $\varphi(v) \neq \varphi(v')$, then $\varphi(e) \in \mathscr{P}(G)$ and the path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(v')$ and
(iii′) if $e = vv' \in E(H)$ and $\varphi(v) = \varphi(v')$, then $\varphi(e) \in \mathscr{C}(G)$ and $\varphi(v)$ is a vertex on cycle $\varphi(e)$.

Writing $H \leqslant_2 G$ if there exists an immersion $\varphi : H \to G$ in the weak sense, the modified version of Conjecture A reads as follows.

Conjecture B (Nash-Williams [3]). *If $G_1, G_2, \ldots$ is an infinite sequence of graphs, then there exist positive integers i, j such that $i<j$ and $G_i \leqslant_2 G_j$.*

In [4], Robertson and Seymour announced a proof of Conjecture A for the case of finite graphs $G_1, G_2, \ldots$ (which, of course, implies the truth of Conjecture B for finite graphs). It is the purpose of the present paper to show that, in the general case of arbitrary graphs, counterexamples to Conjectures A and B exist. The question whether or not the conjectures hold for countable graphs remains open. (Actually, the paper also leaves open the question whether Nash-Williams' conjectures hold for graphs of cardinality less than the first limit cardinal greater than the cardinality of the continuum.)

The paper is organized as follows. In Section 2, we establish a result which, to some extent, clarifies the relationship between the immersion relations $\leqslant_1$ and $\leqslant_2$: we show that for each family of graphs G_i $(i \in I)$ there exists a family of graphs H_i $(i \in I)$ such that, for all $i, j \in I, G_i \leqslant_1 G_j$ if and only if $H_i \leqslant_2 H_j$ (Theorem 1). In particular this means that, in order to disprove Conjecture B, it is enough to find a counterexample to Conjecture A. In Section 3, we modify some of the ideas of Thomas [5] in order to make them compatible with the immersion relation $\leqslant_1$. This, when taken together with Theorem 1 of the present note, implies Theorem 2, which states that there exists a sequence $H_1, H_2, \ldots$ of uncountable graphs such that $H_i \not\leqslant_2 H_j$ for all i,j with $i<j$. Theorem 3 is a sharpened version of Theorem 2 showing that the sequence $H_1, H_2, \ldots$ can be modified to obtain an antichain $H_1', H_2', \ldots$, that is, $H_i' \not\leqslant_2 H_j'$ holds for all i,j with $i \neq j$.

Our terminology is standard. For graph theoretic terms not defined here, we refer to Diestel [1]. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If e is an edge joining vertices v, w, then e is denoted by vw. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. An edge is said to be *pendant* if one of its ends has degree one. A family of graphs G_i $(i \in I)$ is *edge-disjoint* if $E(G_i) \cap E(G_j) = \emptyset$ for all $i, j \in I$ with $i \neq j$. A *path* is a graph P consisting of $n+1$ distinct vertices $a_0, \ldots, a_n$ $(n \geqslant 0)$ and edges $a_i a_{i+1}$ $(i = 0, \ldots, n-1)$. For paths, we use notations like $P = (a_0, \ldots, a_n)$. A path is *trivial* if it consists of just one vertex. For a path $P = (a_0, \ldots, a_n)$, the vertices a_0, a_n are called *endvertices* of P. A path P is an *x, y-path* (or likewise P is said to *connect x with y*) if x and y are its endvertices. In order to stress the difference of the above defined notions of an 'immersion' and an 'immersion in the weak sense', the former will also be referred to as an *immersion in the strong sense*.

A binary relation $\preccurlyeq$ on a set Q is a *quasi-order* if it is reflexive and transitive. We say that Q is *well-quasi-ordered* by $\preccurlyeq$ if the relation $\preccurlyeq$ is a quasi-order on Q and if for each infinite sequence $q_1, q_2, \ldots$ of elements of Q there exist positive integers i,j such that $i<j$ and $q_i \preccurlyeq q_j$. Given a quasi-order $\preccurlyeq$ on Q, a family q_i $(i \in I)$ of elements of Q is called an *antichain* (with respect to $\preccurlyeq$) if $q_i \not\preccurlyeq q_j$ for all $i, j \in I$ with $i \neq j$.

The set $\{0, 1, \ldots\}$ of natural numbers is denoted ω. The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. Cardinals are identified with their initial ordinals.

2. A result linking the two versions of the immersion relation

In this section, we establish a result (Theorem 1) on the relations $\leqslant_1$ and $\leqslant_2$ which in particular shows that, in order to disprove Conjecture B, it is enough to find a counterexample to Conjecture A. Most likely, Theorem 1 can also be applied in other situations to reduce a problem for $\leqslant_2$ to the analogous problem for $\leqslant_1$.

Theorem 1. *For each family of graphs* G_i $(i \in I)$ *there exists a family of graphs* H_i $(i \in I)$ *such that, for all* $i, j, \in I, G_i \leqslant_1 G_j$ *if and only if* $H_i \leqslant_2 H_j$. *In particular,* G_i $(i \in I)$ *is an antichain with respect to* $\leqslant_1$ *if and only if* H_i $(i \in I)$ *is an antichain with respect to* $\leqslant_2$.

Proof. Let G_i $(i \in I)$ be a family of graphs. We may assume that $G_i \cap G_j = \emptyset$ for all $i, j \in I, i \neq j$. Let $\mu, \alpha, \beta, \gamma$ be cardinals such that

$$\omega \leqslant \mu < \alpha < \beta < \gamma \quad \text{and} \quad d_{G_i}(v) < \mu \quad \text{for all } i \in I \text{ and } v \in V(G_i). \tag{1}$$

For the purpose of defining the graphs H_i, we introduce a graph F as follows. Let $V(F)$ be the union of disjoint sets $X_0, X_1, \ldots, X_5$ where X_0 consists of three elements a, b, c and $X_1, \ldots, X_5$ are infinite sets with $|X_1| = \mu, |X_2| = |X_3| = \alpha, |X_4| = \beta, |X_5| = \gamma$. Let further $\xi : X_2 \to X_3$ be a bijection and define the edge set of F by

$$E(F) = \{ax : x \in X_1 \cup X_2 \cup X_3\} \cup \{bx : x \in X_1 \cup X_4 \cup \{c\}\}$$
$$\cup \{cx : x \in X_5\} \cup \{xx' : x \in X_2, x' = \xi(x)\}.$$

The graph F is displayed in Fig. 1. In particular, observe that in F there are

- μ edge-disjoint paths of length two connecting a with b;
- α edge-disjoint triangles attached at a;
- β pendant edges attached at b and
- γ pendant edges attached at c.

Let $V = \bigcup V(G_i)$, where the union is taken over all $i \in I$. For each $v \in V$, let F_v be a copy of F such that $F_v \cap F_w = \emptyset$ whenever $v \neq w$ and such that $F_v \cap G_j = \emptyset$ for all $v \in V$ and $j \in I$. For each $v \in V$, let $f_v : F \to F_v$ be an isomorphism and put $a_v := f_v(a), b_v := f_v(b), c_v := f_v(c)$. For each $i \in I$, let H_i be the graph that results from G_i and the graphs $F_v(v \in V(G_i))$ by identifying each vertex $v \in G_i$ with vertex $a_v \in F_v$.

Since the so-defined graphs H_i are disjoint, we may write $d(x)$ rather than $d_{H_i}(x)$ to denote the degree of a vertex x in H_i. For each $i \in I$, we put $A_i := V(G_i)(=$

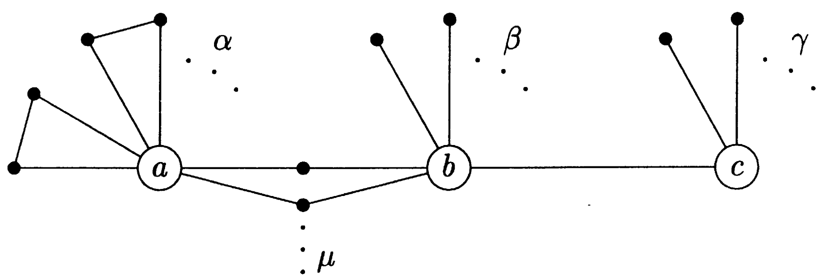

Fig. 1. The graph F occurring in the proof of Theorem 1.

$\{a_v : v \in V(G_i)\})$, $B_i := \{b_v : v \in V(G_i)\}$, and $C_i := \{c_v : v \in V(G_i)\}$. Further, we write $x \sim y$ to indicate that x and y are vertices of the same graph F_v.

Clearly, if $G_i \leqslant_1 G_j$ then $H_i \leqslant_1 H_j$ and thus $H_i \leqslant_2 H_j$. For the proof of the converse, let $i, j \in I$ and assume that $h : H_i \to H_j$ is an immersion in the weak sense. Let $g : G_i \to H_j$ denote the restriction of h to G_i. Clearly, Theorem 1 is proved if we show that

$$g \text{ is an immersion } G_i \to G_j \text{ in the strong sense.} \tag{2}$$

For the proof of (2), first observe that

$$h(c) \in C_j \quad \text{for all } c \in C_i. \tag{3}$$

Indeed, as a consequence of the construction of the graphs H_i and H_j, together with assumption (1), one obtains $d(c) = \gamma > d(v)$ for all $c \in C_i$ and all $v \in V(H_j) \backslash C_j$. Hence (3). For similar reasons, we have $h(b) \in B_j \cup C_j$ for all $b \in B_i$. However, for $b \in B_i$, $h(b) \in C_j$ is impossible since there is no cycle of H_j passing through a vertex of C_j, while each vertex of B_i is on a cycle of H_i. Hence

$$h(b) \in B_j \quad \text{for all } b \in B_i. \tag{4}$$

Now, let $a \in A_i$. Note that there are α edge-disjoint cycles passing through a and, consequently, the same must be true for $h(a)$. From this, together with the fact that $\alpha > \mu$, one concludes that $h(a) \in A_j$, and thus we have found that

$$h(a) \in A_j \quad \text{for all } a \in A_i. \tag{5}$$

We next show that the following holds.

$$\text{Let } a \in A_i,\ c \in C_i \text{ with } a \sim c. \text{ Then } h(a) \sim h(c). \tag{6}$$

For the proof, consider $b \in B_i$ with $a \sim b \sim c$. We first show $h(a) \sim h(b)$. Suppose that $h(a) \sim h(b)$ does not hold. Note that there exists a family $(P_v)_{v < \mu}$ of μ edge-disjoint paths of H_j connecting $h(a)$ with $h(b)$ since there are μ such paths of H_i connecting a with b. For each $v < \mu$, let e_v be the edge of P_v which is incident with $h(a)$. From the supposition that $h(a) \sim h(b)$ does not hold, together with (5), it follows that all edges e_v are in G_j and thus we have $d_{G_j}(h(a)) \geqslant \mu$, which contradicts (1). Hence $h(a) \sim h(b)$. Thus in order to obtain (6) it remains to show $h(b) \sim h(c)$. Suppose that this does not hold. Then one concludes from (4) and (5), together with $h(a) \sim h(b)$, that the path $h(bc)$ contains $h(a)$ as an inner vertex, which is impossible. Hence $h(b) \sim h(c)$.

We claim that

$$h(c) \neq h(c') \quad \text{for all } c, c' \in C_i,\ c \neq c'. \tag{7}$$

Suppose $h(c) = h(c')$ for distinct $c, c' \in C_i$. By (3), we have $h(c) \in C_j$. Denote by D the component of $H_j - h(c)$ containing the vertices of B_j; let e be the unique edge connecting $h(c)$ with D. Let $b, b' \in B_i$ with $b \sim c$, $b' \sim c'$. By (4) we have $h(b), h(b') \in D$ and thus we conclude from $h(c) = h(c')$ that e is an edge of both $h(cb)$ and $h(c'b')$, which is impossible. Hence (7).

As a consequence of (3), (5)–(7) one obtains

$$h(a) \neq h(a') \quad \text{for all } a, a' \in A_i,\ a \neq a'. \tag{8}$$

By (8), g is an immersion $G_i \to H_j$ in the strong sense. Further, by (5), g maps the vertices of G_j onto vertices of G_j. Moreover, note that it follows from the construction of H_j that a path of H_j is completely contained in G_j if its endvertices are in G_j. Hence (2). $\square$

3. Counterexamples to Conjectures A and B

In this section, based on ideas of [5], we construct a counterexample to Conjecture A. The existence of a counterexample to Conjecture B then follows from Theorem 1.

For an ordinal $\alpha, 2^\alpha$ denotes the set of mappings $x : \alpha \to \{0, 1\}$. We put $2^{<\alpha} = \bigcup_{\beta<\alpha} 2^\beta$. If $x \in 2^\alpha$ and $\beta \leqslant \alpha$, then $x \restriction \beta \in 2^\beta$ denotes the restriction of x to β. For $x_0 \in 2^\omega$ and $n \in \omega$, we put

$$U_n(x_0) = \{x \in 2^\omega : x \restriction n = x_0 \restriction n\}.$$

The set 2^ω is considered as a topological space endowed with the product topology, i.e. the topology whose basic open sets are the $U_n(x)$.

Let $\kappa_0, \kappa_1, \ldots$ be cardinals with $\mathfrak{c} \leqslant \kappa_0 < \kappa_1 < \cdots$. For each $x \in 2^\omega$ and $n \in \omega$, let $A_{x,n}$ be a set of cardinality κ_n such that $A_{x,n} \cap 2^{<\omega+1} = \emptyset$ and $A_{x,n} \cap A_{y,m} = \emptyset$ whenever $(x, n) \neq (y, m)$. For each $X \subseteq 2^\omega$, we define a graph G_X by

$$V(G_X) := \{x \restriction \alpha : x \in X, \alpha \in \omega + 1\} \cup \bigcup_{\substack{x \in X \\ n \in \omega}} A_{x,n},$$

$$E(G_X) := \bigcup_{\substack{x \in X \\ n \in \omega}} \{av : a \in A_{x,n}, v \in \{x, x \restriction n\}\}.$$

For $X \subseteq 2^\omega$ and $v \in V(G_X)$, we denote by $d_X(v)$ the degree of v in G_X. As a consequence of the definitions, one obtains $d_X(v) = \sup\{\kappa_i : i \in \omega\}$ if $v \in X, d_X(v) = |U_n(x) \cap X| \cdot \kappa_n = \kappa_n$ if $v = x \restriction n$ for $x \in X$ and $n \in \omega$, and $d_X(v) = 2$, otherwise. Hence, for $X, Y \subseteq 2^\omega$, we have

$$d_X(x) > d_Y(v) \quad \text{for all } x \in X \text{ and } v \in V(G_Y) \setminus Y. \tag{9}$$

Lemma 1. *For $X \subseteq 2^\omega$, let x, x_0 be distinct elements of X. Let further $n \in \omega$. Then there exists a system of κ_n edge-disjoint x, x_0-paths of G_X if and only if $x \in U_n(x_0)$.*

Proof. Assume $x \in U_n(x_0)$. Then $x_0 \restriction n = x \restriction n$. Let $f : A_{x_0,n} \to A_{x,n}$ be a bijection. For each $a \in A_{x_0,n}$, we define a path P_a of length four by putting $P_a = (x_0, a, x_0 \restriction n = x \restriction n, f(a), x)$. Then these paths form a system of κ_n edge-disjoint x, x_0-paths of G_X. For a proof of the 'only if' direction, assume that $x \notin U_n(x_0)$. Let P be an x, x_0-path of G_X. From the definition of G_X, together with the fact that $x \notin U_n(x_0)$, one readily obtains that P must contain a subpath $P' = (y, a, v, b, z)$ with $y \in X \setminus U_n(x_0)$, $z \in U_n(x_0)$, $v = y \restriction m = z \restriction m$, $a \in A_{y,m}$, $b \in A_{z,m}$ for some $m \in \omega$. Because $y \notin U_n(x_0)$ and $z \in U_n(x_0)$, we have $y \restriction n \neq z \restriction n$. Hence $m < n$ and thus we have proved that each x, x_0-path P of G_X must pass through a vertex v with $v = y \restriction m$ for some $y \in X$ and

$m<n$. Note that there are only finitely many vertices of this kind and each such vertex has degree at most κ_{n-1}. Thus there cannot exist κ_n edge-disjoint x, x_0-paths of G_X. □

Lemma 2. *For $X, Y \subseteq 2^\omega$, let $g: G_X \to G_Y$ be an immersion in the strong sense and denote by φ the restriction of g to X. Then φ is a continuous injective function $X \to Y$.*

Proof. By (9), we have $\varphi(X) \subseteq Y$. Clearly φ is injective and thus it remains to show that φ is continuous. For this purpose, let $x_0 \in X$. For some $n \in \omega$, consider the basic open neighborhood $U_n(\varphi(x_0))$. We have to show that there exists a natural number a_n such that

$$\varphi(U_{a_n}(x_0) \cap X) \subseteq U_n(\varphi(x_0)). \tag{10}$$

We show that this is true for $a_n = n$. Let $x \in U_n(x_0) \cap X$. If $x = x_0$, then $\varphi(x) \in U_n(\varphi x_0))$ clearly holds. Hence let $x \neq x_0$. By Lemma 1, x_0 and x are joined by κ_n edge-disjoint paths of G_X, and thus the same must hold for $\varphi(x_0)$ and $\varphi(x)$ in G_Y. Consequently, again by Lemma 1, $\varphi(x) \in U_n(\varphi(x_0))$. Hence (10). □

By a result of Thomas [5, Theorem 2] there exists a sequence $X_1, X_2, \ldots$ of subsets of 2^ω (each of cardinality $\mathfrak{c}$) such that for all i,j with $i<j$ there is no continuous injective function $X_i \to X_j$. By combining this with Theorem 1 and Lemma 2 of the present paper, we obtain the following result.

Theorem 2. *There exists a sequence $H_1, H_2, \ldots$ of uncountable graphs such that $H_i \nleq_2 H_j$ for all positive integers i,j with $i<j$.*

Proof. Let $X_1, X_2, \ldots$ be as given above. Put $G_i := G_{X_i}$ $(i = 1, 2, \ldots)$. Then, by Lemma 2, we have $G_i \nleq_1 G_j$ for all i,j with $i<j$ and thus Theorem 2 follows by application of Theorem 1. (It immediately follows from the proof of Theorem 1 that the resulting graphs $H_1, H_2, \ldots$ are uncountable.) □

4. Antichains

The following is a sharpened version of Theorem 2.

Theorem 3. *There exists a sequence $H'_1, H'_2, \ldots$ of uncountable graphs such that $H'_i \nleq_2 H'_j$ for all positive integers i,j with $i \neq j$.*

Proof. Let G_i $(i = 1, 2, \ldots)$ be as in the proof of Theorem 2. Then $G_i \nleq_1 G_j$ for all i,j with $i<j$. Note that a graph G_X (as defined in Section 3) is infinite and connected if X is non-empty. Consequently, the G_i are infinite connected graphs. Let $\lambda_1 < \lambda_2 \ldots$ be cardinals with $\lambda_1 > |V(G_i)|$ $(i = 1, 2, \ldots)$. For $i = 1, 2, \ldots,$ let G'_i result from G_i by adding λ_i new vertices of degree 0. Then $|V(G'_j)| < |V(G'_i)|$ for all j, i with $j < i$, and thus $G_i' \leqslant_1 G_j'$ is impossible if $j<i$. On the other hand, $G_i' \leqslant_1 G_j'$ is also impossible if $i<j$ since this clearly would imply $G_i \leqslant_1 G_j$. Hence $G_i' \nleqslant_1 G_j'$ for all i,j with $i \neq j$. Theorem 3 then follows from Theorem 1. (By the proof of Theorem 1, the resulting graphs $H_1', H_2', \ldots$ are uncountable.) $\square$

Note that the above proof of Theorem 3 yields disconnected graphs $H_1', H_2', \ldots$. In order to obtain *connected* graphs showing the same, one can proceed as in the proof of Theorem 3 with just one slight modification when defining G_i': just pick one of the λ_i new vertices and join it by edges to all other vertices of G_i'. Let $a_i \in V(G_i')$ be this vertex which now has degree λ_i $(i = 1, 2, \ldots)$. Then any immersion $G_i' \to G_j'$ (in the strong sense) would map a_i to a_j (because of their degrees), and hence G_i to G_j. But the former is impossible for $i>j$, while the latter is impossible for $i<j$. Application of Theorem 1 then yields the desired connected graphs $H_1', H_2', \ldots$. (Note that, by the proof of Theorem 1, the connectedness of the graphs H_i' is an immediate consequence of the connectedness of the graphs G_i'.)

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